Teaching Notes

Betting games: higher-order thinking projects for calculus students

Introduction

In this Note we discuss an activity, conducted at the beginning of the introductory Calculus I course, that has produced favourable results at our University; one of the largest US State Universities, by student enrollment. The project promotes vivid discussions engaging students into higher order thinking skills activities, including *evaluation*, arguably the highest cognitive domain, according to Bloom's taxonomy.

Data analysis supporting our findings are aligned with the ones from [1, 2, 3], and it will be the subject of a forthcoming paper. Also, the activity has helped set the tone of the course, focused more on *understanding* ideas and concepts of Calculus, rather than bluntly memorizing formulas and carrying out calculations.

Activity Description: Do you want to play a game? If so, let's make it interesting; shall we bet one dollar? The game is simple, here are the rules:

- You put one dollar down to the betting pool.
- Every minute that passes by, I will add 1% to the pool's value.
- You can stop any time; however there is a catch (there's always a catch!). You have to divide the total pool's value by the minutes you've played.

To *feel* the game, maybe one should make a sample table:

Minutes Played	Pool's value	Your share
1	\$1.01	\$1.01
2	\$1.0201	\$0.51
3	\$1.030301	\$0.34
4	\$1.040604	\$0.26
5	\$1.05101	\$0.21
10	\$1.1046	\$0.11
60	\$1.8167	\$0.03

It doesn't look very promising to you, does it? Should you cut the losses and stop playing?

Students love a bet, and what is interesting about this game is that one cannot make any reasonable prediction without the aid of mathematics. Asking the right question is a mathematical ability often left out from the classrooms, [4], and in this game students are triggered by the following questions:

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Q1 What is the mathematical formula for one's share in terms of the minutes played?

Q2 What happens if one plays forever?

Q3 If you are the pool owner, how should you stipulate the maximum duration of this game?

Q4 If you are a regulatory organisation, how would you stipulate the maximum duration of this game?

Discussions

Question 1: modelling the game.

Question Q1 pertains to modelling. If n represents the number of minutes played, then the corresponding pool's value, $P(n)$, is equal to $P(n) = \$1 \times (1.01)^n$. Consequently, per the game's rule, if you decide to stop playing after n minutes, the formula for your share, $S(n)$, is:

$$S(n) = \frac{1.01^n}{n}.$$

Upon modelling, the other questions can now be investigated with the lens of Calculus.

Question 2: confronting student's premisses

Students are often intrigued by the expression “play forever”; what does it really mean? After some discussions, students realise that what one is really interested in is understanding what happens with one's share as n becomes arbitrarily large. This is relevant, critical information for the dynamics of the game and the *concept* of limit appears naturally in such a discussion. Mathematically, the question leads to understanding the limit: $\lim_{n \to \infty} \frac{1.01^n}{n}$, which, as opposite to the initial impression given by the table, is infinity!

Questions 3 *and* 4: discussions beyond the realm of mathematics

After some debate, most students agree with the following answer for Q3: *If you are the pool owner, your best strategy is to stipulate the maximum duration of the game by finding the absolute minimum of the value* $S(n)$. Finding global maxima and minima of functions are one of the cornerstone goals of Calculus. In the project, it appears naturally after *evaluation*.

As for question Q4, students often agree that: *as a regulatory organisation, at least it should be allowed for the player to be able to recover his or her initial money*. Some students feel this is not yet fair. Should a regulatory organisation allow for the player to earn more? If so, how much more?

Back to the Calculus of the game: to answer Q3, one is led to compute the minimum value of the real-valued function, $f(x) = \frac{1.01^x}{x}$ over the

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interval $[1, \infty)$. Such a discussion introduces a practical optimisation problem and motivates the technique. This is easily solved with typical Calculus tools, i.e. finding the critical point, to be $x_0 = \dfrac{1}{\ln 1.01} \approx 100$.

Once agreed upon the maximum value allowed for a player to earn in the game, say $a \geqslant 1$, question Q4 leads to the problem of finding n such that $\dfrac{1.01^n}{n} = a$. The solution of such an algebraic equation would most likely not be an integer, and thus one needs to consider the real-variable version of the problem. The existence of such a solution alludes to the Intermediary Value Theorem (IVT) for continuous functions.

According to Q2, $\lim\limits_{n \to \infty} S(n) = +\infty$, hence, the *continuous* version of the function, defined over $[1, \infty)$, is entitled to the IVT, which assures any value a greater than or equal to its minimum value (pertaining to Q3) can be achieved.

Dynamic versions and further projects

The game, as described above, is essentially static, in the sense that the player simply puts a dollar down and waits. To make it more iterative, we can redesign the game as to include a fair dice.

Say at each turn, the player rolls a dice which determines the increment to the pool's value. For instance, if the player gets a 3, then the current pool value will the increased by 3%.

The game gets more interesting and the modelling a bit more challenging. If $d(n) \in \{1, 2, 3, 4, 5, 6\}$ denotes the value the player obtains in the n th roll, then the pool's value $P(n)$ is:

$$\$1 \times 1.01^{d(1)} \times 1.01^{d(2)} \times \dots \times 1.01^{d(n)} = \prod_{j=1}^{n} 1.01^{d(j)} = 1.01^{\sum_{j=1}^{n} d(j)}.$$

Even without a solid understanding about probability and/or fully grasping the more involved formula that models the player's share, students often arrive to the conclusion that this version of the game is more favourable to the player than the previous one. The reasoning is that, in the *worst case scenario*, the player will get all 1s, adding 1% to the pool, which is exactly what he/she would get in the static version of the game anyway. Hence, by *estimation reasonings*, students conclude this new game is better for the player than the previous one.

At this point, we can make things a bit more interesting again, by adding yet another rule. Namely, we impose that, once the player decides to stop, he/she rolls the dice one last time, which shall determine the power of n to divide the pool's total value. Say if the player decides to stop after rolling n dice and at the last roll gets a 5, then final share of he/she will be $\frac{1}{n^5} P(n)$, where $P(n)$ is the pool's value when the player decided to stop, computed in (1).

The follow-up question is then: does adding this final extra rule change the answer on what happens if one plays forever? The answer is 'no' the expected gain is still infinity for a player willing to play forever. Students are led to argue by estimation again. Namely in the worst case scenario, the player gets all straight 1s and in the last roll a whopping 6. In this (unfortunate to the player) scenario, his or her share would be $\frac{1}{n^6}1.01^n$, which still goes to infinity as $n \to \infty$.

Other variants of the game can be designed. For instance, a competitive game between two players and a fair coin. At each turn, a player flips a coin and, say, if he/she gets a head a 1% is increased, but if he/she gets tail, the pool's value decreases. We often ask students to come up with different games by themselves, try to model their games and make further questions about what to expect.

References

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Two quick direct proofs of the irrationality of tan15°

Figure 1 depicts a square *ACDE* of side length $2p$ on one side of which is constructed an equilateral triangle *ABC*. Line *BHGF* is an axis of symmetry with $BF = q$, and triangle *CBG* is isosceles with $BG = BC = 2p$. A short angle-chase shows that the two angles marked α are both 15°.

In triangle *BFE* we have

$$\tan 15^\circ = \frac{p}{q}. \qquad (1)$$