Pythagorean triples using the relativistic velocity addition formula

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1. *The relativistic velocity addition formula*

In the special theory of relativity, there is an unusual formula for addition of velocities. We will use this formula to generate Pythagorean triples. We define a Pythagorean triple in the usual manner as a triple of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. The numbers a and b are called legs, and c is called the hypotenuse. Later, we will allow b to take on the value of any integer. A *primitive* Pythagorean triple is a Pythagorean triple for which a, b and c are relatively prime.

When students first learn about the special theory of relativity they are taught that there is no preferred inertial frame of reference. They are also taught that no object with a speed less than c, the speed of light, can ever have a speed greater than or equal to c. At this point, the question often arises: What if two spaceships are moving away from each other along a line, and both have a speed of $\frac{2}{3}c$ (with respect to an observer not on either ship)? See Figure 1.

FIGURE 1: Two spaceships moving away from each other

Then, in the frame of reference of one of the ships, isn't the other one moving at a speed of $\frac{2}{3}c + \frac{2}{3}c = \frac{4}{3}c$, which is greater than the speed of light, and *that* is not allowed?

The answer is that there is a weird addition formula for this situation. Specifically, if two objects are moving away from each other with speeds u and v, respectively, then, in the frame of reference of one of the objects, the other object is moving at a speed of

$$u \oplus v = \frac{u + v}{1 + \frac{uv}{c^2}}. \tag{1}$$

And so, if we set both u and v in (1) equal to $\frac{2}{3}c$, we get

$$\frac{2}{3}c \oplus \frac{2}{3}c = \frac{\frac{2}{3}c + \frac{2}{3}c}{1 + \frac{(\frac{2}{3}c)^2}{c^2}} = \frac{\frac{4}{3}c}{1 + \frac{4}{9}} = \frac{\frac{4}{3}c}{\frac{13}{9}} = \frac{12}{13}c. \tag{2}$$

Note that, as desired, this speed is less than the speed of light.

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When dealing with *non*-relativistic speeds, i.e. speeds in everyday life, u and v are much less than c, and therefore the $\frac{uv}{c^2}$ term in (1) can be neglected, and so (1) reduces to ordinary addition, as we would expect.

2. *Generating Pythagorean triples*

Note that the result we got in (2), ignoring the factor of c, is $\frac{12}{13}$, and the numerator and denominator of this fraction give a leg and the hypotenuse of the Pythagorean triple (5, 12, 13). This is not a coincidence.

Since we will be working with Pythagorean triples, which are often represented by (a, b, c), we will avoid using the letter c to represent the speed of light. We can do this by taking speeds to be multiples of the speed of light. For instance, a speed of 1 is the speed of light, a speed of $\frac{1}{2}$ is half the speed of light, and so forth. Then (1) may be written

$$u \oplus v = \frac{u + v}{1 + uv}. \tag{3}$$

Let us do another example. If we let $u = v = \frac{3}{4}$, we get

$$\frac{3}{4} \oplus \frac{3}{4} = \frac{\frac{3}{4} + \frac{3}{4}}{1 + \left(\frac{3}{4}\right)^2} = \frac{\frac{6}{4}}{\frac{25}{16}} = \frac{24}{25},$$

and the numerator and denominator of this fraction give a leg and the hypotenuse of the Pythagorean triple (7, 24, 25).

Let us generalise. We will let $u = v = \frac{n}{m}$, where m and n are positive integers and $m > n$. Substituting into (3), we get

$$\frac{n}{m} \oplus \frac{n}{m} = \frac{\frac{n}{m} + \frac{n}{m}}{1 + \left(\frac{n}{m}\right)^2} = \frac{\frac{2n}{m}}{1 + \frac{n^2}{m^2}} = \frac{2mn}{m^2 + n^2}.$$

The numerator and denominator give a leg and the hypotenuse of the well-known (see, for example, [1]) parametric form of a Pythagorean triple,

$$\left(m^2 - n^2,\ 2mn, m^2 + n^2\right). \tag{4}$$

So far, we have taken the case for which u and v are equal in (3). We can also take the case where they are not equal, but now, if we want the result to be associated with a Pythagorean triple, u and v must be fractions of the form

$$u = \frac{b_1}{c_1}; \qquad v = \frac{b_2}{c_2}, \tag{5}$$

where (a_1, b_1, c_1) and (a_2, b_2, c_2) are Pythagorean triples. This was first shown by Antonio Di Lorenzo in [2].

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For instance, we let $u = \frac{4}{5}$ and $v = \frac{5}{13}$. Then (3) gives

$$\frac{4}{5} \oplus \frac{5}{13} = \frac{\frac{4}{5} + \frac{5}{13}}{1 + \frac{4}{5} \cdot \frac{5}{13}} = \frac{\frac{77}{65}}{\frac{85}{65}} = \frac{77}{85}.$$

and this fraction gives a leg and the hypotenuse of the Pythagorean triple (36, 77, 85). In general, we substitute (5) into (3) and get

$$\frac{b_1}{c_1} \oplus \frac{b_2}{c_2} = \frac{\frac{b_1}{c_1} + \frac{b_2}{c_2}}{1 + \frac{b_1}{c_1} \cdot \frac{b_2}{c_2}} = \frac{b_1 c_2 + b_2 c_1}{c_1 c_2 + b_1 b_2}. \tag{6}$$

To show that the numerator is a leg and the denominator is the hypotenuse of a Pythagorean triple, we need to show that the difference of their squares is a perfect square, specifically that

$$(c_1 c_2 + b_1 b_2)^2 - (b_1 c_2 + b_2 c_1)^2 = (a_1 a_2)^2. \tag{7}$$

This is left as an exercise for the reader.

And so we have a binary operation that sends the Pythagorean triples (a_1, b_1, c_1) and (a_2, b_2, c_2) to the Pythagorean triple

$$(a_1 a_2, b_1 c_2 + b_2 c_1, c_1 c_2 + b_1 b_2). \tag{8}$$

It turns out that we can construct an abelian group consisting of fractions of the form $\frac{b}{c}$, where

$$(a, b, c) \in \{(a, b, c) : a^2 + b^2 = c^2,\ b \text{ is an integer, and } a \text{ and } c \text{ are positive integers}\}.$$

We consider any Pythagorean triple to be equivalent to the primitive triple of which it is a multiple. For instance, the triple (6, 8, 10) is equivalent to the primitive triple (3, 4, 5). The group operation is given by (6). The operation $\oplus$ in (3) is commutative and associative. The identity element is $\frac{0}{1} = 0$. The inverse of $\frac{b}{c}$ is $\frac{-b}{c}$. The proofs are left as an exercise for the reader.

3. *Variation of the addition formula*

Let us now return to the image of two spaceships moving along a line. Equation (3) gives the speed of one ship in the reference frame of the other if they are moving in opposite directions. We now imagine one of spaceships reverses direction, so that both ships are now moving in the same direction. If $u > v$ then the speed of one ship relative to the other is now given by

$$\frac{u - v}{1 - uv}. \tag{9}$$

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We now attempt to generate Pythagorean triples using (9). We substitute (5) into (9) and get

$$\frac{\frac{b_1}{c_1} - \frac{b_2}{c_2}}{1 - \frac{b_1}{c_1} \cdot \frac{b_2}{c_2}} = \frac{b_1c_2 - b_2c_1}{c_1c_2 - b_1b_2}.$$

By a computation analogous to (7) and (8), we get the Pythagorean triple $(a_1a_2, b_1c_2 - b_2c_1, c_1c_2 - b_1b_2)$.

4. *Another variation of the addition formula*

We now consider another variation of (3) and (9):

$$u \hat{\oplus} v = \frac{u + v}{1 - uv}. \tag{10}$$

This formula is purely mathematical, and does not represent a speed or velocity. It is reminiscent of the formula for the tangent of a sum of angles, $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$. We will use it to generate Pythagorean triples. For instance, if $u = v = \frac{2}{3}$, then (10) gives

$$\frac{2}{3} \hat{\oplus} \frac{2}{3} = \frac{\frac{2}{3} + \frac{2}{3}}{1 - \left(\frac{2}{3}\right)^2} = \frac{\frac{4}{3}}{\frac{5}{9}} = \frac{12}{5}.$$

The numerator and denominator of this fraction are the legs of the Pythagorean triple (5, 12, 13).

Let us generalise. We will let $u = v = \frac{n}{m}$, where m and n are positive integers and $m > n$. Substituting into (10), we get

$$\frac{n}{m} \hat{\oplus} \frac{n}{m} = \frac{\frac{n}{m} + \frac{n}{m}}{1 - \left(\frac{n}{m}\right)^2} = \frac{\frac{2n}{m}}{1 - \frac{n^2}{m^2}} = \frac{2mn}{m^2 - n^2}.$$

The numerator and denominator give the two legs of the Pythagorean triple given in (4).

So far we have taken the case for which u and v are equal in (10). We can also take the case where they are not equal, but now they must be fractions of the form

$$u = \frac{b_1}{a_1};\ v = \frac{b_2}{a_2}, \tag{11}$$

where (a_1, b_1, c_1) and (a_2, b_2, c_2) are Pythagorean triples.

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We substitute (11) into (10) and get

$$\frac{b_1}{a_1} \hat{\oplus} \frac{b_2}{a_2} = \frac{\dfrac{b_1}{a_1} + \dfrac{b_2}{a_2}}{1 - \dfrac{b_1}{a_1} \cdot \dfrac{b_2}{a_2}} = \frac{a_1 b_2 + a_2 b_1}{a_1 a_2 - b_1 b_2}. \tag{12}$$

To show that the numerator and denominator are legs of a Pythagorean triple, we need to show that the sum of their squares is a perfect square, specifically that $(a_1a_2 - b_1b_2)^2 + (a_1b_2 + a_2b_1)^2 = (c_1c_2)^2$. This is left as an exercise for the reader.

We have a binary operation that sends the Pythagorean triples (a_1, b_1, c_1) and (a_2, b_2, c_2) to the Pythagorean triple

$$(a_1a_2 - b_1b_2, a_1b_2 + a_2b_1, c_1c_2). \tag{13}$$

It turns out that we can construct an abelian group consisting of fractions of the form $\frac{b}{a}$, where

$(a, b, c) \in \{(a, b, c) : a^2 + b^2 = c^2,$ b is an integer, and a and c are positive integers$\}$.

As above, we consider any Pythagorean triple to be equivalent to the primitive triple of which it is a multiple. The group operation is given by (12). The operation $\hat{\oplus}$ in (10) is commutative and associative. The identity element is $\frac{0}{1} = 0$. The inverse of $\frac{b}{a}$ is $\frac{-b}{a}$. The proofs are left as an exercise for the reader.

5. *Yet another variation of the addition formula*

Our last variation of (3), (9), and (10) is

$$\frac{u - v}{1 + uv}. \tag{14}$$

It does not represent a speed or velocity. However, it is reminiscent of the formula for the angle between two lines having slopes m_1 and m_2 : $\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2}$.

It turns out that (10) and (14) generate the same Pythagorean triples. Let $u = \frac{a_1}{b_1}$ and $v = \frac{b_2}{a_2}$, where (a_1, b_1, c_1) and (a_2, b_2, c_2) are Pythagorean triples. Substituting into (14) gives

$$\frac{\dfrac{a_1}{b_1} - \dfrac{b_2}{a_2}}{1 + \dfrac{a_1}{b_1} \cdot \dfrac{b_2}{a_2}} = \frac{a_1a_2 - b_1b_2}{a_1b_2 + a_2b_1},$$

and this fraction is the reciprocal of the fraction in (12). So we get the Pythagorean triple given in (13), as desired.

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References

1. 'Pythagorean triple', Wikipedia, accessed September 2022.
2. A. Di Lorenzo, A relation between pythagorean triples and the special theory of relativity, accessed September 2022 at https://www.academia.edu/37574176/A_relation_between_pythagorean_triples_and_the_special_theory_of_relativity

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The answers to the *Nemo* page from March 2024 on momentum were:

1.	John Galsworthy	The Man of Property	Part 2, Chapter 8
2.	Henry James	The Wings of the Dove	Book 5, Chapter 2
3	Joseph Conrad	An Outcast of the Islands	Chapter 6
4.	Jack London	White Fang	Chapter 4
5.	George Eliot	Daniel Deronda	Chapter 21
6.	Mark Twain	Life on the Mississippi	Chapter 51

Congratulations to Martin Lukarevski on tracking all of these down. This month, Nemo revisits types of curve. Quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 23rd September 2024.

1. The vehicle had a square black tilt which nodded with the motion of the wheels, and at a point in it over the driver's head was a hook to which the reins were hitched at times, when they formed a catenary curve from the horse's shoulders.

2. One sees that it is rare --
   that striking grasp of opposites
   opposed each to the other, not to unity,
   which in cycloid inclusiveness
   has dwarfed the demonstration
   of Columbus with the egg --
   a triumph of simplicity --

3. My uncle Toby understood the nature of a parabola as well as any man in England – but was not quite such a master of the cycloid – he talked however about it every day

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