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ON TIME DEPENDENT MULTISTEP DYNAMIC PROCESSES

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The discrete time scale Liapunov theory is extended to time dependent, higher order, nonlinear difference equations in a partially ordered topological space. The monotone convergence of the solution is examined and the speed of convergence is estimated.

1. INTRODUCTION

Iteration processes are extremely important in solving optimisation problems, linear and nonlinear equations, and in general, they are used in all fields of applied mathematics.

In the context of nonlinear programming Zangwill [**15**] presented a general theory on convergence of iteration processes based on point-to-set mappings. He investigated only one-step stationary iterations, and he proved that the process either terminates after a finite number of steps or the limit of any convergent subsequence is a solution. Special but practically useful criteria were derived for example by Brock and Scheinkman [**1**], Fujimoto [**4**], [**5**], Szidarovszky and Okuguchi [**10**] based on special selections of the Liapunov function.

In this paper the convergence theorem of Zangwill is generalised and extended to nonstationary multistep iteration processes in partially ordered topological spaces. In addition, monotone convergence and the speed of the convergence of the processes are examined.

Before presenting our convergence results, a brief summary of the concepts of special topological spaces is presented.

Let X be a linear space. A subset K of X is called a *cone* if $K + K \subset K$ and $\alpha K \subset K$ for all $\alpha > 0$. The cone K is *proper* if $K \cap \{-K\} = 0$. The relation "$\leqslant$" defined by

$$x \leqslant y \text{ if and only if } y - x \in K$$

is a partial ordering on K which is compatible with the linear structure of this space. Two elements x and y of X are called *comparable* if either $x \leqslant y$ or $y \leqslant x$ holds. The space X endowed with the above relation is called a *partially ordered linear space*

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(POL-space). If X has a topology compatible with its linear structure and if the cone K is closed in that topology then X is called a *partially ordered topological space* (POTL-space).

We remark that in a POTL-space the intervals $[a, b] = \{x; a \leqslant x \leqslant b\}$ are closed sets. A lot of examples show that the closedness of the nonnegative cone is not, in general, a strong enough connection between the ordering and the topology (see for example Vandergraft, [**13**]).

DEFINITION 1: A POTL-space is called *normal* if, given a local base U for the topology, there exists a positive number η such that $0 \leqslant z \in U$ implies $[0, z] \subset \eta U$.

DEFINITION 2: A POTL-space is called *regular* if every order bounded increasing sequence has a limit.

If the topology of a POTL-space is given by a norm then this space is called a *partially ordered normed space* (PON-space). If a PON-space is complete with respect to its topology then it is called a *partially ordered Banach space*(POB-space). According to Definition 1 a PON-space is normal if and only if there exists a positive number α such that

$$\|x\| \leqslant \alpha \|y\| \text{ for all } x, y \in X \text{ with } 0 \leqslant x \leqslant y.$$

Note that any regular POB-space is normal. The reverse is not true. For example, the space $C[0, 1]$ of all continuous real functions defined on $[0, 1]$, ordered by the cone of nonnegative functions, is normal but is not regular. All finite dimensional spaces are both normal and regular.

2. CONVERGENCE THEOREMS

Let $S \subset X$ be a set such that $u^* \in \overline{S}$, and for $k \geqslant 0$ the point-to-set mappings $f(k; \cdot)$ are defined on

$$S^\ell = S \times S \times S \cdots \times S,$$

and for all $t^{(1)}, \cdots, t^{(\ell)} \in S$ and $k \geqslant \ell - 1$, $f\big(k; t^{(1)}, \cdots, t^{(\ell)}\big)$ is nonempty in S. Define the iteration sequence

$$x_{k+1} \in f(k_{k-\ell+1}, x_{k-\ell+2}, \cdots, x_k) \tag{1}$$

where $k \geqslant \ell - 1$, $x_0, x_1, \cdots, x_{\ell-1} \in S$, and an arbitrary element from the set can be selected as the successor of x_k.

DEFINITION 3: A function $V : S^\ell \to R_+$ is called the *Liapunov function* of process (1), if for arbitrary $t^{(i)} \in S$ $\big(i = 1, 2, \cdots, \ell, t^{(\ell)} \neq u^*\big)$ and $y \in f\big(k; t^{(1)}, \cdots, t^{(\ell)}\big)$ $(k \geqslant \ell - 1)$,

$$V\Big(t^{(2)}, \cdots, t^{(\ell)}, y\Big) < V\Big(t^{(1)}, t^{(2)}, \cdots, t^{(\ell)}\Big).$$

DEFINITION 4: The Liapunov-function V is called *closed*, if it is defined on $\overline{S} = \overline{S} \times \overline{S} \times \cdots \times \overline{S}$, furthermore, if $k_i \to \infty$, $t_i^{(j)} \to t^{(j)^*}$

$$\left(t_i^{(j)} \in S \text{ for } i \geqslant 0 \text{ and } j = 1, 2, \cdots, \ell, t^{(\ell)^*} \neq u^*\right),$$

$y_i \in f\left(k_i; t_i^{(1)}, \cdots, t_i^{(\ell)}\right)$ $(i \geqslant 0)$ and $y_i \to y^*$, then

$$V\left(t^{(2)^*}, \cdots, t^{(\ell)^*}, y^*\right) < V\left(t^{(1)^*}, t^{(2)^*}, \cdots, t^{(\ell)^*}\right).$$

REMARK. Assume that $f(k; \cdot) \equiv f(\cdot)$ for all k, $S = \overline{S}$, and mapping $f(\cdot)$ is closed (for the definition of closed mappings see for example Zangwill, [**15**], p.88), then any Liapunov–function is also closed.

Our main convergence result can be formulated as follows:

THEOREM 1. *Assume that X is a topological space and*

(A) *for all $k \geqslant \ell - 1$, $f\left(k; t^{(1)}, \cdots, t^{(\ell)}, u^*\right) = \{u^*\}$ with arbitrary $t^{(1)}, \cdots, t^{(\ell-1)} \in S$, if $u^* \in S \subseteq X$;*

(B) *the iteration process (1) has a continuous, closed Liapunov function;*

(C) *there exists a compact set C in X and that for all $k \geqslant \ell - 1$, $x_k \in C$. Then $x_k \to u^*$ as $k \to \infty$.*

PROOF: Condition (A) implies that if for $k \geqslant \ell - 1$, $x_k = u^*$, then all successors of x_k are also equal to u^*. Hence we may assume that $x_k \neq u^*$ $(k \geqslant \ell - 1)$. Assume that $x_k \not\to u^*$; then since the sequence is in a compact set there is a subsequence x_{k_i} which tends to $x^* \neq u^*$. The construction of the iteration sequence and the definition of the Liapunov function imply that for all $i \geqslant 0$,

$$\begin{aligned}(2) \qquad V\left(x_{k_{i+1}-\ell+1}, \cdots, x_{k_{i+1}}\right) &\leqslant V\left(x_{k_i-\ell+2}, \cdots, x_{k_i}, x_{k_i+1}\right) \\ &\leqslant V\left(x_{k_i-\ell+1}, \cdots, x_{k_i}\right).\end{aligned}$$

Without loss of generality assume that all sequences $\{x_{k_i-\ell+1}\}$, $\{x_{i-\ell+2}\}, \cdots, \{x_{k_i}\}$, and $\{x_{k_i+1}\}$ are also convergent, otherwise take further subsequences of $\{x_{k_i}\}$. Let $x_{\ell-1}^*, \cdots, x_1^*$ and y^* denote the limits of the above subsequences; then the continuity of the Liapunov–function and relation (2) imply that

$$V\left(x_{\ell-2}^*, \cdots, x_1^*, x^*, y^*\right) = V\left(x_{\ell-1}^*, \cdots, x_1^*, x^*\right).$$

Since the Liapunov–function is closed, strict inequality must hold in the above relation. This contradiction completes the proof. ∎

REMARK 1. Assumption $u^* \in \overline{S}$ is needed in order to obtain u^* as the limit of sequences from S. Assumption (A) guarantees that if at any iteration step the solution u^* is obtained, then the process remains at the solution. We may also show that the existence of a Liapunov function is not a too strong assumption. Consider the special case when X is a normed space and f is point-to-point from S to S, and assume that starting from arbitrary initial sulution $x_0 \in S$ the process converges to the solution u^* of equation $x = f(x)$. Let $V\colon S \to R_+$ be constructed as follows. With selecting $x_0 = x$ consider sequence $x_{k+1} = f(x_k)$ $(k \geqslant 0)$, and define

$$V(x) = \begin{cases} 0 & \text{if } x = u^* \\ \max\limits_k \|x_k - u^*\| & \text{otherwise.} \end{cases}$$

Obviously $V(f(x)) \leqslant V(x)$ for all $x \in S$. The continuity-type assumptions in (B) are also natural, since without certain continuity conditions no convergence can be established. Assumption (C) is necessarily satisfied, for example, if $x = R^n$, and either S is bounded or if for every $B > 0$ there exists a $Q > 0$ such that $t^{(1)}, \cdots, t^{(\ell)} \in S$ and $\left\|t^{(j)}\right\| > Q$ (for at least one index j) imply relation

$$V\left(t^{(1)}, \cdots, t^{(\ell)}\right) > B.$$

In the case of one-step processes (that is, if $\ell = 1$) this last condition can be reformulated as

$$\lim_{\substack{\|x\| \to \infty \\ x \in S}} V(x) = \infty.$$

REMARK 2. Iteration processes in this general form have real practical importance. Note first that one of the most popular solvers of nonlinear equations is the secant method, which is actually a two-step process. Many dynamic economic processes are based on the selection of optimal strategies by the participants at each time period. If the optimal solution is not unique, then the strategy for the next period can be selected from the set of optimal solutions. Hence the iteration is based on a set-valued mapping. In addition, if the participants' decisions are based on extrapolative expectations on the other's behaviour, then the process becomes multistep. Time dependency of the process follows from price changes, technological development, et cetera. For the description of such models in the oliopoly theory see Okuguchi and Szidarovszky [7].

Assume next that the iteration process is stationary, that is, in recursion (1) function f does not depend on k. In this case Theorem 1 reduces to the following.

THEOREM 2. *Assume that X is a topological space, $S \subset X$; furthermore*

(i) $S = \overline{S}$;

(ii) *for all* $t^{(1)}, \cdots, t^{(\ell-1)} \in S$,

$$f\left(t^{(1)}, \cdots, t^{(\ell-1)}, u^*\right) = \{u^*\};$$

(iii) *function* f *is closed on* S;

(iv) *the iteration process has a continuous Liapunov function;*

(v) *there exists a compact set* $C \subset X$ *such that for all* $k \geqslant \ell - 1$, $x_k \in C$.

Then $x_k \to u^*$ *as* $k \to \infty$.

REMARK. This result in the special case of $\ell = 1$ can be considered as the discrete-time-scale counterpart of the famous stability result of Uzawa [**12**].

3. MONOTONE CONVERGENCE

In this section sufficient conditions will be given for the monotone convergence of the iteration scheme (1). Assume now that X is a partially ordered topological space, and $S \subset X$.

DEFINITION 5: The *sequence of point-to-set mappings* $f(k;\, \cdot)$ from S to S is called *increasingly isotone* on S if for arbitrary $k \geqslant \ell - 1$, $t^{(i)} \in S$ $(i = 1, 2, \cdots, \ell+1)$ such that $t^{(\ell+1)} \geqslant t^{(\ell)} \geqslant \cdots \geqslant t^{(2)} \geqslant t^{(1)}$ and for any $y_1 \in f\left(k;\, t^{(1)}, \cdots, t^{(\rho)}\right)$ and $y_2 \in f\left(k+1;\, t^{(2)}, \cdots, t^{(\rho+1)}\right)$, $y_1 \leqslant y_2$.

DEFINITION 6: *Point-to-set mapping* $f(k;\, \cdot)\colon S^\ell \to S$, for a fixed k $(k \geqslant \ell - 1)$, is called *increasingly isotone* if $t^{(i)} \in S$ $(i = 1, 2, \cdots, \ell+1)$ such that $t^{(\ell+1)} \geqslant t^{(\ell)} \geqslant \cdots \geqslant t^{(2)} \geqslant t^{(1)}$ and $y_1 \in f\left(k;\, t^{(1)}, \cdots, t^{(\ell)}\right)$ and $y_2 \in f\left(k;\, t^{(2)}, \cdots, t^{(\ell+1)}\right)$ imply that $y_1 \leqslant y_2$.

REMARK 1. Note that if $f(k;\, \cdot)$ does not depend on k, then Definitions 5 and 6 are equivalent.

REMARK 2. In the literature *a point-to-set mapping* $f(k;\, \cdot)$ is called *isotone* if for all $t^{(i)} \in S$, $s^{(i)} \in S$ such that $t^{(i)} \leqslant s^{(i)}$ $(i = 1, 2, \cdots, \ell)$, $y_1 \leqslant y_2$ for all $y_1 \in f\left(k;\, t^{(1)}, \cdots, t^{(\ell)}\right)$, $y_2 \in f\left(k;\, s^{(1)}, \cdots, s^{(\ell)}\right)$. It is obvious that an isotone mapping is increasingly isotone, but the reverse is not necessarily true, as the example of set $S = [0, 1] \subset R^1$ and function

$$g\left(t^{(1)} t^{(2)}\right) = \begin{cases} t^{(2)} & \text{if } t^{(1)} \geqslant 2t^{(2)} - 1 \\ t^{(1)} - t^{(2)} + 1, & \text{if } t^{(1)} < 2t^{(2)} - 1 \end{cases}$$

illustrates. Let the partial order $\leqslant$ be defined as $\left(s^{(1)}, s^{(2)}\right) \leqslant \left(t^{(1)}, t^{(2)}\right)$ if and only if $s^{(1)} \leqslant t^{(1)}$ and $s^{(2)} \leqslant t^{(2)}$. First we show that g is increasingly monotone.

Select $t^{(1)} \leqslant t^{(2)} \leqslant t^{(3)}$. Note first that $g(t^{(1)}, t^{(2)}) \leqslant t^{(2)}$. If $t^{(1)} \geqslant 2t^{(2)} - 1$, then $g(t^{(1)}, t^{(2)}) = t^{(2)}$; and if $t^{(1)} < 2t^{(2)} - 1$, then

$$g\left(t^{(1)}, t^{(2)}\right) = t^{(1)} - t^{(2)} + 1 < 2t^{(2)} - 1 - t^{(2)} + 1 = t^{(2)}.$$

Note next that $g(t^{(2)}, t^{(3)}) \geqslant t^{(2)}$. If $t^{(2)} \geqslant 2t^{(3)} - 1$, then $g(t^{(2)}, t^{(3)}) = t^{(3)} \geqslant t^{(2)}$; and if $t^{(2)} < 2t^{(3)} - 1$, then

$$g\left(t^{(2)}, t^{(3)}\right) = t^{(2)} - t^{(3)} + 1 \geqslant t^{(2)}.$$

Hence

$$g\left(t^{(1)}, t^{(2)}\right) \leqslant t^{(2)} \leqslant g\left(t^{(2)}, t^{(3)}\right).$$

We can also verify that the mapping g is not isotone on S. Consider points $(t, 1)$ and $(t, 1-\varepsilon)$ $(t, \varepsilon > 0; t + 2\varepsilon < 1)$. Then $g(t, 1) = t$ and $g(t, 1-\varepsilon) = t - (1-\varepsilon) + 1 = t + \varepsilon > g(t, 1)$. Hence g is not isotone.

Theorem 3. *Assume that in iteration (1) the sequence of mappings $f(k; \cdot)$ is increasingly isotone; furthermore $x_i \in S$ $(0 \leqslant i < \ell - 1)$ and $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{\ell-1} \leqslant x_\ell$. Then for all $k \geqslant 0$, $x_{k+1} \geqslant x_k$.*

Proof: By induction, assume that for i $(i < k)$, $x_{i+1} \geqslant x_i$. Then relations $x_k \in f(k-1, x_{k-\ell}, \cdots, x_{k-1})$, $x_{k+1} \in f(k, x_{k-\ell+1}, \cdots, x_k)$ and the definition of increasingly monotone family of mappings imply that $x_k \leqslant x_{k+1}$. Since this inequality holds for $k = 0, 1, \cdots, \ell - 1$, the proof is completed. □

Consider next the modified iteration scheme

$$y_{k+1} \in f(k; y_k, y_{k-1}, \cdots, y_{k-\ell+1}). \tag{3}$$

Using finite induction, similarly to Theorem 3, we may prove the following:

Theorem 4. *Assume that the sequence of mappings $f(k; \cdot)$ is increasingly isotone, furthermore $y_i \in S(0 \leqslant i \leqslant \ell - 1)$ and*

$$y_0 \geqslant y_1 \geqslant \cdots \geqslant y_{\ell-1} \geqslant y_\ell.$$

Then for all $k \geqslant 0$, $y_{k+1} \geqslant y_k$.

Corollary. *Assume that $X = R^n$ and for $k \to \infty$, the sequences $\{x_k\}$ and $\{y_k\}$ have the same limit u^*, and $\leqslant$ is the usual partial order of vectors. (That is, $a = a(i) \leqslant b = b(i)$ if and only if $a(i) \leqslant b(i)$ for all i.) Under the conditions of Theorems 3 and 4, for all $k \geqslant 0$,*

$$x_k \leqslant u^* \leqslant y_k.$$

This relation is very useful in the error analysis of the iteration methods (1) and (3), since for all coordinates $x_k(i)$, $y_k(i)$ and $u^*(i)$ of vectors x_k, y_k, u^*, respectively,

$$0 \leqslant u^*(i) - x_k(i) \leqslant y_k(i) - x_k(i)$$

and

$$0 \leqslant y_k(i) - u^*(i) \leqslant y_k(i) - x_k(i).$$

Furthermore, we can show:

THEOREM 5. *Assume that X is a regular POB-space, $S \subset X$ and*

(A) *the sequence of mappings $f(k;\cdot)$ is increasingly isotone in iteration (1) with $x_i \in S$ $(0 \leqslant i \leqslant \ell - 1)$ and*

$$x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{\ell-1} \leqslant x_\ell;$$

(B) *there exists a set H_1 defined by $H_1 = \{x \in S;\, x \leqslant x_0\}$ with the property that if for any points $t^{(1)}, t^{(2)}, \dots, t^{(\ell)}$ in H_1 with*

$$t^{(1)} \leqslant t^{(2)} \leqslant \cdots \leqslant t^{(\ell)} \leqslant x_0,$$

then

$$x_{k+1} \leqslant x_0 \text{ for any } x_{k+1} \in f\left(k; t^{(1)}, t^{(2)}, \dots, t^{(\ell)}\right),\ k \geqslant \ell - 1.$$

Then the sequence $\{x_n\}$, $n \geqslant 0$ generated by the iteration (1) *process* (1) *is monotonically increasing, remains in H_1 and converges to some $u^* \in H_1$.*

PROOF: From (A) and Theorem 3 it follows that the sequence $\{x_n\}$, $n \geqslant 0$ is monotonically increasing, whereas from (B) we get that the sequence is bounded above by x_0. Since X is a regular POB-space the sequence $\{x_n\}$, $n \geqslant 0$, converges to some u^* with $u^* \leqslant x_0$. Hence $u^* \in H_1$.

That completes the proof of the theorem. □

These monotonic properties of the iteration processes are very useful, but in cases where the convergence is very slow the above methods have only very limited practical importance. In the next section of this paper the convergence speed of the above iteration schemes is estimated and practical error estimates are derived.

4. Estimates on the speed of convergence

We can now formulate the following theorem.

THEOREM 6. *Assume that X is a normal POB-space, $S \subset X$ and*

(A) *the sequence of mappings $f(k;\cdot)$ is increasing isotone in iteration (1) with $x_i \in S$ $(0 \leqslant i \leqslant \ell - 1)$ and $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{\ell-1} \leqslant x_\ell$;*

(B) *There exists a constant b with $0 \leqslant b < 1$ such that*

(4) $$x_{n+2} - x_{n+1} \leqslant b(x_{n+1} - x_n), \text{ for all } n \geqslant 0.$$

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Then the sequence $\{x_n\}$, $n \geqslant 0$, generated by the iteration process (1) is monotonically increasing and converges to some u^* with

$$\|x_n - u^*\| \leqslant \frac{\alpha \|x_1 - x_0\|}{1-b} b^n, \quad n \geqslant 0. \tag{5}$$

PROOF: From (A) and Theorem 3 it follows that the sequence $\{x_n\}$ is monotonically increasing and inequality (4) can be rewritten as

$$0 \leqslant x_{n+2} - x_{n+1} \leqslant b(x_{n+1} - x_n), \quad n \geqslant 0.$$

Using the above inequality we get

$$0 \leqslant x_{n+p} - x_n = \sum_{i=0}^{p-1} (x_{n+i+1} - x_{n+i}) \leqslant \frac{x_1 - x_0}{1-b} b^n, \quad p \geqslant 0.$$

Since X is normal we deduce

$$\|x_{n+p} - x_n\| \leqslant \alpha \frac{\|x_1 - x_0\|}{1-b} b^n.$$

It nor follows that the sequence $\{x_n\}$, $n \geqslant 0$, is a Cauchy sequence in a Banach space and as such it converges to some u^*. By letting $p \to \infty$ we obtain (5).

That completes the proof of the theorem. □

Note that an identical theorem can be proved if the assumptions (A) and (B) in the above theorem are replaced with the condition

$$0 \leqslant x_{n+2} - x_{n+1} \leqslant b(x_{n+1} - x_n),$$

for all $n \geqslant 0$ and some b, $0 \leqslant b < 1$. Let us define the set H_2 by $H_2 = \{x \in S;\, x_0 \leqslant x,\, \|x - x_0\| \leqslant h\}$ for some $h > 0$.

Then we can show the following theorem.

THEOREM 7. *Let X be a normal POB-space, $S \subset X$ and assume that the following conditions are satisfied:*

$$\alpha \frac{\|x_1 - x_0\|}{1 - c_2} \leqslant h, \tag{6}$$

$$c_1(x_n - x_{n-1}) \leqslant x_{n+1} - x_n \leqslant c_2(x_n - x_{n-1}),\ n \geqslant 1,\ 0 \leqslant c_1 \leqslant c_2 < 1, \tag{7}$$

$$x_0 \leqslant x_n, \tag{8}$$

$$\|x_n - x_0\| \leqslant h \tag{9}$$

and

$$0 \leqslant x_{n+1} - x_n \leqslant c_2^n (x_1 - x_0) \text{ for all } n = 0, 1, 2, \cdots, \ell - 1. \tag{10}$$

Then the sequence $\{x_n\}$, $n \geqslant 0$, *generated by the iteration process (1) is monotonically increasing and converges to some* u^* *with*

$$\|x_n - u^*\| \leqslant \alpha \frac{\|x_1 - x_0\|}{1 - c_2} c_2^n \text{ for all } n \geqslant 0. \tag{11}$$

PROOF: We will show that the estimates (8), (9) and (10) are true for all $n \geqslant 0$. For $n = 0, 1, 2, \cdots, \ell - 1$, they hold by hypothesis. Let us suppose that they are true for $n = 0, 1, 2, \cdots, k$ with $k \geqslant \ell - 1$. From (8) and (10) for $n = k$ it follows that

$$x_0 \leqslant x_k \leqslant x_{k+1}$$

and thus (8) is true for $n = k + 1$.

Using (10), the above inequality, and the properties of the partial order $\leqslant$, we have successively:

$$0 \leqslant x_{k+1} - x_0 = \sum_{i=0}^{k} (x_{i+1} - x_i) \leqslant (x_1 - x_0) \sum_{i=1}^{k} c_2^i \leqslant \frac{x_1 - x_0}{1 - c_2}$$

where from (6) we deduce that (9) is true for $n = k + 1$.

From (7), (10) and the induction hypothesis we get

$$0 \leqslant x_{k+2} - x_{k+1} \leqslant c_2 c_2^k (x_1 - x_0) = c_2^{k+1} (x_1 - x_0).$$

It now follows that (10) is true for $n = k + 1$. Moreover for $p \geqslant 0$ we get

$$0 \leqslant x_{n+p} - x_n = \sum_{i=0}^{p-1} (x_{n+i+1} - x_{n+i}) \leqslant \frac{x_1 - x_0}{1 - c_2} c_2^n$$

from which we obtain

$$\|x_{n+p} - x_n\| \leqslant \alpha \frac{\|x_1 - x_0\|}{1 - c_2} c_2^n \leqslant h c_2^n. \tag{12}$$

The above inequality shows that the sequence $\{x_n\}$, $n \geqslant 0$ is Cauchy in a POB-space and as such it converges to some u^*. By letting $p \to \infty$ in (12) we obtain (11).

That completes the proof of the theorem. ∎

REMARK 1. Note that a similar theorem can be proved if the condition (7) is replaced by the relation

$$0 \leqslant x_{n+1} - x_n \leqslant c_2(x_n - x_{n-1}),\ n \geqslant 1,\ 0 \leqslant c_2 < 1.$$

REMARK 2. Assume that there exists a sequence $c_{2n}^{(n)}$, $n \geqslant 0$, such that more generally

$$0 \leqslant x_{n+1} - x_n \leqslant c_2^{(n)}(x_n - x_{n-1}),\ n \geqslant 1,\ 0 \leqslant c_2^{(n)} \leqslant q < 1.$$

Then similarly to (12) we have that

$$\begin{aligned} x_{n+p} - x_n &= \sum_{i=0}^{p-1} (x_{n+i+1} - x_{n+i}) \\ &\leqslant (x_1 - x_0)(v_n + v_{n+1} + \cdots + v_{n+p-1}), \end{aligned}$$

where

$$v_n = c_2^{(1)} c_2^{(2)} \cdots c_2^{(n)}.$$

Hence

$$\begin{aligned} x_{n+p} - x_n &\leqslant (x_1 - x_0) v_n \big(1 + q + q^2 + \cdots\big) \\ &= \frac{x_1 - x_0}{1 - q} v_n, \end{aligned}$$

and therefore (11) is modified as

$$\|x_n - u^*\| \leqslant \alpha \frac{\|x_1 - x_0\|}{1 - q} v_n.$$

In the special case when $c_2^{(n)}$ is a decreasing sequence we may select $q = c_2^{(1)}$.

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