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INDEFINITE FINSLER SPACES AND TIMELIKE SPACES

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1. Introduction. In this paper we investigate indefinite Finsler spaces in which the metric tensor has signature $n - 2$. These spaces are a generalization of Lorentz manifolds. Locally a partial ordering may be defined such that the reverse triangle inequality holds for this partial ordering. Consequently, the spaces we study may be made into what Busemann [3] terms locally timelike spaces. Furthermore, sufficient conditions are obtained for an indefinite Finsler space to be a doubly timelike surface (see [**2**; **4**]). In particular, all two-dimensional pseudo-Riemannian spaces are shown to be doubly timelike surfaces.

2. Indefinite Finsler spaces. Let M be a connected paracompact differentiable manifold of dimension n and class C^∞. Denote the local coordinates of a point x on M by $x^1, x^2, \ldots, x^n$. In the tangent space $T(x)$ at x we take a natural frame and denote the components of a vector y in $T(x)$ by $y^1, \ldots, y^n$. Let $L(x, y)$ be a function on the tangent bundle $T(M)$ of M which has the following properties:

(A) The function $L(x, y)$ is of class C^4 whenever $y \neq 0$;

(B) $L(x, ky) = k^2 L(x, y)$ for all $k > 0$;

(C) The metric tensor $g_{ij}(x, y) = \frac{1}{2}(\partial^2 L/\partial y^i \partial y^j)$ has $n - 1$ positive eigenvalues and one negative eigenvalue for all (x, y) with $y \neq 0$.

If M is a Lorentz manifold with $ds^2 = g_{ij}(x)dx^i dx^j$, then $L(x, y) = g_{ij}(x)y^i y^j$ satisfies the above conditions.

For each fixed (x, y) the tangent vectors u at x are separated into three classes, spacelike, null, and timelike according to whether $g_{ij}(x, y)u^i u^j$ is (respectively) positive, zero or negative.

Define $F(x, y) = |L(x, y)|^{\frac{1}{2}}$. Then $F(x, y)$ is of class C^4 if $L(x, y) \neq 0$. In general, $F(x, y)$ is not differentiable when $L(x, y) = 0$. If $F(x, -y) = +F(x, y)$, then F is called symmetric.

For fixed x and a non-zero constant c let S be a component of $\{y|\ L(x, y) = c\}$. Then S is an $(n - 1)$-dimensional surface in the tangent space $T(x)$. Let $y_0 \in S$ and define $H(y) = g_{ij}(x, y_0)y^i y^j$. Then $H(y) = \pm c$ consists of two conjugate quadrics. Let S_1 be the component of $H(y) = c$ that contains y_0.

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Because of the homogeneity of $g_{ij}(x, y)$ in the second variable we have

$$\frac{\partial g_{ij}(x, y_0)y_0^i}{\partial y^k} = 0.$$

Therefore,

$$\frac{\partial H(y_0)}{\partial y^k} = \frac{\partial L(x, y_0)}{\partial y^k} = 2g_{ik}(x, y_0)y_0^i.$$

Consequently, S and S_1 have a common tangent hyperplane P at y_0. Let y be a vector parallel to P. Let $\partial H/\partial y^i = H_{yi}$ and $\partial L/\partial y^i = L_{yi}$. Then

$$H_{yi}(y_0)y^i = L_{yi}(x, y_0)y^i = 0.$$

The normal curvature of S in the direction y is

$$k_n = \frac{2g_{ij}(x, y_0)y^i y^j}{[\sum (y^i)^2][\sum L_{yi}^2(x, y_0)]^{\frac{1}{2}}} = \frac{2H(y)}{[\sum (y^i)^2][\sum L_{yi}^2(x, y_0)]^{\frac{1}{2}}}.$$

LEMMA 1. *Let S be a component of $\{y|\ L(x, y) = c$ with $c < 0\}$. Then S is a strictly convex surface whose principal curvatures are all positive.*

Proof. Using the above notation let $y_0 \in S$ and let y be parallel to the tangent plane P at y_0. Then $H(y) > 0$ and hence $k_n > 0$. Therefore, all of the principal curvatures of S at y_0 are positive.

Let $S^0 = \{y|\ y = \lambda u$ for $\lambda \geqq 1$ and $u \in S\}$. Then S^0 is closed, connected, and strongly locally convex. Consequently, S^0 is convex. The set S is the boundary of S^0 and must be strictly convex. If $n = 2$, then S is a strictly convex curve with non-zero curvature at each point.

By consideration of the normal curvature as in Lemma 1 we can establish the following result.

LEMMA 2. *Let S be a component of $\{y|\ L(x, y) = c$ and $c > 0\}$. Then at each point, S has one negative principal curvature and $n - 2$ positive principal curvatures.*

3. The extremals.

In this section and the next let F be symmetric. The Christoffel symbols are defined by

$$\gamma_h{}^j{}_k(x, y) = \frac{g^{ij}}{2}\left[\frac{\partial g_{hi}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^h} - \frac{\partial g_{hk}}{\partial x^i}\right],$$

where the tensor $g^{ij}(x, y)$ is determined by $g^{ik}g_{kj} = \delta_j{}^i$. The extremals are given by

$$(*) \qquad \frac{d^2x^j}{ds^2} + \gamma_h{}^j{}_k \frac{dx^k}{ds}\frac{dx^h}{ds} = 0.$$

This defines a space of paths. Consequently, simple convex neighbourhoods exist in which every pair of distinct points x, y determine a unique solution of $(*)$; see **[5]**. If $U(x_0)$ is a simple convex neighbourhood about x_0 and $p, q \in U(x_0)$, let $\alpha(p, q)$ denote the unique extremal in $U(x_0)$ from p to q.

Let $\dot{x}_0$ be given with $L(x_0, \dot{x}_0) = -1$. Construct a simple convex neighbourhood $U(x_0)$ in M such that for all $x \in U(x_0)$ we have $L(x, \dot{x}_0) < 0$. Let $K_1(x)$ be the component of $\{y|\ L(x, y) = -1\}$ that contains $\lambda\dot{x}_0$ for some $\lambda > 0$. Define $B(x) = \{\theta y|\ y \in K_1(x) \text{ and } \theta > 0\}$.

An indefinite metric is defined on $U(x_0)$ by $\rho(p, q) = \int_0^1 F(x, \dot{x})\, dt$, where $x(t)$ represents $\alpha(p, q)$ with $x(0) = p$ and $x(1) = q$. Since $F(x, -y) = F(x, y)$, it follows that $\rho(p, q) = \rho(q, p)$.

The Weierstrass E-function for the above integral is given by

$$E(x, \dot{x}, u) = F(x, u) - u^i F_{\dot{x}^i}(x, \dot{x}).$$

LEMMA 4. *Let $x \in U(x_0)$ and $\dot{x}, u \in B(x)$. Then $E(x, \dot{x}, u) \leqq 0$ and equality holds if and only if $u = \lambda\dot{x}$.*

Proof. The Weierstrass E-function is homogeneous in both u and $\dot{x}$. Without loss of generality we may assume that $\dot{x} \in K_1(x)$ and u is in the tangent plane to $K_1(x)$ at $\dot{x}$. Then

$$F_{\dot{x}^i}(x, \dot{x})(u^i - \dot{x}^i) = 0 \quad \text{and} \quad F(x, \dot{x}) = 1.$$

Since $K_1(x)$ is convex, we have $F(x, u) < 1$. Thus,

$$E(x, \dot{x}, u) = F(x, u) - 1 < 0.$$

4. Timelike spaces. For $p, q \in U(x_0)$ define $p < q$ if there is a solution $x(s)$ of $(*)$ in $U(x_0)$ with $x(0) = p$, $x(s_0) = q$, and $x'(0) \in B(p)$.

THEOREM 5. *The relation $<$ is a partial ordering. Furthermore, if $p < q < r$, then $\rho(p, r) \geqq \rho(p, q) + \rho(q, r)$ and equality holds if and only if $q \in \alpha(p, r)$.*

Proof. Let $p < q$ and $q < r$. It is necessary to show that $p < r$. Consider the Meyer field obtained by taking the extremals through p. This field covers a convex neighbourhood of $U(p)$ and covers $U(p) - p$ simply. We may assume that $q, r \in U(p)$. At each point $v \in \alpha(q, r)$ there is a tangent vector v_p to $\alpha(p, v)$ determined by the field. Traversing $\alpha(q, r)$ from q toward r we obtain a tangent vector v_q to $\alpha(q, r)$ at v. For v close to q we have $v_p, v_q \in B(v)$. Thus,

$$E(v, v_p, v_q) \leqq 0.$$

Consequently, $\rho(p, v) \geqq \rho(p, q) + \rho(q, v)$ with equality if and only if $q \in \alpha(p, v)$. It follows that $\rho(p, v)$ is non-decreasing as v traverses $\alpha(q, r)$ from q to r. Therefore, $p < r$ and the theorem is established.

This theorem states that locally a timelike space can be obtained from an indefinite Finsler space of signature $n - 2$.

In the special case of $n = 2$, Lemma 1 holds for components of $\{y|\ F(x, y) = 1\}$. Using this it is not hard to show the following theorem.

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THEOREM 6. *Let M be a two-dimensional space and let $F(x, y)$ be symmetric. If at each point of M there are exactly two (linearly independent) null directions, M is a doubly timelike surface.*

It is only necessary to prove Theorem 6 locally for simple convex sets U_i. Then we cover M with sufficiently small convex neighbourhoods U_i such that $U_i \cap U_j$ is always a simple convex set and such that $\bar{U}_i$ is always compact. Then define $\Pi = \{(p, q) | p \neq q$ and both points belong to a common $U_i\}$. The external $\alpha(p, q)$ and indefinite distance $\rho(p, q)$ are as previously defined.

COROLLARY 7. *If M is a two-dimensional pseudo-Riemannian space, then locally M is a doubly timelike surface.*

5. The Minkowski case. If M is the space of real n-tuples and L depends only on y and not on x, the space is called a Minkowski space. The extremals are the ordinary straight lines (even in the non-symmetric case).

Since F and L only depend on y, we write $F(y)$ and $L(y)$. Let z denote the origin and identify the tangent space at z with M. Consider F to be a function from M to the non-negative reals. If p and q are points of M, then

$$\rho(p, q) = \int F(\dot{x})\, dt = F(q - p).$$

Here the integral is from p to q along the segment $\alpha(p, q)$.

Using the above identification, it is clear that $F(x)$ is just $\rho(z, x)$, the distance from z to x. The unit sphere K is $\{x | F(x) = 1\}$ and the light cone C is $\{x | F(x) = 0\}$. The light cone must consist of a union of half lines from z.

LEMMA 8. *If $x_0 \in C - z$, then x_0 and $(L_{x^1}(x_0), \ldots, L_{x^n}(x_0))$ are linearly independent.*

Proof. Let $H(h) = g_{ij}(x_0)h^i h^j$. Then

$$\frac{\partial H(x_0)}{\partial h^k} = \frac{\partial L(x_0)}{\partial x^k} = 2g_{ik}(x_0){x_0}^i.$$

Therefore, $(L_{x^1}(x_0), \ldots, L_{x^n}(x_0)) \neq 0$. By Euler's Theorem we have

$$L_{x^i}(x_0){x_0}^i = 2L(x_0) = 0.$$

This establishes the lemma.

It now follows that if $n = 2$, the light cone consists of only a finite number of half lines. These half lines separate the space into open components $S_1, \ldots, S_r$. Exactly one component of K lies in each S_i. We may assume that the components S_i are labeled consecutively around z. If S_i and S_{i+1} are adjacent components, then, by Lemma 8, $L(x)$ is positive on one and negative on the other. Traversing a circle about z, the function $L(x)$ must alternate each time a half line of C is crossed. Consequently, there must be an even number of components S_i.

THEOREM 9. *Let M be Minkowskian and $n = 2$. Then the unit sphere K has an even number r of components. Furthermore, if $L(-x) = -L(x)$, then r is not divisible by 4. If $L(-x) = L(x)$, then r is divisible by 4.*

Proof. The fact that r is even follows from the above remarks. Let $t = \frac{1}{2}r + 1$. Let $L(-x) = -L(x)$. If S_1 is a component on which $L(x) > 0$, then $L(x) < 0$ on S_t and $\frac{1}{2}r$ must be odd. By similar reasoning, if $L(-x) = L(x)$, then $\frac{1}{2}r$ is even.

We now give two examples which are neither pseudo-Riemannian nor Minkowskian general G-spaces; compare [**1**]. They are both two-dimensional.

Example 1. Let

$$L(x) = \frac{(x^1)^3 - x^1(x^2)^2}{[(x^1)^2 + (x^2)^2]^{\frac{1}{2}}}.$$

Then $L(-x) = -L(x)$. The unit sphere K has six components.

Example 2. Let

$$L(x) = \frac{(x^1)^3x^2 - x^1(x^2)^3}{(x^1)^2 + (x^2)^2}.$$

Then $L(-x) = L(x)$. The unit sphere K has eight components.

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REFERENCES

1. J. K. Beem, *Indefinite Minkowski spaces*, Pacific J. Math. *33* (1970), 29–42.

2. J. K. Beem and P. Y. Woo, *Doubly timelike surfaces*, Mem. Amer. Math. Soc. No. 92, 1969.

3. H. Busemann, *Timelike spaces*, Dissertationes Math. Rozprawy Mat. *53* (1967), 52 pp.

4. H. Busemann and J. K. Beem, *Axioms for indefinite metrics*, Rend. Circ. Mat. Palermo (2) *15* (1966), 223–246.

5. J. H. C. Whitehead, *Convex regions in the geometry of paths—Addendum*, Quart. J. Math. Oxford Ser. *4* (1933), 226–227.

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