Can. J. Math., Vol. XXXI, No. 4, 1979, pp. 845–857

ON THE OPERATOR IDENTITY $\sum A_k X B_k \equiv 0$

C. K. FONG AND A. R. SOUROUR

Let A_j and B_j $(1 \leqq j \leqq m)$ be bounded operators on a Banach space $\mathfrak{X}$ and let Φ be the mapping on $\mathfrak{L}(\mathfrak{X})$, the algebra of bounded operators on $\mathfrak{X}$, defined by

$$(1) \quad \Phi(X) = A_1XB_1 + \ldots + A_mXB_m.$$

We give necessary and sufficient conditions for Φ to be identically zero or to be a compact map or (in the Hilbert space case) for the induced mapping on the Calkin algebra to be identically zero. These results are then used to obtain some results about inner derivations and, more generally, about mappings of the form

$$C(S, T) : X \rightarrow SX - XT.$$

For example, it is shown that the commutant of the range of $C(S, T)$ is "small" unless S and T are scalars.

1. Main results. Consider the mapping Φ defined by (1). We may arrange the operators A_j and B_j in such a way that, for some $n \leqq m$, the operators $B_1, \ldots, B_n$ form a maximal linearly independent subset of $B_1, \ldots, B_m$. Therefore there are constants c_{kj} $(1 \leqq k \leqq n$ and $n + 1 \leqq j \leqq m)$, such that

$$(2) \quad B_j = \sum_{k=1}^{n} c_{kj}B_k \qquad (n + 1 \leqq j \leqq m).$$

Our first result gives a necessary and a sufficient condition for Φ to be identically zero.

THEOREM 1. *The mapping Φ is identically zero if and only if:*

$$(3) \quad A_k = -\sum_{j=n+1}^{m} c_{kj}A_j \qquad (1 \leqq k \leqq n).$$

(In case $m = n$, the identity (2) becomes vacuous and the condition (3) should be interpretated as $A_1 = A_2 = \ldots = A_m = 0$.)

Vala [**9**] proved that, if A and B are nonzero operators in $\mathfrak{L}(\mathfrak{X})$, then the linear mapping which sends X to AXB is compact if and only if both A and B are compact. The following theorem may be regarded as a generalization of this result:

THEOREM 2. (i) *If the linear mapping Φ is compact and if $B_1, \ldots, B_n$ are linearly independent, then*

$$(4) \quad A_k + \sum_{j=n+1}^{m} c_{kj}A_j \qquad (1 \leqq k \leqq n)$$

Received February 6, 1978 and in revised form March 28, 1978. This research was supported by a grant from the National Research Council.

are compact operators. (Again, in case $m = n$, the conclusion should be interpretated as "$A_1, \ldots, A_m$ are compact operators".)

(ii) *The mapping Φ is compact if and only if there are compact operators $C_1, \ldots, C_r$ (respectively, $D_1, \ldots, D_r$) in the linear span of $A_1, \ldots, A_m$ (respectively, $B_1, \ldots, B_m$) such that*

$$\Phi(X) = C_1XD_1 + \ldots + C_rXD_r$$

for all X in $\mathfrak{L}(\mathfrak{X})$.

Now we let $A_1, \ldots, A_m, B_1, \ldots, B_m$ be operators on a separable (infinite dimensional) Hilbert space $\mathfrak{H}$. The mapping Φ from $\mathfrak{L}(\mathfrak{H})$ into itself is still defined by the identity (1). We suppose $\Phi(X)$ compact for all X in $\mathfrak{L}(\mathfrak{H})$. Thus if we write π for the natural projection into the Calkin algebra $\mathfrak{L}(\mathfrak{H})/\mathfrak{K}(\mathfrak{H})$ and $a_1, \ldots, a_m, b_1, \ldots, b_m$ for $\pi(A_1), \ldots, \pi(A_m), \pi(B_1), \ldots, \pi(B_m)$ respectively, then

$$(5) \qquad a_1xb_1 + \ldots + a_mxb_m = 0$$

for all x in the Calkin algebra. As before, we assume that $b_1, \ldots, b_n$ are linearly independent for some n and there are constants c_{kj}, $1 \leqq k \leqq n$ and $n + 1 \leqq j \leqq m$ such that

$$(6) \qquad b_j = \sum_{k=1}^{n} c_{kj}b_k \qquad (n + 1 \leqq j \leqq m).$$

Theorem 1 suggests that the identity (5) holds if and only if

$$a_k = \sum_{j=n+1}^{m} c_{kj}a_j \qquad (1 \leqq k \leqq n).$$

This turns out to be true and we rephrase it formally as follows:

THEOREM 3. *Let $A_1, \ldots, A_m, B_1, \ldots, B_m$ be operators on a separable Hilbert space $\mathfrak{H}$. Suppose $B_1, \ldots, B_n$ $(n \leqq m)$ are linearly independent modulo the compacts and there are constants c_{kj}, $1 \leqq k \leqq n$ and $n + 1 \leqq j \leqq m$, such that*

$$B_j = \sum_{k=1}^{m} c_{kj}B_k \text{ modulo the compacts } (n + 1 \leqq j \leqq m).$$

Then $A_1XB_1 + \ldots + A_mXB_m$ is compact for each X in $\mathfrak{L}(\mathfrak{H})$ if and only if

$$A_k = -\sum_{j=n+1}^{m} c_{kj}A_j \text{ modulo the compacts } (1 \leqq k \leqq n).$$

Here we point out that the proof of the above theorem depends on a beautiful result of Voiculescu [**10**].

2. Proofs. Now we start proving results that we have claimed in the previous section. For the proof of Theorem 1, we need the following technical lemma.

LEMMA 1. *Let $B, B_1, B_2, \ldots, B_n$ be operators in $\mathfrak{L}(\mathfrak{X})$. Then B is not in the linear span of $B_1, \ldots, B_n$ if and only if there are finitely many vectors $x_1, \ldots, x_r$*

in $\mathfrak{X}$ and equally many linear functionals $f_1, \ldots, f_r$ in $\mathfrak{X}^*$ (the dual space of $\mathfrak{X}$) such that

$$\textstyle\sum_{k=1}^{r} f_k(B_j x_k) = 0 \qquad (j = 1, \ldots, n)$$

while $\sum_{k=1}^{r} f_k(Bx_k) \neq 0$.

Proof. Let $\mathfrak{F}$ be the set of all linear functionals on $\mathfrak{L}(\mathfrak{X})$ of the form

$$X \to \textstyle\sum_{k=1}^{r} f_k(Xx_k)$$

where $x_1, \ldots, x_r$ are in $\mathfrak{X}$ and $f_1, \ldots, f_r$ are in $\mathfrak{X}^*$. Then $\mathfrak{F}$ is a linear space in $\mathfrak{L}(\mathfrak{X})^*$ which separates the points of $\mathfrak{L}(\mathfrak{X})$. That is, if $X, Y \in \mathfrak{L}(\mathfrak{X})$ and $X \neq Y$, then there exists a linear functional F in $\mathfrak{F}$ such that $F(X) \neq F(Y)$. By regarding $B, B_1, \ldots, B_n$ as linear functionals on $\mathfrak{F}$, the desired conclusion follows from the following result in [**4**, V. 3.10]: if $g, f_1, \ldots, f_n$ are any linear functionals on a linear space $\mathfrak{X}$ and if $f_i(x) = 0$ for $i = 1, \ldots, n$ implies $g(x) = 0$, then g is a linear combination of the f_i.

For convenience, we write $f \otimes x$ (where f is in $\mathfrak{X}^*$ and x is in $\mathfrak{X}$) for the operator on $\mathfrak{X}$ defined by $(f \otimes x)y = f(y)x$.

Proof of Theorem 1. First we consider the case that $B_1, \ldots, B_m$ are linearly independent. By Lemma 1, there exist finitely many vectors $x_1, \ldots, x_r$ in $\mathfrak{X}$ and linear functionals $f_1, \ldots, f_r$ in $\mathfrak{X}^*$ such that

$$\sum_{k=1}^{r} f_k(B_j x_k) = \begin{cases} 0 & \text{if } j = 2, \ldots, m, \\ 1 & \text{if } j = 1. \end{cases}$$

Then we have

$$0 = \textstyle\sum_{k=1}^{r} \Phi(f_k \otimes x)x_k = \sum_{k=1}^{r} \sum_{j=1}^{m} f_k(B_j x_k) A_j x = A_1 x.$$

Hence $A_1 = 0$. Similarly we have $A_2 = A_3 = \ldots = A_m = 0$.

Now, suppose that $B_1, B_2, \ldots, B_n$ are linearly independent and (2) holds. Then

$$\begin{aligned}\Phi(X) &= \textstyle\sum_{j=1}^{n} A_j X B_j + \sum_{j=n+1}^{m} A_j X (\sum_{k=1}^{n} c_{kj} B_k) = \sum_{k=1}^{n} A_k X B_k \\ &\quad + \textstyle\sum_{k=1}^{n} \sum_{j=n+1}^{m} c_{kj} A_j X B_k = \sum_{k=1}^{n} (A_k + \sum_{j=n+1}^{m} c_{kj} A_j) X B_k.\end{aligned}$$

Since $B_1, \ldots, B_n$ are linearly independent, we must have

$$A_k + \textstyle\sum_{j=n+1}^{m} c_{kj} A_j = 0.$$

The proof of the *if* part is straightforward and hence is omitted.

Proof of Theorem 2. The proof here is similar to that of Theorem 1. We start by considering the case when $B_1, \ldots, B_m$ are linearly independent and use Lemma 1 to obtain $x_1, \ldots, x_r$ in $\mathfrak{X}$ and $f_1, \ldots, f_r$ in $\mathfrak{X}^*$ such that

$$\textstyle\sum_{k=1}^{r} f_k(B_j x_k)$$

https://doi.org/10.4153/CJM-1979-080-x Published online by Cambridge University Press

vanishes for $j = 2, \ldots, m$ and equals 1 for $j = 1$. Again we have

$$\sum_{k=1}^{r} \Phi(f_k \otimes x)x_k = A_1 x.$$

Since Φ is a compact mapping, the set

$$\{\sum_{k=1}^{r} \Phi(f_k \otimes x)x_k : \|x\| \leqq 1\}$$

is precompact in $\mathfrak{X}$ and hence A_1 is a compact operator. The rest of the proof of (i) is similar to that of Theorem 1 and hence is omitted.

To prove the *only if* part of (ii), we observe that Φ can be written in the form $\Phi(X) = C_1XD_1 + \ldots + C_rXD_r$ where $C_1, \ldots, C_r$ (respectively, $D_1, \ldots, D_r$) form a linearly independent set in the linear span of $A_1, \ldots, A_m$ (respectively, of $B_1, \ldots, B_m$). (Here we assume that $\Phi \neq 0$; the case $\Phi = 0$ is trivial.) By (i), $C_1, \ldots, C_r$ are compact. It remains to show that $D_1, \ldots, D_r$ are compact. Since $C_1, \ldots, C_r$ are linearly independent, by Lemma 1, there exist $x_1, \ldots, x_s$ in $\mathfrak{X}$ and $f_1, \ldots, f_s$ in $\mathfrak{X}^*$ such that $\sum_{k=1}^{s} f_k(C_j x_k)$ vanishes for $j = 2, \ldots, r$ and equals 1 for $j = 1$. Then

$$\sum_{k=1}^{s} \Phi(f \otimes x_k)^* f_k = D_1^* f.$$

Since Φ is a compact map, the set

$$\{\sum_{k=1}^{s} \Phi(f \otimes x_k)^* f_k : \|f\| \leqq 1\}$$

is precompact in $\mathfrak{X}^*$. Hence D_1^* is compact and so is D_1. Similarly we can show that $D_2, \ldots, D_r$ are compact.

The *if* part follows from [**9**].

Proof of Theorem 3. The *if* part is easy to check and hence left to the reader. To prove the *only if* part, let $\mathfrak{A}$ be the separable C^*-algebra generated by $I, A_1, \ldots, A_m, B_1, \ldots, B_m$. Its image $\pi(\mathfrak{A})$ in the Calkin algebra is also a separable C^*-algebra. Let ρ be a faithful representation of $\pi(\mathfrak{A})$ on a separable Hilbert space $\mathfrak{H}_\rho$. Note that the direct sum of countably many copies of ρ, denoted by $\rho^{(\infty)}$, is also a representation of $\pi(\mathfrak{A})$ on a separable space, namely, $\mathfrak{H}_\rho^{(\infty)}$.

Now we can apply [**10**; Theorem 1.3] to obtain a unitary transformation $U : \mathfrak{H} \rightarrow \mathfrak{H} \oplus \mathfrak{H}_\rho^{(\infty)}$ such that, for each T in $\mathfrak{A}$, $T \oplus \rho^{(\infty)}(\pi(T)) - UTU^{-1}$ is compact. Since $\Phi(X)$ is compact for every X in $\mathfrak{L}(\mathfrak{H})$, the operator

$$\sum_{j=1}^{m} (A_j \oplus \rho^{(\infty)} \pi(A_j)) Y(B_j \oplus \rho^{(\infty)} \pi(B_j))$$

is compact for every Y in $\mathfrak{L}(\mathfrak{H} \oplus \mathfrak{H}_\rho^{(\infty)})$. By taking $Y = O_{\mathfrak{H}} \oplus Z^{(\infty)}$, where Z is in $\mathfrak{L}(\mathfrak{H}_\rho)$, we see that $W_Z^{(\infty)}$ is compact, where

$$W_Z = \sum_{j=1}^{m} \rho(\pi(A_j))Z\rho(\pi(B_j)).$$

But a compact operator of the form $A^{(\infty)}$ must be zero. Therefore $W_Z = O$ for every Z in $\mathfrak{L}(\mathfrak{H}_\rho)$. Since ρ is a faithful representation, we can complete the proof by applying Theorem 1.

From the proofs of Theorem 1 and Theorem 2, we can establish the following two similar results:

THEOREM 1′. *If* $A_1XB_1 + \ldots + A_mXB_m = 0$ *for all* X *in* $\mathfrak{L}(\mathfrak{X})$ *and* $A_1, \ldots, A_n$ $(n \leqq m)$ *are linearly independent, then* $B_1, \ldots, B_n$ *depend linearly on* $B_{n+1}, \ldots, B_m$.

THEOREM 2′. *If the mapping* $\Phi : \mathfrak{L}(\mathfrak{X}) \to \mathfrak{L}(\mathfrak{X})$ *defined by* $\Phi(X) = A_1XB_1 + \ldots + A_mXB_m$ *is compact and* $A_1, \ldots, A_n$ $(n \leqq m)$ *are linearly independent, then* $B_1, \ldots, B_n$ *are linear combinations of* $B_{n+1}, \ldots, B_m$ *plus compact operators.*

3. Special cases. For the purpose of illustration and affording facility for later applications, we consider some examples of special cases of the mapping Φ investigated in the previous sections. In the present section, we always assume the underlying Banach space is infinite dimensional.

Example 1. Assume that $\Phi_0 : \mathfrak{L}(\mathfrak{X}) \to \mathfrak{L}(\mathfrak{X})$ defined by $\Phi_0(X) = AX - XB$ is compact. If B is not a scalar multiple of I, then I and B are linearly independent and hence, by Theorem 2, A and $-I$ are compact, a contradiction to the fact that I is not compact. Hence B is a scalar multiple of I, say $B = \beta I$. Similarly A is a scalar multiple of I, say $A = \alpha I$. Then $\Phi_0(X) = (\alpha - \beta)X$. Since Φ_0 is compact, $\alpha = \beta$. We have proved the equivalence of the following three statements: (i) Φ_0 is compact, (ii) $\Phi_0 = 0$, and (iii) $A = B =$ scalar multiple of I.

Example 2. Suppose that $\Phi_1 : \mathfrak{L}(\mathfrak{X}) \to \mathfrak{L}(\mathfrak{X})$ defined by $\Phi_1(X) = AX + XB + CXD$ is compact. We may assume that C and D are not scalar multiple of I; otherwise, Φ_1 can be reduced to a map of the form Φ_0 considered in Example 1. Since I and D are linearly independent, by Theorem 2, C must be of the form $\lambda I + K$ for some scalar λ and some compact operator K. Hence $\Phi_1(X)$ becomes $AX + X(B + \lambda D) + KXD$. By our assumption, $K \neq 0$. Hence I and K are independent. By Theorem 2′, D must be of the form $\mu I + J$ for some scalar μ and compact J. Thus

$$\Phi_1(X) = (A + \mu K)X + X(B + \lambda J + \lambda\mu) + KXJ.$$

Since the map $X \to KXJ$ is compact, the map

$$X \to (A + \mu K)X + X(B + \lambda J + \lambda\mu)$$

is also compact. By example 1, we conclude that there are scalars α, β, λ, μ and compact operators K, J such that $A = \alpha I - \mu K$, $B = \beta I - \lambda J$, $C = \lambda I + K$, $D = \mu I + J$ and $\alpha + \beta + \lambda\mu = 0$. It is easy to check that these conditions are sufficient for Φ_1 to be compact.

Example 3. Suppose that the map Φ_1 defined in Example 2 is identically zero. Then, following an argument similar to, but simpler than that in Example 2, we deduce that either A, C are scalars or B, D are scalars.

Example 4. Assume that the mapping $\Phi_2 : \mathfrak{L}(\mathfrak{X}) \to \mathfrak{L}(\mathfrak{X})$ defined by $\Phi_2(X) = A_1X + XB_2 + A_3XB_3 + A_4XB_4$ is identically zero. We apply Theorem 1 and 1′ to this mapping with $m = 4$ and $B_1 = A_2 = I$.

Since $A_2 = I \neq 0$, the set $\mathfrak{B} = \{I, B_2, B_3, B_4\}$ must be linearly dependent. If I, B_2, B_3 are independent, then A_1, I, A_3 must be scalar multiples of A_4 and hence A_1, A_3, A_4 are scalars. In general, if there are three independent elements among $\mathfrak{B}$, then all elements in $\mathfrak{A} = \{A_1, I, A_3, A_4\}$ are scalars. Similarly, if there are three independent elements among $\mathfrak{A}$, then all elements in $\mathfrak{B}$ are scalars. Henceforth we assume that there are at most two independent elements in $\mathfrak{A}$ (respectively, in $\mathfrak{B}$).

If B_4 is a scalar, say $B_4 = \beta_4 I$, then $\Phi_2(X)$ becomes $(A_1 + \beta_4A_4)X + XB_2 + A_3XB_3$ and hence it reduces to a map of the form considered in Example 3. Therefore either B_2, B_3 are scalars or $A_1 + \beta_4A_4, A_3$ are scalars.

Now we assume that B_4 is not a scalar. Then $B_1(= I), B_4$ are independent and hence by our earlier assumption, B_2 and B_3 are linear combinations of I and B_4, say

$$(*) \qquad \begin{aligned} B_2 &= \alpha I + \beta B_4 \\ B_3 &= \gamma I + \delta B_4. \end{aligned}$$

By Theorem 1, we have

$$(**) \qquad \begin{aligned} A_1 &= -\alpha I - \gamma A_3 \\ A_4 &= -\beta I - \delta A_3. \end{aligned}$$

We conclude that if Φ_2 is identically zero, then one of the following four cases must occur: (i) all elements in $\mathfrak{A}$ are scalar multiples of I, (ii) all elements in $\mathfrak{B}$ are scalar multiples of I, (iii) A_3 and B_4 are scalars, say $A_3 = \alpha_3 I$, $B_4 = \beta_4 I$, and $A_1 + \beta_4A_4 = -B_2 - \alpha_3B_3 =$ scalar multiple of I, and (iv) there exist scalars $\alpha, \beta, \gamma, \delta$ such that both (*) and (**) are satisfied.

4. Applications. Let $\mathfrak{X}$ be a Banach space and S, T be operators on $\mathfrak{X}$. We denote by $C(S, T)$ the mapping from $\mathfrak{L}(\mathfrak{X})$ into itself defined by

$$C(S, T)X = SX - XT.$$

Note that in case T is a scalar multiple of I, say $T = \lambda I$, then $C(S, T)$ is the left multiplication by $\lambda I - S$ and in case S is a scalar, say $S = \mu I$, then $C(S, T)$ is the right multiplication by $T - \mu I$. Also note that $C(S, S)$ is the derivation δ_S.

Let $\mathfrak{C} = \{C(S, T) : S, T \in \mathfrak{L}(\mathfrak{X})\}$. It is easy to see that $\mathfrak{C}$ is closed under Lie multiplication; in fact, for S, T, S', T' in $\mathfrak{L}(\mathfrak{X})$, we have

$$[C(S, T), C(S', T')] = C([S, S'], [T, T'])$$

(where $[X, Y]$ is the commutator $XY - YX$). However, in view of the following proposition, the usual product of elements in $\mathfrak{C}$ is rarely in $\mathfrak{C}$.

PROPOSITION 1. *Let S_1, S_2, T_1, T_2 be in $\mathfrak{L}(\mathfrak{X})$. Then*

$$C(S_2, T_2)C(S_1, T_1) = C(S, T)$$

for some S, T if and only if one of the following three cases occurs: (i) *Both $C(S_1, T_1)$ and $C(S_2, T_2)$ are left multiplications,* (ii) *both $C(S_1, T_1)$ and $C(S_2, T_2)$ are right multiplications, and* (iii) *$C(S_1, T_1)$, $C(S_2, -T_2)$ and the identity map I are linearly dependent.*

Proof. Obviously, if either (i) or (ii) is satisfied, then $C(S_2, T_2)C(S_1, T_1)$ is a left or right multiplication. Suppose that (iii) is satisfied. Then there are scalars α, β, γ such that

$$|\alpha| + |\beta| + |\gamma| \neq 0 \quad \text{and}$$
$$\alpha C(S_1, T_1) + \beta C(S_2, -T_2) + \gamma I = 0.$$

If $\alpha = 0$, then $C(S_2, -T_2)$ is a scalar multiplication and hence by Example 1 in § 3 it is easy to see that $C(S_2, T_2)$ is also a scalar multiplication. If $\alpha \neq 0$, then

$$C(S_1, T_1) = \mu I + \lambda C(S_2, -T_2)$$

where $\mu = -\alpha^{-1}\gamma$ and $\lambda = -\alpha^{-1}\beta$ and hence we have

$$\begin{aligned} C(S_2, T_2)C(S_1, T_1) &= \mu C(S_2, T_2) + \lambda C(S_2, -T_2)C(S_2, T_2) \\ &= \mu C(S_2, T_2) + \lambda C(S_2^2, T_2^2) = C(\mu S_2 + \lambda S_2^2, \mu T_2 + \lambda T_2^2). \end{aligned}$$

Thus we have shown the *if* part.

To prove the *only if* part, we assume $C(S_2, T_2)C(S_1, T_1) = C(S, T)$. Applying both sides of this identity to X, we obtain:

$$(S_2S_1 - S)X + X(T_1T_2 + T) - S_1XT_2 - S_2XT_1 = 0.$$

By applying the conclusion of Example 4 in § 3 with $A_1 = S_2S_1 - S$, $A_3 = -S_1$, $A_4 = -S_2$, $B_2 = T_1T_2 + T$, $B_3 = T_2$ and $B_4 = T_1$, we arrive at the following four cases: (a) S_1 and S_2 are scalars, (b) T_1 and T_2 are scalars, (c) S_1 and T_1 are scalars, and (d) there exist scalars β, γ, δ such that $S_2 = \beta I - \delta S_1$ and $T_2 = \gamma I + \delta T_1$. Obviously, case (a) implies (i) and case (b) implies (ii). In case (c), $C(S_1, T_1)$ is a scalar multiplication and hence (iii) holds. In case (d), we have $C(S_2, -T_2) = (\beta + \gamma)I - \delta C(S_1, T_1)$ and hence (iii) holds. The proof is complete.

In what follows, we say that a pair of operators (A, B) intertwines an operator Y if $AY = YB$. For a set $\mathfrak{S}$ of operators, we write $\mathfrak{S}'$ for its commutant, that is, that set of all operators which commute with each operator in $\mathfrak{S}$. For an operator T, we write $\mathfrak{R}T$ for its range. The following consequence of Proposition 1 indicates that, in general, the range of $C(S, T)$ is large.

COROLLARY 1. *Suppose that neither S nor T is a scalar. Then the linear space $\mathfrak{S}$*

of all pairs (A, B) in the product space $\mathfrak{L}(\mathfrak{X}) \times \mathfrak{L}(\mathfrak{X})$ which intertwine every operator in the range of $C(S, T)$ is at most two dimensional.

Proof. Note that $\mathfrak{S} = \{(A, B) : C(A, B)C(S, T) = 0\}$. By Proposition 1, $C(A, -B)$ is a linear combination of $C(S, T)$ and the identity mapping. By Theorem 1, (A, B) is in the subspace spanned by $(S, -T)$, $(I, 0)$ and (I, I). But the pair $(I, 0)$ intertwines only the zero operator, so $\mathfrak{S}$ is a proper subspace of the three dimensional subspace indicated.

For the commutant of the range of $C(S, T)$, we have the following more precise result.

COROLLARY 2. *Suppose that S and T are not both scalars. Then $(\Re C(S, T))'$ is either $\mathbf{C}I$ or $\mathbf{C}I + \mathbf{C}T$. Furthermore, $(\Re C(S, T))' = \mathbf{C}I + \mathbf{C}T$ if and only if the following condition is satisfied:*

(*C*) *There exist scalars λ and α such that $S + T = \lambda I$ and $T^2 - \lambda T + \alpha I = 0$.*

(Note that condition (C) implies $S^2 - \lambda S + \alpha I = 0$.)

Proof. Suppose $A \in (\Re C(S, T))'$ and A is not a scalar. Then $C(A, A)$ cannot be a left multiplication nor a right multiplication. Since $C(A, A)C(S, T) = 0$, by Proposition 1, $C(A, -A)$, $C(S, T)$ and I are linearly dependent. Since $C(A, -A)$ is not a scalar, $C(S, T)$ must be a linear combination of I and $C(A, -A)$. Hence there exist constant β, γ, δ such that $S = \beta I + \delta A$ and $T = \gamma I - \delta A$. Since at least one of S and T is not a scalar, $\delta \neq 0$. Multiplying A by a suitable constant, we may assume that $\delta = 1$. By a straightforward computation, we obtain

$$C(A, A)C(S, T)X = ((\gamma - \beta)A - A^2)X - X((\gamma - \beta)A - A^2).$$

Hence $A^2 - (\gamma - \beta)A + \epsilon I = 0$ for some ϵ. Let $\lambda = \beta + \gamma$. Then it is easy to check that $S + T = \lambda I$ and $S^2 - \lambda S + \alpha I = 0$. Hence condition (C) is satisfied.

Conversely, if condition (C) is satisfied, then it is straightforward to check that $C(T, T)C(S, T) = O$ and hence $\mathbf{C}I + \mathbf{C}T \subset (\Re C(S, T))'$.

COROLLARY 3 [**11**; Corollary 1 in § 1]. *If an operator T is not a scalar multiple of I, then $(\Re\delta_T)' = \mathbf{C}I$.*

COROLLARY 4. *Suppose that S and T are nonscalar operators. Then the following conditions are equivalent:*

(i) *There exist S', T' in $\mathfrak{L}(\mathfrak{X})$ such that*

$$C(S', T')C(S, T) = I.$$

(ii) *There exist operators S'', T'' such that*

$$C(S, T)C(S'', T'') = I.$$

(iii) *There exist scalars λ, α, β with $\alpha \neq \beta$ such that*

$$S^2 - \lambda S + \alpha I = 0 \text{ and } T^2 - \lambda T + \beta I = 0.$$

https://doi.org/10.4153/CJM-1979-080-x Published online by Cambridge University Press

Proof. If condition (iii) is satisfied, then we let $S' = (\beta - \alpha)^{-1}S$ and $T' = (\beta - \alpha)^{-1}(\lambda I - T)$ and it is straightforward to check that both (i) and (ii) are satisfied.

Suppose that (i) is satisfied. By Proposition 1, $C(S', T')$ is a linear combination of $C(S, -T)$ and I, say $C(S', T') = \gamma C(S, -T) + \delta I$ for some scalars γ, δ. Therefore

$$I = \gamma C(S, -T)C(S, T) + \delta C(s, T) = \gamma C(S^2, T^2) + \delta C(S, T)$$
$$= C(\gamma S^2 + \delta S, \gamma T^2 + \delta T).$$

Since S and T are not scalars, we have $\gamma \neq 0$. Now it is easy to check that (iii) holds. In the same way (iii) follows from (ii).

The next proposition generalizes [**11**; Theorem 1] from the separable Hilbert space case to the Banach space case. First we need a technical lemma.

LEMMA 2. *Let A, B be in $\mathfrak{L}(\mathfrak{X})$. If Ax and Bx are linearly dependent for all x in $\mathfrak{X}$, then A, B are linearly dependent.*

The proof of this lemma is elementary and hence omitted.

PROPOSITION 2. *Let δ_1, δ_2 be two derivations on $\mathfrak{L}(\mathfrak{X})$. If $\delta_2\delta_1$ is also a derivation, then either $\delta_1 = 0$ or $\delta_2 = 0$.*

Proof. By the assumption, we have

$$(\delta_2\delta_1)(XY) = (\delta_2\delta_1(X))Y + X\delta_2\delta_1(Y).$$

On the other hand

$$(\delta_2\delta_1)(XY) = \delta_2(\delta_1(X)Y + X\delta_1(Y)) = \delta_2\delta_1(X)Y + \delta_1(X)\delta_2(Y)$$
$$+ \delta_2(X)\delta_1(Y) + X\delta_1\delta_2(Y).$$

Hence we have

$$(*) \quad \delta_1(X)\delta_2(Y) + \delta_2(X)\delta_1(Y) = 0$$

for all X and Y. Replacing X by ZX in (*), we obtain

$$\delta_1(Z)X\delta_2(Y) + Z\delta_1(X)\delta_2(Y) + \delta_2(Z)X\delta_1(Y) + Z\delta_2(X)\delta_1(Y) = 0.$$

Hence, by (*), we arrive at

$$(**) \quad \delta_1(Z)X\delta_2(Y) + \delta_2(Z)X\delta_1(Y) = 0$$

for all X, Y, Z in $\mathfrak{L}(\mathfrak{X})$. (The argument up to here is taken from Theorem 1 in [**11**].)

Now, by using Theorem 1 and Lemma 2, we see that δ_1 and δ_2 are linearly dependent. Thus either $\delta_1 = 0$ or $\delta_2 = \lambda\delta_1$ for some scalar λ. Suppose $\delta_2 = \lambda\delta_1$. Then (**) becomes $2\lambda\delta_1(Z)X\delta_1(Y) = 0$. Hence either $\lambda = 0$ or $\delta_1 = 0$. Therefore $\delta_2 = 0$.

https://doi.org/10.4153/CJM-1979-080-x Published online by Cambridge University Press

In the following proposition and its corollary, we assume that the underlying space is a separable infinite dimensional Hilbert space denoted by $\mathfrak{H}$.

PROPOSITION 3. *If δ_1 and δ_2 are two derivations on the Calkin algebra $\mathfrak{L}(\mathfrak{H})/\mathfrak{K}(\mathfrak{H})$ and $\delta_2\delta_1$ is also a derivation, then either $\delta_1 = 0$ or $\delta_2 = 0$.*

The proof is exactly the same as that of Proposition 2 except using Theorem 2 instead of Theorem 1 and hence omitted.

COROLLARY [**5**; Theorem 1]. *If A, B are nonscalar operators on $\mathfrak{H}$ and $\delta_B\delta_A$ is a compact mapping, then there are constants α, β such that $A - \alpha I$, $B - \beta I$ are compact operators and $(A - \alpha I)(B - \beta I) = (B - \beta I)(A - \alpha I) = 0$.*

(The proof given in [**5**] has a gap.)

Proof. By Theorem 2, $\delta_B\delta_A(X)$ is a compact operator for each X in $\mathfrak{L}(\mathfrak{H})$. Hence, as a mapping on the Calkin algebra, $\delta_b\delta_a$ is identically zero, where $a = \pi(A)$ and $b = \pi(B)$. By Proposition 3, we have either $\delta_a = 0$ or $\delta_b = 0$. On the other hand, since I, A are linearly independent and since

$$\delta_B\delta_A(X) = BAX - BXA - AXB - XBA,$$

from Theorem 2 we see that B must be a linear combination of I and A plus a compact operator. Now it is easy to see that both A and B are scalars modulo the compacts, say $A = \alpha I + K$ and $B = \beta I + J$ where K and J are compact operators. Since

$$\delta_B\delta_A(X) = JKX - JXK - KXJ + XKJ$$

and since the mapping $X \to JXK + KXJ$ is compact, so is the map $X \to JKX + XKJ$. By Example 1 in § 3, we have $JK = KJ = 0$.

It is known that the spectrum $\sigma(C(S, T))$ of $C(S, T)$ is $\sigma(S) - \sigma(T)$. (See [**6**].) Hence if $C(S, T)$ is quasi-nilpotent, then there exists a scalar λ such that $S - \lambda I$ and $T - \lambda I$ are quasinilpotent. The following proposition shows that if, in addition, $C(S, T)$ is nilpotent, then so are $S - \lambda I$ and $T - \lambda I$.

PROPOSITION 4. *Let $\mathfrak{X}$ be an infinite dimensional Banach space and S, T be in $\mathfrak{L}(\mathfrak{X})$. Then the following three conditions are equivalent:*

(i) *$C(S, T)$ is nilpotent.*

(ii) *There exists a positive integer n such that $C(S, T)^n$ is a compact operator.*

(iii) *There exists a scalar λ such that $S - \lambda I$ and $T - \lambda I$ are nilpotent.*

Proof. That (i) implies (ii) is obvious. Suppose (ii) holds. Note that

$$(*) \qquad C(S, T)^n X = \sum_{k=0}^{n} \binom{n}{k} S^k X(-T)^{n-k}$$

where $S^0 = (-T)^0 = I$. By Theorem 2, $I, S, S^2, \ldots, S^n$ must be linearly dependent. Hence S is an algebraic operator. Similarly T is an algebraic operator. To show (iii), it suffices to show that $\sigma(S) = \sigma(T) = \{\lambda\}$ for some scalar

λ. Let λ be an eigenvalue of T such that the spectral manifold $\mathfrak{M}$ associated with λ is infinite dimensional. We have $(T - \lambda I)^n = 0$ on $\mathfrak{M}$. It is easy to see that there is a projection F of infinite rank with its range in $\mathfrak{M}$ such that $FT^kF = \lambda^k F$ for $k = 1, 2, \ldots$. Now let μ be an eigenvalue of S. It is enough to show that $\mu = \lambda$. Let E be a projection onto the eigenspace

$$\{x \in \mathfrak{X} : Sx = \mu x\}.$$

Then $SE = \mu E$. Now

$$\begin{aligned}(C(S, T)^n EXF)F &= \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} S^k EXFT^{n-k}F \\ &= \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\mu^k EXF \,.\, \lambda^{n-k} = (\mu - \lambda)^n EXF.\end{aligned}$$

Since $E \neq 0$ and F is not compact, the compactness of the map

$$X \rightarrow (C(S, T)^n EXF)F$$

forces $\mu - \lambda$ to be zero.

That (iii) implies (i) follows from the identity (*) with S and T replaced by $S - \lambda I$ and $T - \lambda I$ respectively on the right hand side. The proof is complete.

Anderson and Foias [**1**] showed that if P is a self-adjoint projection on a Hilbert space such that $0 \neq P \neq I$, then δ_P is a hermitian operator on $\mathfrak{L}(\mathfrak{H})$ while $\delta_P{}^2$ is not hermitian. The following proposition shows that if A is a non-scalar self-adjoint operator on a Hilbert space, then δ_A is hermitian while $\delta_A{}^2$ is not hermitian. Its proof depends on the fact that an operator on $\mathfrak{L}(\mathfrak{H})$ is a hermitian operator if and only if it is of the form $C(S, T)$ where S and T are self-adjoint operators on $\mathfrak{H}$. This follows from a result of Sinclair [**8**] which characterizes the hermitian elements in a C^*-algebra.

PROPOSITION 5. *If $\mathfrak{H}$ is a Hilbert space, Φ is an operator on $\mathfrak{L}(\mathfrak{H})$ and Φ and Φ^2 are hermitian, then Φ is either a left multiplication or a right multiplication by a self-adjoint operator on $\mathfrak{H}$.*

Proof. Suppose that both Φ and Φ^2 are hermitian. By the aforementioned result of Sinclair, there exist self-adjoint operators S, T, A, B such that $\Phi(X) = SX - XT$ and $\Phi^2(X) = AX - XB$ for all X in $\mathfrak{L}(\mathfrak{H})$. Now $\Phi^2(X) = S^2X - 2SXT + XT^2$. Hence we obtain

$$(S^2 - A)X + X(T^2 + B) - 2SXT = 0.$$

By Example 3 in § 3, we see that either S or T is a scalar.

5. A conjecture. In this final section we raise a question related to Theorem 3. Suppose that $\mathfrak{H}$ is a separable, infinite dimensional Hilbert space, a_1, $a_2, \ldots, a_m, b_1, \ldots, b_m$ are elements in the Calkin algebra $\mathfrak{L}(\mathfrak{H})/\mathfrak{K}(\mathfrak{H})$ and the mapping φ defined on the Calkin algebra by $\varphi(x) = a_1xb_1 + \ldots + a_mxb_m$ is compact. We ask if φ has to be identically zero.

We mention three facts in support of an affirmative answer to this question.

First, if there exist $A_1, \ldots, A_m, B_1, \ldots, B_m$ which are preimages of $a_1, \ldots, a_m, b_1, \ldots, b_m$ in $\mathfrak{L}(\mathfrak{H})$ respectively such that that the mapping Φ defined on $\mathfrak{L}(\mathfrak{H})$ by $\Phi(X) = A_1XB_1 + \ldots + A_mXB_m$ is compact, then, by Theorem 2, $\varphi = 0$.

Second, the conjecture is true for $m = 1$. (To prove this, let A, B be two non-compact operators. We have to show that the mapping φ defined on the Calkin algebra by $\varphi(x) = \pi(A)x\pi(B)$ is not compact. By using polar decompositions, we can assume that both A and B are positive. Since A is not compact, there is a projection E of infinite rank and a positive number λ such that $AE = EA \geqq \lambda E$. Similarly there is a projection F of infinite rank such that $BF = FB \geqq \mu F$ for some $\mu > 0$. Then it is easy to see that $B'B = BB' = F$ and $A'A = AA' = E$ for some self-adjoint operators B' and A' such that $||A'|| \leqq \lambda^{-1}$ and $||B'|| \leqq \mu^{-1}$. Let V be a partial isometry such that $VV^* = E$ and $V^*V = F$. Let $\{E_n\}$ be a sequence of projections of infinite rank such that $E_nE_m = 0$ for $n \neq m$ and $\sum_n E_n = E$. Let X_n be $A'E_nVB'$. Then $\lambda\mu||X_n|| \leqq 1$ and, for $n \neq m$,

$$\begin{aligned}||\varphi(\pi(X_n)) - \varphi(\pi(X_m))|| &= ||\pi(AX_nB) - \pi(AX_mB)|| \\ &= ||\pi(E_nV) - \pi(E_mV)|| = ||\pi(E_n) - \pi(E_m)|| \geqq 1.\end{aligned}$$

Hence φ is not compact.)

Third, if φ is of the form $x \to ax - xb$, then the conjecture is true. (*Proof.* Let e be a self-adjoint idempotent. Then the mapping $x \to -\varphi(x(1-e))e = x(1-e)be$ is compact. By the previous result, $(1-e)be = 0$. Therefore b commutes with each self-adjoint idempotent and hence so does $b + b^*$. If $b + b^*$ is not a scalar, then $b + b^*$ can be written as $\lambda_1e_1 + \lambda_2e_2 + e_3(b + b^*)e_3$ where e_1, e_2 and e_3 are mutually orthogonal self-adjoint idempotents and $\lambda_1 \neq \lambda_2$. Let v be a partial isometry in the Calkin algebra such that $v^*v = e_1$ and $vv^* = e_2$. Let $e = \frac{1}{2}(e_1 + e_2 + v + v^*)$. Then $e^2 = e = e^*$ but $e(b + b^*) \neq (b + b^*)e$. Hence $b + b^*$ is a scalar. Similarly $b - b^*$ is a scalar. Therefore b is a scalar, say, $b = \lambda 1$. Now $\varphi(x)$ becomes $(a - \lambda 1)x$. By the previous result again we have $\varphi = 0$.)

We would like to thank K. R. Davidson for pointing out an error and J. P. Williams for his helpful remarks.

References

1. J. Anderson and C. Foias, *Properties which normal operators share with normal derivations and related operators*, Pac. J. Math., *61* (1975), 313–325.
2. J. W. Calkin, *Two sided ideals and congruences in the ring of bounded operators in Hilbert space*, Ann. of Math., *42* (1941), 839–873.
3. C. Davis and P. Rosenthal, *Solving linear operator equations*, Can. J. Math., *26* (1974), 1384–1389.

4. N. Dunford and J. T. Schwartz, *Linear operators. Part I*, (Interscience, New York, 1958).
5. Y. Ho, *A note on derivations*, Bull. Inst. Math. Acad. Sinica, *5* (1977), 1–5.
6. G. Lumer and M. Rosenblum, *Linear operator equations*, Proc. Amer. Math. Soc., *10* (1959), 32–41.
7. M. Rosenblum, *On the operator equation* $BX - XA = Q$, Duke Math. J. *23* (1956), 263–269.
8. A. M. Sinclair, *Jordan homomorphisms and derivations on semi-simple Banach algebra*, Proc. Amer. Math. Soc., *24* (1970), 209–214.
9. K. Vala, *On compact sets of compact operators*, Ann. Acad. Sci. Fenn. AI No. *351* (1964).
10. D. Voiculescu, *A non-commutative Weyl-von Neumann theorem*, Rev. Roumaine Math. Pures Appl., *21* (1976), 97–113.
11. J. P. Williams, *On the range of a derivation*, Pac. J. Math., *38* (1971), 273–279.

University of Toronto,
Toronto, Ontario;
University of Victoria,
Victoria, British Columbia