Can. J. Math., Vol. XXIII, No. 4, 1971, pp. 644–652

DISCONJUGACY CRITERIA FOR NONSELFADJOINT DIFFERENTIAL EQUATIONS OF EVEN ORDER

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1. Introduction. Disconjugacy criteria have been established for linear selfadjoint differential equations of order $2n$ by Sternberg [**4**] and Ahlbrandt [**1**]. Such differential equations can be written in the form

$$\sum_{k=0}^{n} (-1)^k (P_k(x) v^{(k)})^{(k)} = 0 \tag{1.1}$$

where it is assumed that the coefficients are real and that $P_n(x) \neq 0$. We shall be interested in nontrivial solutions $v(x)$ of (1.1), which satisfy

$$v(\alpha) = v'(\alpha) = \ldots = v^{(n-1)}(\alpha) = 0 = v(\beta) = v'(\beta) = \ldots = v^{(n-1)}(\beta) \tag{1.2}$$

for distinct points α and β. The smallest $\beta > \alpha$ such that (1.2) is satisfied nontrivially by a solution of (1.1), is denoted by $\mu_1(\alpha)$ and called the first conjugate point of $x = \alpha$ with respect to (1.1). If no such conjugate point exists we write $\mu_1(\alpha) = \infty$, and say that (1.1) is disconjugate on $[\alpha, \infty)$.

The principal purpose of this paper is to generalize these disconjugacy criteria to the general linear *nonselfadjoint* differential equation of the form

$$\sum_{k=0}^{n} (-1)^k (p_k(x) u^{(k)})^{(k)} + \sum_{k=0}^{n-1} (-1)^k (q_k(x) u^{(k+1)})^{(k)} = 0. \tag{1.3}$$

The smallest $\gamma > \alpha$ such that

$$u(\alpha) = \ldots = u^{(n-1)}(\alpha) = 0 = u(\gamma) = \ldots = u^{(n-1)}(\gamma) \tag{1.4}$$

is satisfied nontrivially by a solution of (1.3) is denoted by $\eta_1(\alpha)$ and called the first conjugate point of α with respect to (1.3). Our disconjugacy criteria for (1.3) will follow from a comparison theorem for conjugate points which establishes conditions on the coefficients of (1.1) and (1.3) assuring that $\eta_1(\alpha) \geqq \mu_1(\alpha)$.

In order to establish such a comparison theorem for conjugate points, it is necessary to transform (1.1) and (1.3) into first order vector systems of the type studied extensively by W. T. Reid. Such systems are considered in § 2 where a nonoscillation criterion of Reid [**6**] is generalized to nonselfadjoint vector systems. This nonoscillation theorem is the basis for the disconjugacy criteria established in § 3.

Received October 9, 1970 and in revised form, February 1, 1971. This research was supported by a grant of the National Science Foundation, NSF GP-11219.

2. First order vector and matrix systems. Reid [**6**] has established necessary and sufficient conditions for the nonoscillation of differential systems of the form

$$\begin{aligned} \mathbf{v}' &= A(x)\mathbf{v} + B(x)\mathbf{z} \\ \mathbf{z}' &= C(x)\mathbf{v} + D(x)\mathbf{z} \end{aligned} \tag{2.1}$$

under the selfadjointness conditions

$$B(x) \equiv B^*(x),\ C(x) \equiv C^*(x),\ A(x) \equiv -D^*(x). \tag{2.2}$$

Here $\mathbf{v}(x)$ and $\mathbf{z}(x)$ are n-dimensional vector functions and $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are to be real integrable $n \times n$ matrices (i.e., with real integrable elements) defined on some interval X of the real line. Two distinct points α and β on X are said to be *mutually conjugate* with respect to (2.1) if there exists a solution $(\mathbf{v}(x), \mathbf{z}(x))$ of (2.1) with $\mathbf{v}(\alpha) = \mathbf{v}(\beta) = 0$ and $\mathbf{v}(x) \not\equiv 0$ on (α, β). The system (1.1) is said to be *nonoscillatory* on a subinterval X_0 if no two distinct points of X_0 are mutually conjugate.

The principal result of this section is a comparison theorem for conjugate points which allows one to generalize Reid's nonoscillation criteria to vector systems of the form

$$\begin{aligned} \mathbf{u}' &= a(x)\mathbf{u} + b(x)\mathbf{w} \\ \mathbf{w}' &= c(x)\mathbf{u} + d(x)\mathbf{w} \end{aligned} \tag{2.3}$$

where $a(x)$, $b(x)$, $c(x)$, and $d(x)$ are to be real integrable $n \times n$ matrices which need not satisfy selfadjointness conditions such as (2.2).

In addition to (2.1) and (2.3), we shall make reference to selfadjoint matrix systems of the form

$$\begin{aligned} V' &= A(x)V + B(x)Z, \\ Z' &= C(x)V - A^*(x)Z, \end{aligned} \tag{2.4}$$

whose coefficients satisfy the selfadjointness conditions (2.2). Here $V(x)$ and $Z(x)$ are to be $n \times n$ matrices for $x \in X$. Letting $\mathbf{v}_i(x)$ and $\mathbf{z}_i(x)$ denote the ith columns of $V(x)$ and $Z(x)$, respectively, it follows that $(V(x), Z(x))$ is a solution of (2.4) if and only if $(\mathbf{v}_i(x), \mathbf{z}_i(x))$ is a solution of (2.1) for $i = 1, \ldots, n$. The following well known results for (2.4) will also be required.

2.1 LEMMA. *If the coefficients of* (2.4) *satisfy* (2.2) *on* X, *then for any solution* $(V(x), Z(x))$ *of* (2.4), $V^*(x)Z(x) - Z^*(x)V(x)$ *is constant on* X.

If $V^*(x)Z(x) \equiv Z^*(x)V(x)$, then the solution $(V(x), Z(x))$ of (2.4) is said to be *conjoined*. Under the hypotheses of Lemma 2.1, a solution is conjoined if $V^*(x_0)Z(x_0) = Z^*(x_0)V(x_0)$ for some $x_0 \in X$.

2.2. LEMMA. *If* $(V(x), Z(x))$ *is a conjoined solution of* (2.4) *for which* $V(x)$ *is nonsingular, then*

$$ZV^{-1} = V^{-1*}Z^*. \tag{2.5}$$

In studying selfadjoint vector and matrix systems, most authors use a transformation due to Sternberg [7] to reduce a system such as (2.1) to the form

$$\tilde{\mathbf{v}}' = E\tilde{\mathbf{z}};\ \tilde{\mathbf{z}}' = F\tilde{\mathbf{v}},$$

where $\tilde{\mathbf{v}}(x)$ and $\tilde{\mathbf{z}}(x)$ are again n-dimensional vectors, and $E(x)$ and $F(x)$ are $n \times n$ matrices. However, in [5] the author established a matrix analogue of the classical Picone identity for selfadjoint systems of the form (2.3) and (2.4). This identity made it possible to circumvent the transformation of Sternberg and to establish comparison theorems for general selfadjoint systems in a substantially more direct manner. In order to establish an appropriate generalization of the Picone identity to the case where (2.3) need not be selfadjoint, we shall consider a generalized notion of matrix inversion. Consider a symmetric nonnegative definite matrix B whose null space is orthogonal to the range of b. Then the range of b is contained in the range of B, and there exists a matrix k such that $b = Bk$. If $\mathbf{s}$ is in the range of b and $\mathbf{t}$ is such that $\mathbf{s} = b\mathbf{t}$, then we shall define $B^i b$ by $B^i bt = kt$. If B is singular, then k is only defined modulo a matrix whose columns are a basis for the null space of B, and B^i is not unique. If B is nonsingular, then $B^i = B^{-1}$.

Consider now the following ordering among matrices.

2.3. *Definition.* Given a symmetric nonnegative definite matrix B, we shall write $b \prec B$ in case

(i) the range of b is orthogonal to the null space of B, and
(ii) B has an integrable generalized inverse B^i for which the symmetric part of $b^* - b^*B^i b$ is nonnegative definite.

2.4. THEOREM. *Suppose B is nonnegative definite and that the coefficients of* (2.1) *satisfy* (2.2). *If $b \prec B$,* $(\mathbf{u}, \mathbf{w})$ *is a solution of* (2.3), *and* $(V(x), Z(x))$ *is a solution of* (2.4) *for which $V(x)$ is nonsingular, then*

$$\begin{aligned}(2.6)\quad \frac{d}{dx}[\mathbf{u}^*\mathbf{w} - \mathbf{u}^*ZV^{-1}\mathbf{u}] &= \mathbf{u}^*(c - C)\mathbf{u} + \mathbf{u}^*[a^* + d + (a^* - A^*)B^i b]\mathbf{w}\\ &\quad + \mathbf{w}^*(B^i b)^*(A - a)\mathbf{u} + \mathbf{w}^*(b^* - b^*B^i b)\mathbf{w} + \mathbf{u}^*(a^* - A^*)\mathbf{y}\\ &\quad + \mathbf{y}^*(a - A)\mathbf{u} + \mathbf{y}^*B\mathbf{y},\end{aligned}$$

where $\mathbf{y} = B^i b\mathbf{w} - ZV^{-1}\mathbf{u}$.

Proof. Expanding the left side of (2.6) yields

$$\begin{aligned}\frac{d}{dx}[\mathbf{u}^*\mathbf{w} - \mathbf{u}^*ZV^{-1}\mathbf{u}] &= \mathbf{u}^*\mathbf{w}' - \mathbf{u}^*Z'V^{-1}\mathbf{u} + \mathbf{u}^{*\prime}\mathbf{w}\\ &\quad - \mathbf{u}^{*\prime}ZV^{-1}\mathbf{u} - \mathbf{u}^*ZV^{-1}\mathbf{u}' + \mathbf{u}^*ZV^{-1}V'V^{-1}\mathbf{u}.\end{aligned}$$

Using the differential equations (2.3), (2.4) and "completing the square" by adding and subtracting the following expressions:

$$\mathbf{u}^*(a^* - A^*)B^i b\mathbf{w},\quad \mathbf{w}^*(B^i b)^*(a - A)\mathbf{u},\quad \mathbf{w}^*b^*B^i b\mathbf{w}$$

yields the desired result.

Theorem 2.4 can be used to establish nonoscillation criteria for the non-selfadjoint system (2.3). By [**6**, Theorem 5.2], (2.1) is nonoscillatory on X_0 if and only if (2.4) has a conjoined solution $(V(x), Z(x))$ with $V(x)$ nonsingular on X_0, in which case (2.6) is valid for any solution $(\mathbf{u}(x), \mathbf{w}(x))$ of (2.3). If (2.3) is oscillatory on X_0 and $(\mathbf{u}(x), \mathbf{w}(x))$ is a solution of (2.3) satisfying $\mathbf{u}(\alpha) = \mathbf{u}(\beta) = \mathbf{0}$ for distinct points $\alpha < \beta$ on X_0, then we can integrate (2.6) from α to β, and the integral of the left side will vanish. Therefore any hypotheses which assure that the integral of the right side of (2.6) is positive will lead to a contradiction and assure that (2.3) is nonoscillatory on X.

Suppose now that we choose the coefficients of (2.4) to satisfy the following relations:

$$\begin{aligned} A &= a \\ b^* - b^*B^ib &= R_1 \\ C &= c - R_2 \end{aligned} \tag{2.7}$$

where R_1 and R_2 are $n \times n$ matrices to be specified later. Then (2.6) becomes

$$\frac{d}{dx}[\mathbf{u}^*\mathbf{w} - \mathbf{u}^*ZV^{-1}\mathbf{u}] = \mathbf{u}^*R_2\mathbf{u} + \mathbf{u}^*(a^* + d)\mathbf{w} + \mathbf{w}^*b^*R_1b\mathbf{w} + \mathbf{y}^*B\mathbf{y}. \tag{2.8}$$

The above discussion leads to the following nonoscillation criterion for (2.3).

2.5. THEOREM. *Suppose there exist matrices R_1 and R_2 such that the matrices A, B, and C of* (2.7) *satisfy the hypotheses of Theorem* 2.4, *and that in addition the symmetric part of matrix*

$$\begin{pmatrix} R_2 & (a^* + d) \\ 0 & R_1 \end{pmatrix}$$

is nonnegative definite on X_0. If (2.1) *is nonoscillatory on X_0 then* (2.3) *is also nonoscillatory on X_0.*

Proof. Suppose (2.3) is oscillatory on X_0 and that $(\mathbf{u}(x), \mathbf{w}(x))$ is a solution of (2.3) for which $\mathbf{u}(\alpha) = \mathbf{u}(\beta) = 0$, but $\mathbf{u}(x) \not\equiv 0$ on (α, β). Let $(V(x), Z(x))$ be the conjoined solution of (2.4) whose existence is assured by [**6**, Theorem 5.2]. Upon integrating both sides of (2.6) from α to β, the left side vanishes while the integral of each of the terms of the right side is nonnegative. Therefore we must have $y^*By = 0$, or

$$\sqrt{B^i}b\mathbf{w} - \sqrt{B^i}BZV^{-1}\mathbf{u} = 0. \tag{2.9}$$

Since B^i is nonsingular on the range of b, (2.9) is satisfied if and only if

$$\begin{aligned} b\mathbf{w} - BZV^{-1}\mathbf{u} &= 0, \\ (\mathbf{u}' - a\mathbf{u}) - (V' - AV)V^{-1}\mathbf{u} &= 0, \\ \mathbf{u}' - V'V^{-1}\mathbf{u} &= 0, \end{aligned}$$

or finally, if and only if

$$(V^{-1}\mathbf{u})' = \mathbf{0}.$$

https://doi.org/10.4153/CJM-1971-071-7 Published online by Cambridge University Press

This last equation implies that $\mathbf{u} = V\mathbf{k}$ for some nonzero constant vector $\mathbf{k}$ and that $V(x)$ is singular at α and β. This is a contradiction and shows that (2.3) is nonoscillatory on X_0.

3. Differential equations of even order. The general linear homogeneous differential equation of even order (1.3) can also be defined recursively by

$$u_n = p_n u^{(n)},$$

$$u_{n-k} = -u'_{n-k+1} + q_{n-k}u^{(n-k+1)} + p_{n-k}u^{(n-k)},\ k = 1, 2, \ldots, n-1, \tag{3.1}$$

$$u_1' - q_0 u' - p_0 u = 0.$$

While the form (1.3) requires the coefficients $p_k(x)$ and $q_k(x)$ to be k-times differentiable, the recursive definition (3.1) allows one to consider the case where the coefficients are merely integrable functions. The form (3.1) also leads to the vector system formulation

$$\begin{aligned} \mathbf{u}' &= a(x)\mathbf{u} + b(x)\mathbf{w} \\ \mathbf{w}' &= c(x)\mathbf{u} + d(x)\mathbf{w}, \end{aligned} \tag{2.3}$$

where $\mathbf{u}(x)$ and $\mathbf{w}(x)$ are column vectors defined by

$$\mathbf{u}(x) = \operatorname{col}(u(x), u'(x), \ldots, u^{(n-1)}(x)),\ \mathbf{w}(x) = \operatorname{col}(u_1(x), \ldots, u_n(x))$$

and

$$a = (a_{ij}) \text{ where } a_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$b = \operatorname{diag}(0, \ldots, 0, 1/p_n),$$

$$c = (c_{ij}) \text{ where } c_{ij} = \begin{cases} p_{i-1} & \text{if } j = i \\ q_{i-1} & \text{if } j = i+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$d = (d_{ij}) \text{ where } d_{ij} = \begin{cases} -1 & \text{if } j = i-1 \\ q_{n-1}/p_n & \text{if } j = i = n \\ 0 & \text{otherwise.} \end{cases}$$

Equation (1.1) also allows a recursive formulation

$$\begin{aligned} v_n &= P_n v^{(n)}, \\ v_{n-k} &= -v'_{n-k+1} + P_{n-k}v^{(n-k)}, \quad k = 1, 2, \ldots, n-1, \\ v_1' - P_0 v &= 0, \end{aligned} \tag{3.2}$$

which leads to the vector system

$$\begin{aligned} \mathbf{v}' &= A(x)\mathbf{v} + B(x)\mathbf{z} \\ \mathbf{z}' &= C(x)\mathbf{v} - A^*(x)\mathbf{z}, \end{aligned} \tag{2.1}$$

https://doi.org/10.4153/CJM-1971-071-7 Published online by Cambridge University Press

where

$$\mathbf{v}(x) = \operatorname{col}(v(x), v'(x), \ldots, v^{(n-1)}(x)), \mathbf{z}(x) = \operatorname{col}(v_1(x), \ldots, v_n(x))$$

and
$$A = (A_{ij}) \text{ where } A_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$B = \operatorname{diag}(0, \ldots, 0, 1/P_n),$$
$$C = \operatorname{diag}(P_0, P_1, \ldots, P_{n-1}).$$

There is an important relation between the oscillatory behaviour of (3.2) and a particular solution of the selfadjoint matrix system

$$\begin{aligned} V' &= A(x)V + B(x)Z \\ Z' &= C(x)V - A^*(x)Z, \end{aligned} \tag{2.4}$$

(see [**2**; **3**]). For if $v_1(x), \ldots, v_n(x)$ is a fundamental system of solutions of (3.2) satisfying

$$v_i^{(j-1)}(\lambda) = 0, \quad i, j = 1, \ldots, n,$$
$$v_{ij}(\lambda) = \delta_{ij}, \quad i, j = 1, \ldots, n,$$

then every solution $v(x)$ of (3.2) satisfying

$$v(\lambda) = v'(\lambda) = \ldots = v^{(n-1)}(\lambda) = 0$$

is a linear combination of $v_1(x), \ldots, v_n(x)$. Therefore, if we consider the matrices $V(x)$ and $Z(x)$ whose ith columns are given by $\operatorname{col}(v_i, v_i', \ldots, v_i^{(n-1)})$ and $\operatorname{col}(v_{i1}, \ldots, v_{in})$, respectively, then $(V(x), Z(x))$ is a solution of (2.4). It follows readily that a necessary and sufficient condition for the existence of a nontrivial solution of (3.2) satisfying

$$v(\lambda) = \ldots = v^{(n-1)}(\lambda) = 0 = v(\mu) = \ldots = v^{(n-1)}(\mu)$$

for some $\mu \neq \lambda$ is that $V(\mu)$ be singular. Since $V(\lambda) = 0$ and (2.4) is selfadjoint, $(V(x), Z(x))$ is a conjoined solution of (2.4) and Theorem 2.5 can be applied.

Writing the coefficients of the comparison equation (3.2) in the form

$$P_k(x) = p_k(x) - r_k(x),$$

it will be possible to formulate disconjugacy criteria for (3.1) in terms of the coefficients of (3.1) and the $r_k(x)$. In order to make use of Theorem 2.5 we note that

$$b^* - b^*B^ib = \operatorname{diag}(0, \ldots, 0, r_n/p_n^2),$$

$$c - C = (\gamma_{ij}) \text{ where } \gamma_{ij} = \begin{cases} r_{i-1} & \text{if } j = i \\ q_{i-1} & \text{if } j = i + 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$a^* + d = \operatorname{diag}(0, \ldots, 0, q_{n-1}/p_n).$$

These observations make it possible to formulate our principal result.

https://doi.org/10.4153/CJM-1971-071-7 Published online by Cambridge University Press

3.1. THEOREM. *If $\mu_1(\alpha)$ is the first conjugate point of α with respect to* (3.2), *and if for $\alpha \leqq x \leqq \mu_1(\alpha)$*

(i) *$r_n(x) \equiv p_n(x) - P_n(x) \geqq 0$ and $r_n(x) > 0$ on $\{x | q_{n-1}(x) \neq 0\}$, and*

(ii) *$c - C - \operatorname{diag}(0, \ldots, 0, q^2_{n-1}/4(p_n - P_n))$ has a nonnegative definite symmetric part,*

then $\eta_1(\alpha) \geqq \mu_1(\alpha)$.

Proof. Suppose that $\eta_1(\alpha) < \mu_1(\alpha)$, so that (3.2) is disconjugate on $[\alpha, \eta_1(\alpha)]$. Then the particular solution $(V(x), Z(x))$ of (2.4) constructed above satisfies $V(\alpha) = 0$ and $\det V(x) \neq 0$ for $\alpha < x < \eta_1(\alpha)$. By continuity, it is possible to choose $\lambda < \alpha$ such that $V(\lambda) = 0$, but $\det V(x) \neq 0$ for $\alpha \leqq x \leqq \eta_1(\alpha)$. Given this particular solution $(V(x), Z(x))$ of (2.4), we shall apply Theorem 2.5 to obtain the necessary contradiction.

If $u(x)$ is a nontrivial solution of (3.1) with nth order zeros at $\alpha < \beta \equiv \eta_1(\alpha)$, then the vectors $\mathbf{u}(x) = \operatorname{col}(u(x), u'(x), \ldots, u^{(n-1)}(x))$ and $\mathbf{w}(x) = (u_1(x), \ldots, u_n(x))$ satisfy (1.3) and $\mathbf{u}(\alpha) = \mathbf{u}(\beta) = 0$. According to Theorem 2.5, we obtain the necessary contradiction if it can be shown that the symmetric part of the matrix

$$\begin{pmatrix} c - C & a^* + d \\ 0 & b^* - b^* B^t b \end{pmatrix} \tag{3.3}$$

is nonnegative definite on $[\alpha, \beta]$. Specific criteria for the nonnegative definiteness of (3.3) can be derived by considering two cases.

If $q_{n-1}(x) = 0$, then (3.3) is nonnegative definite if $r_n(x) \geqq 0$ and $c - C$ is nonnegative definite. The latter clearly requires that $r_k(x) \geqq 0$ for $k = 0, \ldots, n - 1$.

If $q_n(x) \neq 0$, then we consider a matrix $K = \operatorname{diag}(0, \ldots, 0, k)$ and require that

(i) $c - C - K$ have a nonnegative definite symmetric part, and

(ii) the matrix

$$\begin{pmatrix} K & a^* + d \\ 0 & b^* - b^* B^t b \end{pmatrix}$$

have a nonnegative definite symmetric part.

The latter condition requires that $r_n(x) > 0$, and in this case the most propitious choice of k is

$$k = \frac{q^2_{n-1}}{4r_n} = \frac{q^2_{n-1}}{4(p_n - P_n)}.$$

This completes the proof.

Since specific criteria are known (see [**1**; **4**]) for the disconjugacy of (3.2), the above comparison theorem yields disconjugacy criteria for (3.1).

Remarks. As in the selfadjoint case [**5**], these results are subject to a number of generalizations which will be sketched out below.

1. Hypothesis (ii) of Theorem 3.1 can be replaced by a weaker integral inequality which assures that the integral from α to $\mu_1(\alpha)$ of the first term on the right side of (2.8) is nonnegative. If $u(x)$ is the nontrivial solution of (3.1) which realizes the conjugate point $\mu_1(\alpha)$, this integral inequality becomes

$$\int_\alpha^{\mu_1(\alpha)} \Bigg[\sum_{k=0}^{n-2} (p_k - P_k)(u^{(k)})^2 + \sum_{k=0}^{n-2} q_k u^{(k)} u^{(k+1)} \\ + (p_n - P_n - q^2_{n-1}/4(p_n - P_n))(u^{(n-1)})^2\Bigg] dx \geqq 0.$$

Hypothesis (i) can also be replaced by an integral inequality.

2. In the proof of Theorem 2.4, no assumptions need be made regarding the linearity of the matrix functions appearing in (2.1) and (2.3). Accordingly, the coefficients of (3.1) and (3.2) may take the form

$$p_k(x, u, \ldots, u^{(n-1)}, u_1, \ldots, u_n),\ P_k(x, v, \ldots, v^{(n-1)}, v_1, \ldots, v_n), \text{ etc.,}$$

as long as the hypotheses of Theorem 3.1 hold for $\alpha \leqq x \leqq \mu_1(\alpha)$, and for all values of the other arguments of the coefficients.

3. In case (2.3) is replaced by the system

$$\mathbf{u}' = a\mathbf{u} + b\mathbf{w},$$
$$\mathbf{u}^*\mathbf{w}' \geqq \mathbf{u}^*c\mathbf{u} + \mathbf{u}^*d\mathbf{w},$$

Theorem 2.4 yields an inequality in place of (2.6) which suffices to prove Theorem 3.1. As a result, we may replace the equation

$$u_1' - q_0u' - p_0u = 0$$

in (3.1) by the inequality

$$uu_1' - q_0uu' - p_0u^2 \geqq 0.$$

4. In case the matrix $C(x)$ constructed in Theorem 2.5 is nonnegative definite and symmetric, we can interchange the roles of $\mathbf{u}$ and $\mathbf{w}$ and of V and Z. This makes it possible to establish a comparison theorem for generalized focal points defined by

$$u(\alpha) = u'(\alpha) = \ldots = u^{(n-1)}(\alpha) = 0 = u_1(\gamma) = \ldots = u_n(\gamma).$$

Such a theorem constitutes a generalization of [**2**, Theorem 4.1].

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