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ON THE SOLUTION OF MOSER'S PROBLEM IN FOUR DIMENSIONS

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ABSTRACT. The problem of finding the largest set of nodes in a d-cube of side 3 such that no three nodes are collinear was proposed by Moser. Small values of d (viz., $d \leq 3$) resulted in elegant symmetric solutions. It is shown that this does not remain the case in 4 dimensions where at most 43 nodes can be chosen, and these must not include the center node.

1. **Introduction.** Given a d-dimensional board of side three, a *solution* is a set of nodes of the board not containing any three collinear nodes. Moser [2], [3] proposed the problem of finding the cardinality $F(d)$ of the largest such set. We have

$F(1) = 2$	(two maximal solutions modulo rotation),
$F(2) = 6$	(unique maximal solution modulo rotation), and
$F(3) = 16$	(unique maximal solution modulo rotation—Figure 1).

It is easy to see that $40 \leq F(4) \leq 46$. Chvátal [1] demonstrated a lower bound for $F(d)$ that gives $F(4) \geq 42$, and, in general, $F(d) > c3^d/\sqrt{d}$. He also showed that for $d=4$ there exists a solution having 43 nodes.

Maximal solutions in one, two and three dimensions have the property that at least one in each case is symmetric about the center, leading one to hope that there might exist such "nice" maximal solutions for all dimensions. Unfortunately, this is not true for the four dimension case. It is shown that any maximal solution in 4 dimensions has 43 nodes, and the center node is not occupied, i.e., it cannot be symmetric about the center.

2. **Some results for two and three dimensions.** The following results can be easily verified, and are stated without proof. Nodes in a 3×3 plane will be referred to by the adjectives "center," "side," and "corner."

(1) The unique solution for $F(2)$ occupies all four side nodes and two opposite corner nodes.

There are five solutions for a two-dimensional board with 5 occupied nodes (modulo rotation and mirror image). These are shown in Figure 2, and will subsequently be referred to as a, b, c, d, e.

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(2) For a three-dimensional board, the unique best solution has 16 nodes distributed 6, 4, 6 in the three parallel planes (along major axes) as presented in Figure 1.

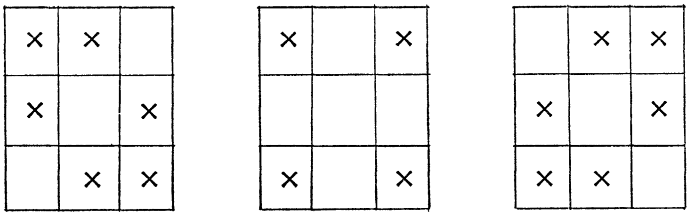

Figure 1
The 16-node solution in three dimensions

(3) For a three-dimensional board, if 6 nodes are occupied in the middle plane, the best solution has 14 occupied nodes.

(4) If a solution for the 3-D problem has 6, 5, 4 occupied nodes in parallel planes then the middle five must be of type e, and of the 4, one must be a center node in the plane.

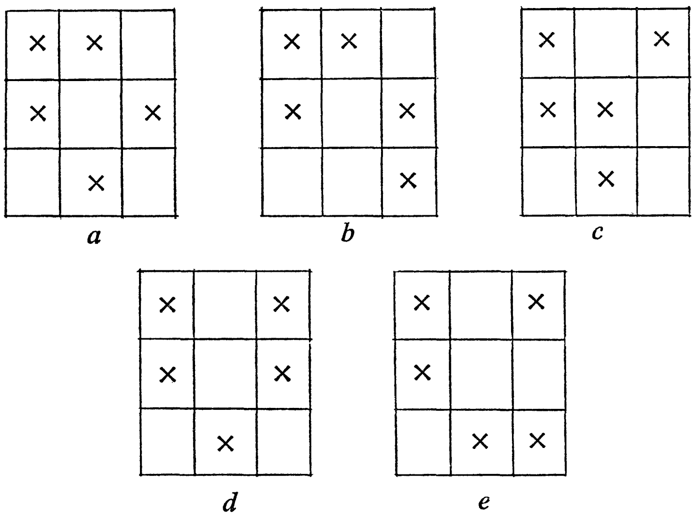

Figure 2
The five-node solutions in two dimensions

(5) If a solution for the 3-D problem has 5, 5, 5 occupied nodes, the configuration must be (a, e, c) or (a, e, e).

(6) If the center node is occupied in a solution for the 3-D problem then no more than 14 nodes can be occupied. This follows from the general result that if the center node is occupied in a solution for d dimensions then the solution can have at most $(3^d+1)/2$ nodes.

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(7) If the left plane in a 3-D solution has 6 occupied nodes and the right plane has either 5 in configuration e or the 4 corners then the middle plane cannot have 4 occupied nodes.

(8) There exists no 5, 3, 5 solution in 3-D where the two 5's are in configuration e (in any relative orientation).

3. **The proof of** $F(4)\leq 43$. A 4-D board may be represented by a tableau of 9 planes each containing nine nodes. The planes will be referred to as $A, B, \ldots, I$ as below.

$$\begin{matrix} A & B & C \\ D & E & F \\ G & H & I \end{matrix}$$

$|A|$ will represent the number of occupied nodes in A, etc. In addition, implicit use will be made of the symmetries of the problem. Planes A, C, G and I will be called corner-planes, etc. "Mid-row" will refer to D, E, F, similarly for "mid col", etc. |Mid-row| stands for the number of occupied nodes in the middle row, and so on. The row-vector of a solution refers to the number of occupied nodes in the three columns, e.g., (15, 14, 13) means |left col|$=|A|+|D|+|G|=15$, etc.; and similarly for the column vector (the first element refers to the top row).

In the proof below it is assumed that there is a solution with 44 nodes and a contradiction is obtained by case analysis. The cases where $|E|\leq 3$ and $|E|=6$ are easy and are disposed of first.

$|E|\leq 3$ *in a Solution with* 44 *nodes*

Both $|D|$ and $|F|$ cannot be 6, otherwise the best possible row vector is (14, 15, 14) by (3) and (2) (since $|E|\neq 4$) and that sums to only 43.

If |mid row|≥ 15 it must be distributed 6, 3, 6—contradiction.

If |mid row|$=14$, i.e., 6, 3, 5 the best row vector is (14, 14, 15) since the middle column also can't contain 15 nodes (by the previous case).

If |mid row|$=13$, i.e., 5, 3, 5, 6, 3, 4 or 6, 2, 5 the best row vectors are (15, 13, 15), (14, 13, 16) and (14, 13, 15) respectively.

If |mid row|$=12$, i.e., not both $|D|$ and $|F|$ are 4, then a row vector (16, 12, 16) is impossible.

If |mid row|≤ 11 the best row vector is (16, 11, 16).

$|E|=6$ *in a Solution with* 44 *Nodes*

By (3), $|A|+|I|\leq 8$, $|B|+|H|\leq 8$, $|C|+|G|\leq 8$, $|D|+|F|\leq 8$, which gives a maximum possible solution of only 38 nodes.

$|E|=5$ *in a Solution with* 44 *Nodes*

Case 1*:* |*mid row*|$=15$

(i) If the mid row is 5, 5, 5 and the column vector is (16, 15, 13). Then D is a, E is e and $|F|=5$ by (5), and $|A|=|C|=6$. Since $|A|=6$ and D is a, $|G|\leq 3$ by (4). Since $|C|=6$ and $|F|=5$, $|I|\leq 4$ by (2). As |bot row|$=13$, $|H|=6$, but this

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is impossible because in B all four corners are occupied and in $E(=e)$ three are occupied.

(ii) If mid row is 5, 5, 5 and the column vector is (15, 15, 14). Then D is a, E is e, $|F|=5$ as before. The best row vector is then (14, 15, 15) for which F is e by (4), (5).

If $|C|=6$ then by (4), $|G|\leq 4$, $|I|\leq 4$, and since |bot row|=14, $|H|=6$. But from H, E and B and by (4) the center node of B must be occupied, which implies that |top row|$\leq$14 by (6)—a contradiction.

If $|C|=5$ then $|I|=5$ and $|A|=5$ (since $|A|\leq 5$ by A, E, I and if $|A|<5$ then |top row|<15). Now if we look at the triangle formed by A, C and I, each line is distributed 5, 5, 5 which means that one end of each line must be configuration a, and the other not an a, by (5); and that is clearly impossible.

If $|C|\leq 4$ then $|I|=6$, $|A|=6$ since |third col|=|top row|=15; but that is impossible (A, E, I).

(iii) If mid row is 6, 5, 4, i.e., $|F|=4$, then the center node of F is occupied by (4), and the best possible row vector is (14, 15, 14) by (3), (2), and (6).

Case 2: *|mid row|*$\leq$14, *and |mid col|*$\leq$14

Now $|D|+|F|\leq 9$ and $|B|+|H|\leq 9$ as $|E|=5$. Also, $|A|+|I|\leq 10$, and $|C|+|G|\leq 10$ by (2); hence the solution has no more than 43 nodes.

$|E|=4$ *in a Solution with* 44 *Nodes*

Case 1: *|mid row|*=16

By (2), $|D|=|F|=6$, and E has the four corner nodes occupied. By (3), |left col|, |right col|$\leq$14, leaving $|B|=|H|=6$. It follows that |left col|=|right col|= |top row|=|bot row|=14. Now consider the planes A, C, G and I. Since all side nodes in B, D, F and H are occupied, at most 4 side nodes of A and C together can be occupied; and similarly for G and I. Also, as all 4 corner nodes of E are occupied, A and I together can have at most 4 occupied corner nodes; and likewise for C and G. This, together with the four center nodes of A, C, G and I gives a total of 20. We want 16 of these nodes to be occupied.

(i) If in any corner plane, say A, two "adjacent" corner nodes are occupied, then there can be no corner nodes in C or G. Hence, to total to 16, all four centers of A, C, G and I are occupied, A has only two corner nodes and I has two, and a pair of adjacent side nodes are occupied in each of A, C, G and I. Further, the orientations of the side nodes in A and I must be the same, but this conflicts with the corners occupied in either A or I.

(ii) If in A two opposite corner nodes are occupied, say top-right and bottom-left (abbreviated tr and bl), then the tr, bl nodes in I cannot be occupied. If any of the other two corner nodes in I is occupied then no corner node in C or G can be occupied. And, if no corner node in I is occupied then only the tr, bl nodes in C, G can be occupied, and at most 2 of these can be taken. Either way, the maximum possible is only 4 corners+3 centers+8 sides=15.

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(iii) Hence each of A, C, G and I must have exactly 1 corner node occupied (to total 16). But this cannot be done owing to the orientation of the corner nodes in B, D, F and H and the fact that all their side nodes are occupied (see Figure 1).

Case 2: $|mid\ row|=15$, *and* $|mid\ col|\leq 15$

$|D|=6$, $|F|=5$. Thus $|\text{left col}|\leq 14$ by (3), and as $|\text{mid col}|\leq 15$ we must have $|\text{right col}|\geq 15$, i.e., F is e by (4), (5); but a 6, 4, e (D, E, F) is not a solution in 3-D by (7).

Case 3: $|mid\ row|=14$, *and* $|mid\ col|\leq 14$

If the mid row is 6, 4, 4, i.e., $|D|=6$, then $|\text{left col}|\leq 14$ implying $|\text{mid col}|=14$ and $|\text{right col}|=16$, i.e., F has four corner nodes occupied; but this is impossible (D, E, F) by (7).

If the mid row is 5, 4, 5 then $|\text{left col}|\leq 15$ and $|\text{right col}|\leq 15$ and as $|\text{mid col}|\leq 14$ all are satisfied with equalities. Thus D and F are both of type e by (4), (5) and D, E, F is impossible by (8).

Case 4: $|mid\ row|\leq 13$, *and* $|mid\ col|\leq 13$

One row and one column must have 16—say the top row and the left column. Then $|A|=|C|=|G|=6$. Now looking at the triangle A, C, G, each line is distributed 6, 4, 6, and by (2) the orientation of the two 6's is opposite in each line. And this is clearly impossible for the triangle.

This exhausts all possibilities, implying that there is no solution for the 4-D problem with 44 nodes. Thus, solutions with 43 nodes are optimal.

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