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Vanishing of the higher direct images of the structure sheaf

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Abstract

We prove that the higher direct images of the structure sheaf under a birational and projective morphism between excellent and regular schemes vanish.

1. Introduction

In this article we prove the following theorem.

Theorem 1.1. *Let $f : X \to Y$ be a projective and birational morphism between excellent and regular schemes. Then the higher direct images of $\mathcal{O}_X$ under f vanish, i.e.,*

$$R^i f_* \mathcal{O}_X = 0 \quad \text{for all } i \geqslant 1.$$

In the case where Y is of finite type over a characteristic zero field, this theorem was proved by Hironaka as a corollary of his work on the existence of resolutions of singularities, see [Hir64, (2), p. 144]. In a similar way, one can prove Theorem 1.1 for $\dim Y = 2$, see [Lip69, Proposition 1.2]. If Y is of finite type over a perfect field, then the theorem holds by [CR11, Corollary 3.2.10]. The proof in [CR11, Corollary 3.2.10] uses the action of correspondences on Hodge cohomology. These methods do not seem to generalize to an arithmetic setup. Instead, in this article, we give a more direct proof, which relies on Grothendieck–Serre duality.

In view of [Lip94] and [San84], we obtain the following application in commutative algebra.

Theorem 1.2. *Let R be an excellent regular local ring, and let $I \subset R$ be an ideal such that the blow up $X = \operatorname{Proj} \bigoplus_{n\geqslant 0} I^n$ is regular. The following statements hold.*

(i) *There is $e > 0$ such that the Rees algebra $\bigoplus_{n\geqslant 0} I^{en}$ is Cohen–Macaulay.*

(ii) *There is $e > 0$ such that the associated graded algebra $\bigoplus_{n\geqslant 0} I^{en}/I^{e(n+1)}$ is Cohen–Macaulay.*

It is worth noting that assertion (i), or equivalently assertion (ii), implies the vanishing $H^i(X, \mathcal{O}_X) = 0$, for all $i > 0$, and hence Theorem 1.1 after an easy reduction to the local case.

By using the main result of [BBE07], we obtain the following application, which was known for X and Y defined over a finite field [FR05, Theorem 1.1].

Theorem 1.3. *Let $f : X \to Y$ be as in Theorem 1.1. Let k be a finite field and let $s : \operatorname{Spec} k \to Y$ be a morphism. Denote by $X_s = X \times_Y \operatorname{Spec} k$ the base change of f along s. Then the number of k-rational points of X_s is congruent to 1 modulo the cardinality of k, i.e.,*

$$|X_s(k)| \equiv 1 \bmod |k|.$$

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See § 5 for a proof of this theorem. Theorem 1.1 is a consequence of Theorem 1.4 below. We have to introduce some notation to state it. From now on, all schemes in this introduction will be assumed to be separated, noetherian and excellent (see [EGAIV, 7.8]) and will admit a dualizing complex (see [Har66, V]). Let $f : X \to Y$ be a finite type morphism between integral schemes that is dominant and generically finite and has a regular target Y. Then we define a morphism (see Proposition 2.6)

$$c_f : \mathcal{O}_X \to f^!\mathcal{O}_Y \quad \text{in } D(Y).$$

It is a version of the fundamental class constructed for flat morphisms in [ElZ78], [AE78] and [AJL14]. If f is a proper complete intersection morphism of virtual dimension 0, the morphism c_f corresponds by adjunction to the trace morphism $\tau_f : Rf_*\mathcal{O}_X \to \mathcal{O}_Y$ constructed in [BER12, Theorem 3.1].

Furthermore, let σ_A be a commutative square

$$\begin{tikzcd} V \arrow[d, "f"'] \arrow[dr, phantom, "\sigma_A"] & A \arrow[l, "g_A"'] \arrow[d, "f_A"] \\ Y & Z \arrow[l, "g"] \end{tikzcd}$$

in which f is a morphism of finite type and g, g_A are proper. From duality theory, we obtain a natural transformation of functors $D^b_{\mathrm{c}}(Z) \to D^+_{\mathrm{c}}(V)$,

$$\xi_{\sigma_A} : Rg_{A*}f_A^! \to f^!Rg_*.$$

Theorem 1.4. *Consider the following diagram*

$$\begin{tikzcd} V \arrow[d, "f"'] & \\ Y & Z \arrow[l, "g"] \end{tikzcd}$$

where we assume that the following conditions are satisfied:

(i) *V, Y, Z are integral schemes and Y, Z are regular;*

(ii) *f is of finite type, dominant and generically finite, and the base change $V \times_Y Z \to Z$ is generically finite;*

(iii) *g is projective.*

Then the following equality holds in $\mathrm{Hom}_{D(V)}(\mathcal{O}_V, f^!Rg_*\mathcal{O}_Z)$*:*

$$\begin{aligned}&\Big[\mathcal{O}_V \xrightarrow{c_f} f^!\mathcal{O}_Y \xrightarrow{f^!(g^*)} f^!Rg_*\mathcal{O}_Z\Big] \\ &\quad = \sum_A \ell_A \cdot \Big[\mathcal{O}_V \xrightarrow{g_A^*} Rg_{A*}\mathcal{O}_A \xrightarrow{Rg_{A*}(c_{f_A})} Rg_{A*}f_A^!\mathcal{O}_Z \xrightarrow{\xi_{\sigma_A}} f^!Rg_*\mathcal{O}_Z\Big],\end{aligned}$$

where the sum runs over all irreducible components A of $V \times_Y Z$ that dominate *Z, ℓ_A is the multiplicity of A in the generic fiber over Z, and σ_A is a commutative diagram as above, where g_A and f_A are induced by the composition of the closed immersion $A \hookrightarrow V \times_Y Z$ followed by the projection to V and Z, respectively.*

The formulation of this theorem is reminiscent of intersection theory. Indeed, methods from intersection theory are used implicitly in the proof, which occupies most of the paper. In § 2 we define the map c_f and establish its main properties; in § 3 we give the definition and main properties for the map ξ_σ; in § 4 we give the main reduction steps for the proof of Theorem 1.4; in § 5 we prove the above theorems.

Let us conclude with a list of open questions to which we hope to come back in the future:

(i) Is Theorem 1.1 or Theorem 1.4 still true if one replaces 'projective' by 'proper'?

(ii) Let S be an excellent scheme and let $f : X \to S$ and $g : Y \to S$ be regular S-schemes. Assume X and Y are properly birational over S, i.e., there exist birational and proper S-morphisms $V \to X$, $V \to Y$. Do we have an isomorphism $Rf_*\mathcal{O}_X \cong Rg_*\mathcal{O}_Y$? (If S is separated and of finite type over a perfect field this holds by [CR11, Theorem 1].)

(iii) What kind of singularities can we allow for Y in order that the vanishing of Theorem 1.1 still holds?

Conventions

All schemes in §§ 2–4 are assumed to be separated and noetherian and to admit a dualizing complex. We say a morphism $f : X \to Y$ is projective if it can be factored as a closed immersion $X \hookrightarrow \mathbb{P}^n_Y$ followed by the projection $\mathbb{P}^n_Y \to Y$. For a scheme X, we use the notation $D^*(X)$ and $D^*_{\mathrm{c}}(X)$, with $* \in \{-, +, b\}$, as in [Har66], [Con00].

2. Fundamental class

Let $f : X \to Y$ be a map of finite type. By [Con00, § 3.3] we have

$$f^! : D^+_{\mathrm{c}}(Y) \to D^+_{\mathrm{c}}(X)$$

at our disposal. Whenever f is proper, the trace $\mathrm{Tr}_f : Rf_* \circ f^! \to 1$ induces an isomorphism

$$\mathrm{Hom}_{D(X)}(F, f^!G) \cong \mathrm{Hom}_{D(Y)}(Rf_*F, G), \tag{2.0.1}$$

provided that $F \in D^-_{\mathrm{c}}(X)$ and $G \in D^+_{\mathrm{c}}(Y)$. In particular, we obtain a natural transformation $\mathrm{Tr}^\vee_f : 1 \to f^! \circ Rf_*$ of functors $D^b_{\mathrm{c}}(X) \to D^+_{\mathrm{c}}(X)$ satisfying

$$\mathrm{id}_{Rf_*(F)} = \mathrm{Tr}_f(Rf_*F) \circ Rf_*(\mathrm{Tr}^\vee_f(F)), \tag{2.0.2}$$

$$\mathrm{id}_{f^!(G)} = f^!\mathrm{Tr}_f(G) \circ \mathrm{Tr}^\vee_f(f^!G), \tag{2.0.3}$$

for $F \in D^b_{\mathrm{c}}(X)$ and $G \in D^b_{\mathrm{c}}(Y)$ such that $f^!G \in D^b_{\mathrm{c}}(X)$.

For a closed immersion f the functor $f^!$ is right adjoint to Rf_* when considered as functors on D^+_{c}.

2.1 Fundamental class

Recall from [SGA6, VIII] that a morphism $f : X \to Y$ is a *complete intersection of virtual relative dimension* 0 (ci0 for short) if any point $x \in X$ has an open neighborhood $U \subset X$ such that the restriction of f to U factors as

$$f_{|U} = \pi \circ i, \tag{2.1.1}$$

where $i : U \hookrightarrow P$ is a regular closed immersion of codimension $\mathrm{codim}_P(U) =: n$ and $\pi : P \to Y$ is a smooth morphism of relative dimension $\mathrm{codim}_P(U)$. We collect the following facts from [BER12].

Let $f : X \to Y$ be a ci0 morphism. There exists an invertible sheaf on X, called *the canonical dualizing sheaf* and denoted by $\omega_{X/Y}$, which on any open subset $U \subset X$ on which f factors as in (2.1.1) is isomorphic to

$$(\omega_{X/Y})_{|U} \cong \bigwedge^n (\mathcal{I}/\mathcal{I}^2)^\vee \otimes_{\mathcal{O}_X} i^*\Omega^n_{P/Y}, \tag{2.1.2}$$

where $\mathcal{I}$ is the ideal sheaf of the regular closed immersion $U \hookrightarrow P$ [BER12, A.2]. If $g : Z \to X$ is another ci0 morphism, then there is a canonical isomorphism (see [BER12, A.5])

$$\zeta'_{g,f} : \omega_{Z/Y} \xrightarrow{\simeq} \omega_{Z/X} \otimes_{\mathcal{O}_Z} g^*\omega_{X/Y}.$$

There is a canonical isomorphism

$$\lambda_f : \omega_{X/Y} \xrightarrow{\simeq} f^!\mathcal{O}_Y$$

in $D(X)$ (see [BER12, B1]), and a global section

$$\delta_f \in H^0(X, \omega_{X/Y})$$

with the following property. For any open subset $U \subset X$ on which f factors as in (2.1.1) and such that the ideal sheaf $\mathcal{I}$ of $i : U \hookrightarrow P$ is generated by a regular sequence $t_1, \ldots, t_n$, δ_f is mapped to the following element under the isomorphism (2.1.2):

$$\delta_{f|U} = (\bar{t}_1^\vee \wedge \cdots \wedge \bar{t}_n^\vee) \otimes i^*(dt_n \wedge \cdots \wedge dt_1), \tag{2.1.3}$$

where $(\bar{t}_j^\vee)_j$ is the dual of the basis $(i^*(t_j))_j$ of $\mathcal{I}/\mathcal{I}^2$ [BER12, A.7]. (If $\mathcal{I}$ is the zero ideal, this means that $\delta_{f|U} = 1 \in \mathcal{O}_U$.)

Definition 2.2. Let $f : X \to Y$ be a ci0 morphism. Then we define the morphism

$$c_f : \mathcal{O}_X \to f^!\mathcal{O}_Y$$

to be the composition $\lambda_f \circ \delta_f$.

Proposition 2.3. *Assume $f : X \to Y$ is a ci0 morphism.*

(i) *Let $u : U \to X$ be an étale morphism. Then $f \circ u : U \to Y$ is ci0 and*

$$c_{f\circ u} = u^* c_f,$$

where u^* is the morphism

$$u^* : H^0(X, f^!\mathcal{O}_Y) \to H^0(X, Ru_*u^*f^!\mathcal{O}_Y) = H^0(U, (f \circ u)^!\mathcal{O}_Y).$$

(ii) *Assume $g : Z \to X$ is another ci0 morphism. Then $c_{f\circ g}$ is equal to the composition*

$$\mathcal{O}_Z \xrightarrow{c_g} g^!\mathcal{O}_X \xrightarrow{g^!(c_f)} g^!f^!\mathcal{O}_Y = (f \circ g)^!\mathcal{O}_Y.$$

(iii) *Assume f is finite and flat. Then the composition*

$$f_*\mathcal{O}_X \xrightarrow{f_*(c_f)} f_*f^!\mathcal{O}_Y \xrightarrow{\mathrm{Tr}_f} \mathcal{O}_Y$$

equals the classical trace morphism $\mathrm{trace}_{X/Y} : f_*\mathcal{O}_X \to \mathcal{O}_Y$.

(iv) *Suppose X is integral. If f is not dominant then $c_f = 0$.*

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Proof. (ii) follows from [BER12, Proposition A.8(i) and Proposition B.2]. Now (i) follows from (ii) and the fact that the composition

$$\mathcal{O}_U \xrightarrow{c_u} u^!\mathcal{O}_X \cong u^*\mathcal{O}_X \cong \mathcal{O}_U$$

is equal to the identity, which follows directly from (2.1.3), taking the factorization $i = \mathrm{id}$ and $\pi = u$. The composition in (iii) equals by its very definition the morphism τ_f constructed in [BER12, (B.7.3)] and hence the statement of (iii) follows from [BER12, Theorem 3.1(iii)].

Finally, for (iv) we can assume that f factors as $f = \pi \circ i$ with $i : X \hookrightarrow P$ a regular closed immersion of codimension n and $\pi : P \to Y$ a smooth morphism of relative dimension n. Let $x \in P$ be the generic point of X, and set $y := \pi(x) \in Y$. Since X is regular at x, so is P, and thus Y is regular at y. By assumption, $c := \dim \mathcal{O}_{Y,y} \geqslant 1$. Let $t_1, \ldots, t_c$ be a regular system of parameters which generates the maximal ideal in $\mathcal{O}_{Y,y}$. Further, $\mathcal{O}_{P,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y) = \mathcal{O}_{\pi^{-1}(y),x}$ is a regular local ring of dimension $n - c$, and hence we find elements $s_{c+1}, \ldots, s_n$ in the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{P,x}$ lifting a regular sequence of parameters of $\mathcal{O}_{P,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$. We see that the sequence

$$\pi^*(t_1), \ldots, \pi^*(t_c), s_{c+1}, \ldots s_n$$

generates $\mathfrak{m}_x$ and is thus a regular sequence of parameters for the regular and n-dimensional ring $\mathcal{O}_{P,x}$. Therefore, after shrinking X and Y further we may assume that the ideal sheaf $\mathcal{I}$ of i is generated by a sequence as above. Since $d\pi^*(t_1) = 0$ in $i^*\Omega^1_{P/Y}$, it follows immediately from the description of δ_f in (2.1.3) that c_f vanishes. □

Proposition 2.4. *Let Y be a regular scheme and $f : X \to Y$ a morphism of finite type with the following property:*

$$\dim \mathcal{O}_{Y,f(\eta)} = \operatorname{trdeg}(k(\eta)/k(f(\eta))) \quad \text{for any generic point } \eta \in X. \tag{2.4.1}$$

For every closed subscheme $Z \subset X$ of codimension $\geqslant c$ we have

$$\mathcal{H}^i(R\underline{\Gamma}_Z f^!\mathcal{O}_Y) = 0 \quad \text{for all } i < c.$$

In particular, $f^!\mathcal{O}_Y = \tau_{\geqslant 0} f^!\mathcal{O}_Y$ and the restriction morphism

$$\operatorname{Hom}(\mathcal{O}_X, f^!\mathcal{O}_Y) \to \operatorname{Hom}(\mathcal{O}_U, f^!_{|U}\mathcal{O}_Y)$$

is injective for all dense open subsets $U \subset X$ and is an isomorphism if $\operatorname{codim}_X(X \backslash U) \geqslant 2$.

Proof. Since $f^!\mathcal{O}_Y$ is a dualizing complex, it is a CM complex for a shifted codimension filtration (see [Har66, IV, § 3]). By assumption (2.4.1), [Con00, (3.2.4), p. 129] and [Con00, (3.3.36), p. 145], this shift is 0. □

Remark 2.5. (i) Let $f : X \to Y$ be a morphism of finite type between irreducible schemes which is dominant and generically finite. Then f satisfies condition (2.4.1).

(ii) If $f : X \to Y$ is a finite-type morphism between regular schemes that satisfies condition (2.4.1), then it is a ci0 morphism. (It is clear that f is a complete intersection morphism and it follows from [EGAIV, Proposition (5.6.4)] that its virtual relative dimension is 0.)

Proposition 2.6. *Let $f : X \to Y$ be a finite type morphism between integral and excellent schemes satisfying condition (2.4.1). Assume that Y is regular. There is a unique morphism in $D(X)$,*

$$c_f : \mathcal{O}_X \to f^!\mathcal{O}_Y,$$

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such that the restriction to the open subset of regular points X_{reg} is the class from Definition 2.2 for the ci0 morphism $f_{|X_{\mathrm{reg}}}$. Furthermore, c_f satisfies the analog of the properties (i)–(iv) of Proposition 2.3.

Proof. Uniqueness follows from Proposition 2.4. For the construction, let $\nu : \tilde{X} \to X$ be the normalization. We may construct $c_{f\circ\nu}$ by restricting to the regular locus and applying Proposition 2.4. We set

$$c_f = \left[\mathcal{O}_X \xrightarrow{\nu^*} \nu_*\mathcal{O}_{\tilde{X}} \xrightarrow{\nu_*(c_{f\circ\nu})} \nu_*(f\circ\nu)^!\mathcal{O}_Y = \nu_*\nu^! f^!\mathcal{O}_Y \xrightarrow{\mathrm{Tr}_\nu} f^!\mathcal{O}_Y\right]. \tag{2.6.1}$$

The second statement follows from Proposition 2.4. □

Lemma 2.7. *Let the assumptions be as in Proposition 2.6. Let $g : Z \to X$ be a proper morphism between integral schemes. Assume that $f\circ g : Z \to Y$ satisfies condition (2.4.1). Then*

$$\deg(Z/X)\cdot c_f = \left[\mathcal{O}_X \xrightarrow{g^*} Rg_*\mathcal{O}_Z \xrightarrow{Rg_*(c_{f\circ g})} Rg_*g^! f^!\mathcal{O}_Y \xrightarrow{\mathrm{Tr}_g} f^!\mathcal{O}_Y\right].$$

Proof. We may assume that f and g are dominant and hence generically finite, because both sides vanish otherwise. In view of Proposition 2.4 it suffices to prove the assertion after restricting to a dense open subset of X. We can therefore assume that X, Z are regular and g is finite and flat. Then the statement follows from Proposition 2.3(ii), (iii). □

3. The twisted base change map

Definition 3.1. Let σ be a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & A \arrow[l, "g_1"'] \arrow[d, "f_1"] \\ Y & Z \arrow[l, "g"] \end{tikzcd} \tag{3.1.1}$$

of finite type morphisms. We call σ an *admissible square* if g and g_1 are proper. We define the natural transformation of functors $D^b_{\mathrm{c}}(Z) \to D^+_{\mathrm{c}}(X)$

$$\xi_\sigma : Rg_{1*}f_1^! \to f^! Rg_*$$

to be the composition

$$\xi_\sigma : Rg_{1*}f_1^! \xrightarrow{\mathrm{Tr}_g^\vee} Rg_{1*}f_1^! g^! Rg_* \xrightarrow{=} Rg_{1*}g_1^! f^! Rg_* \xrightarrow{\mathrm{Tr}_{g_1}} f^! Rg_*.$$

Lemma 3.2. *Let σ be an admissible square as in (3.1.1).*

(i) *Let*

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & A \arrow[l, "g_1"'] \arrow[d, "f_1"] \arrow[dr, phantom, "\sigma_1"] & B \arrow[l, "h_1"'] \arrow[d, "f_2"] \\ Y & Z \arrow[l, "g"] & V \arrow[l, "h"] \end{tikzcd}$$

be two admissible squares and denote by σ_2 their composition. We have

$$\xi_{\sigma_2}(F) = \xi_\sigma(Rh_*(F)) \circ Rg_{1*}(\xi_{\sigma_1}(F)) \quad F \in D^b_{\mathrm{c}}(V).$$

(ii) *Let*

$$\begin{tikzcd} W \arrow[d, "e"'] & B \arrow[l, "g_2"'] \arrow[d, "e_1"] \arrow[dl, phantom, "\sigma_1"] \\ X \arrow[d, "f"'] & A \arrow[l, "g_1"'] \arrow[d, "f_1"] \arrow[dl, phantom, "\sigma"] \\ Y & Z \arrow[l, "g"] \end{tikzcd}$$

be two admissible squares and denote by σ_2 *their composition.*

If $G \in D^b_c(Z)$ *and* $f_1^! G, f_1^! g^! Rg_* G \in D^b(A)$, *then we have*

$$\xi_{\sigma_2}(G) = e^!(\xi_\sigma(G)) \circ \xi_{\sigma_1}(f_1^!(G)).$$

(iii) *Assume the morphisms* f, f_1 *are proper. If* $G \in D^b_c(Z)$ *and* $f_1^! G \in D^b(A)$, *then* ξ_σ *equals the composition*

$$Rg_{1*} f_1^!(G) \xrightarrow{\mathrm{Tr}_f^\vee} f^! Rf_* Rg_{1*} f_1^!(G) \xrightarrow{=} f^! Rg_* Rf_{1*} f_1^!(G) \xrightarrow{f^! Rg_*(\mathrm{Tr}_{f_1})} f^! Rg_*(G). \tag{3.2.1}$$

(iv) *Assume* σ *is cartesian and* f *is étale. Then* ξ_σ *equals the base change isomorphism* $Rg_{1*} f_1^* \xrightarrow{\simeq} f^* Rg_*$.

(v) *Assume the morphisms* f, f_1 *are closed immersions. Then* $f_*(\xi_\sigma)$ *is the morphism*

$$Rg_* R\mathcal{H}\mathrm{om}_Z(f_{1*}\mathcal{O}_A, -) \to Rg_* R\mathcal{H}\mathrm{om}(Lg^* f_* \mathcal{O}_X, -) \cong R\mathcal{H}\mathrm{om}_Y(f_* \mathcal{O}_X, Rg_*(-)),$$

where the first map is induced by the natural map $Lg^* f_* \mathcal{O}_X \to f_{1*}\mathcal{O}_A$.

(vi) *Assume that the square* σ *is tor-independent. If* $G \in D^b_c(Z)$ *satisfies* $g^! Rg_*(G) \in D^b(Z)$ *then* $\xi_\sigma(G)$ *is an isomorphism.*

(vii) *Suppose* $j : U \to Y$ *is an open immersion. Denote by* σ_U *the commutative square*

$$\begin{tikzcd} f^{-1}(U) \arrow[d, "\bar{f}"'] & (f \circ g_1)^{-1}(U) \arrow[l, "\bar{g}_1"'] \arrow[d, "\bar{f}_1"] \arrow[dl, phantom, "\sigma_U"] \\ U & g^{-1}(U) \arrow[l, "\bar{g}"] \end{tikzcd}$$

induced by σ. *Then* $\xi_{\sigma_U} = j^* \xi_\sigma$, *where we use the natural transformations* $j^* Rg_{1*} f_1^! = R\bar{g}_{1*} \bar{f}_1^! j^*$ *and* $j^* f^! Rg_* = \bar{f}^! R\bar{g}_* j^*$.

Proof. Claim (i) follows directly from [Con00, Lemma 3.4.3 (TRA1)]. Claim (ii) follows from the definition and (2.0.3) by a straightforward computation.

Assume f and f_1 are proper. In order to prove Claim (iii), note that (3.2.1) is adjoint to $Rg_*(\mathrm{Tr}_{f_1}(G))$. It follows easily from [Con00, Lemma 3.4.3 (TRA1)] and (2.0.2) that $\xi_\sigma(G)$ has the same property.

Claims (iv) and (vii) follow from [Con00, Lemma 3.4.3 (TRA4)] and (2.0.2). Claim (v) follows in the same way as (iii) by using the fact that Tr_{f_1} on the left-hand side of the composition in (v) is given by precomposition with $\mathcal{O}_Z \to f_{1*}\mathcal{O}_A$ and Tr_f on the right-hand side is given by precomposition with $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

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Let us prove (vi). The question is local on X; thus we can assume that f factors as $X \xrightarrow{i} U \xrightarrow{j} P := \mathbb{P}^n_Y \xrightarrow{\pi} Y$, where i is a closed immersion, j is an open immersion and π is the projection. We can factor σ into three admissible cartesian squares, as in the following diagram.

$$\begin{tikzcd} X \arrow[d, "i"'] \arrow[dr, phantom, "\sigma_i"] & A \arrow[l, "g_1"'] \arrow[d, "i_A"] \\ U \arrow[d, "j"'] \arrow[dr, phantom, "\sigma_j"] & U_Z \arrow[l, "g_U"'] \arrow[d, "j_A"] \\ P \arrow[d, "\pi"'] \arrow[dr, phantom, "\bar\sigma"] & P_Z \arrow[l, "\bar g_1"'] \arrow[d, "\pi_1"] \\ Y & Z \arrow[l, "g"] \end{tikzcd}$$

The square σ_i is tor-independent and hence ξ_{σ_i} is an isomorphism by (v). It follows from (iv) that ξ_{σ_j} is an isomorphism. Since π_1 and $\pi_1 \circ j_A$ are smooth (and hence bounded complexes are mapped to bounded complexes via $(\cdot)^!$), we may use (ii) and prove that $\xi_{\bar\sigma}(G)$ is an isomorphism. Set $\omega_\pi := \Omega^n_{P/Y}$; we define the isomorphism $\xi'_{\bar\sigma} : R\bar g_{1*}\pi_1^! \to \pi^! Rg_*$ as the composition

$$\begin{aligned} R\bar g_{1*}\pi_1^! &\cong R\bar g_{1*}(\omega_{\pi_1}[n] \otimes \pi_1^*(-)) \cong R\bar g_{1*}(\bar g_1^*\omega_\pi[n] \otimes \pi_1^*(-)) \\ &\cong \omega_\pi[n] \otimes R\bar g_{1*}\pi_1^* \cong \omega_\pi[n] \otimes \pi^* Rg_* \cong \pi^! Rg_*. \end{aligned}$$

It suffices to show $\xi'_{\bar\sigma} = \xi_{\bar\sigma}$, which by (iii) is equivalent to

$$\mathrm{Tr}_\pi(Rg_*G) \circ R\pi_*(\xi'_{\bar\sigma}(G)) = Rg_*(\mathrm{Tr}_{\pi_1}(G)) \quad G \in D^b_c(Z).$$

This follows directly from the definition of the projective trace (see [Con00, 2.3]) and [Con00, (2.4.1)]. □

4. Reductions

4.1 Setup

We consider the diagram

$$\begin{tikzcd} V \arrow[d, "f"'] & \\ Y & Z \arrow[l, "g"] \end{tikzcd}$$

from Theorem 1.4. Recall that:

(i) V, Y, Z are integral noetherian excellent schemes and Y, Z are regular;

(ii) f is of finite type and is dominant and generically finite, and the base change $V \times_Y Z \to Z$ is generically finite;

(iii) g is projective.

For an irreducible component A of $V \times_Y Z$, we denote by ℓ_A the multiplicity of A in the generic fiber over Z; if A does not dominate Z then $\ell_A = 0$. We denote by g_A and f_A the composition of the closed immersion $A \hookrightarrow V \times_Y Z$ with the projection to V and Z, respectively, and by σ_A the corresponding admissible square.

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Notation 4.2. With the above notation and assumptions, we say $E(V \to Y \leftarrow Z)$ *holds* if the following equality holds in $\mathrm{Hom}_{D(V)}(\mathcal{O}_V, f^! Rg_*\mathcal{O}_Z)$:

$$\Big[\mathcal{O}_V \xrightarrow{c_f} f^!\mathcal{O}_Y \xrightarrow{f^!(g^*)} f^! Rg_*\mathcal{O}_Z\Big] = \sum_A \ell_A \cdot \Big[\mathcal{O}_V \xrightarrow{g_A^*} Rg_{A*}\mathcal{O}_A \xrightarrow{Rg_{A*}(c_{f_A})} Rg_{A*}f_A^!\mathcal{O}_Z \xrightarrow{\xi_{\sigma_A}} f^! Rg_*\mathcal{O}_Z\Big],$$

where the sum runs over all irreducible components of A with $\ell_A \neq 0$. (Notice that by conditions (i) and (ii) above, c_f and c_{f_A} are defined.)

LEMMA 4.3. *If f is finite and flat, then $E(V \xrightarrow{f} Y \xleftarrow{g} Z)$ holds.*

Proof. The cartesian square

$$\begin{tikzcd} V \arrow[d, "f"'] & V \times_Y Z \arrow[l, "g_1"'] \arrow[d, "f_1"] \\ Y & Z \arrow[l, "g"] \end{tikzcd}$$

is tor-independent, and $g^!$ preserves bounded complexes (since Y, Z are regular). Therefore we can use Lemma 3.2(vi) and Proposition 2.4 to obtain an injective map

$$\mathrm{Hom}(\mathcal{O}_V, f^! Rg_*\mathcal{O}_Z) = \Gamma(V, g_{1*}\mathcal{H}^0(f_1^!\mathcal{O}_Z)) \hookrightarrow \Gamma(f^{-1}(U) \times_U g^{-1}(U), g_{1*}\mathcal{H}^0(f_1^!\mathcal{O}_Z))$$

for every open $U \subset Y$ such that $g^{-1}(U) \neq \emptyset$. Therefore we may replace Y by any such U. In particular, we may suppose that for every irreducible component A of $V \times_Y Z$ the induced map f_A is flat.

By Proposition 2.6(iii), the maps c_f and c_{f_A} are adjoint to the trace maps. By Lemma 3.2(iii) we have to show

$$\Big[f_*\mathcal{O}_V \xrightarrow{\mathrm{Trace}} \mathcal{O}_Y \to g_*\mathcal{O}_Z\Big] = \sum_A \ell_A \cdot \Big[f_*\mathcal{O}_V \xrightarrow{f_*(g_A^*)} g_*f_{A*}\mathcal{O}_A \xrightarrow{g_*(\mathrm{Trace})} g_*\mathcal{O}_Z\Big],$$

which is a straightforward computation. □

PROPOSITION 4.4. *If $\mathcal{O}_Y \xrightarrow{g^*} Rg_*\mathcal{O}_Z$ is an isomorphism in $D_c^b(Y)$, then $E(V \to Y \leftarrow Z)$ holds.*

Proof. For every non-empty open $U \subset Y$, the map

$$\mathrm{Hom}_{D_c^b(V)}(\mathcal{O}_V, f^!\mathcal{O}_Y) \to \mathrm{Hom}_{D_c^b(f^{-1}(U))}(\mathcal{O}_{f^{-1}(U)}, f^!\mathcal{O}_U)$$

is injective (Proposition 2.4). Hence, we may assume that f is finite and flat and the statement follows from Lemma 4.3. □

LEMMA 4.5. *Consider a commutative diagram*

$$\begin{tikzcd} V \arrow[d] & P_V \arrow[l] \arrow[d] & \\ Y & P \arrow[l] & Z \arrow[l] \end{tikzcd}$$

where $P \to Y$ is projective and surjective, P is regular, P_V is integral, $P_V \to V \times_Y P$ is a closed immersion and P_V is the only irreducible component of $V \times_Y P$ dominating P. Assume $E(V \to Y \leftarrow P)$ and $E(P_V \to P \leftarrow Z)$ hold. Then $E(V \to Y \leftarrow Z)$ holds.

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Proof. Note that the irreducible components of $V\times_Y Z$ dominating Z are exactly the irreducible components of $P_V\times_P Z$ dominating Z. Thus the statement follows via a direct computation from Lemma 3.2(i). □

Corollary 4.6. *Assume $E(V\xrightarrow{f}Y\xleftarrow{g}Z)$ holds for all closed immersions g. Then $E(V\xrightarrow{f}Y\xleftarrow{g}Z)$ also holds for all projective morphisms g.*

Proof. This follows directly from Proposition 4.4, Lemma 4.5 and the equality $R\pi_*\mathcal{O}_{\mathbb{P}^n_Y}=\mathcal{O}_Y$, where $\pi:\mathbb{P}^n_Y\to Y$ is the projection. □

Proposition 4.7. *Assume $E(V\xrightarrow{f}Y\xleftarrow{g}Z)$ holds for any closed immersion g of codimension* 1. *Then $E(V\xrightarrow{f}Y\xleftarrow{g}Z)$ holds for any closed immersion g.*

Proof. Assume $g:Z\hookrightarrow Y$ is a closed immersion. Since Y and Z are regular, g is a regular closed immersion. Let $\tilde{Y}$ and $\tilde{V}$ denote the blow up of Y in Z and V in $V\times_Y Z$, respectively. We form the commutative diagram

$$\begin{tikzcd} V \arrow[d, "f"'] & \tilde{V} \arrow[l] \arrow[d] & \\ Y & \tilde{Y} \arrow[l, "\pi"] & E \arrow[l] \end{tikzcd}$$

where E is the exceptional divisor. Denote by $\pi_E:E\to Z$ the base change of π along g. As is well known we have $R\pi_*\mathcal{O}_{\tilde{Y}}=\mathcal{O}_Y$. Thus by Proposition 4.4 $E(V\to Y\leftarrow\tilde{Y})$ holds and by assumption $E(\tilde{V}\to\tilde{Y}\leftarrow E)$ also holds. Hence $E(V\xrightarrow{f}Y\xleftarrow{g\circ\pi_E}E)$ holds by Lemma 4.5.

As $\pi_E:E\to Z$ is a projective bundle we have $R\pi_{E*}\mathcal{O}_E\cong\mathcal{O}_Z$ and the irreducible components of $V\times_Y Z$ correspond via $A\mapsto A\times_Z E$ to the irreducible components of $V\times_Y E$; further, $\ell_A=\ell_{A\times_Z E}$. Set $E_A:=A\times_Z E$ and form the admissible squares shown in the following diagram.

$$\begin{tikzcd} V \arrow[d, "f"'] \arrow[dr, phantom, "\sigma_A"] & A \arrow[l, "g_A"'] \arrow[d, "f_A"] \arrow[dr, phantom, "\sigma_{E_A}"] & E_A \arrow[l, "\pi_{E_A}"'] \arrow[d, "f_{E_A}"] \\ Y & Z \arrow[l, "g"] & E \arrow[l, "\pi_E"] \end{tikzcd}$$

We denote the big outer admissible square by σ_{V,E_A}. Let A be an irreducible component of $V\times_Y Z$ dominating Z. Proposition 4.4 implies that $E(A\xrightarrow{f_A}Z\xleftarrow{\pi_E}E)$ holds, i.e.,

$$c_{f_A}=\Big[\mathcal{O}_A=R\pi_{E_A*}\mathcal{O}_{E_A}\xrightarrow{R\pi_{E_A*}(c_{f_{E_A}})}R\pi_{E_A*}f^!_{E_A}\mathcal{O}_E\xrightarrow{\xi_{\sigma_{E_A}}}f^!_AR\pi_{E*}\mathcal{O}_E=f^!_A\mathcal{O}_Z\Big]. \tag{4.7.1}$$

We obtain

$$\begin{aligned}&\Big[\mathcal{O}_V\to g_{A*}R\pi_{E_A*}\mathcal{O}_{E_A}\xrightarrow{g_{A*}R\pi_{E_A*}(c_{f_{E_A}})}g_{A*}R\pi_{E_A*}f^!_{E_A}\mathcal{O}_E\xrightarrow{\xi_{\sigma_{V,E_A}}}f^!g_*\mathcal{O}_Z\Big]\\ &=\Big[\mathcal{O}_V\to g_{A*}R\pi_{E_A*}\mathcal{O}_{E_A}\to g_{A*}R\pi_{E_A*}f^!_{E_A}\mathcal{O}_E\xrightarrow{g_{A*}\xi_{\sigma_{E_A}}}g_{A*}f^!_A\mathcal{O}_Z\xrightarrow{\xi_{\sigma_A}}f^!g_*\mathcal{O}_Z\Big]\\ &=\Big[\mathcal{O}_V\to g_{A*}\mathcal{O}_A\xrightarrow{c_{f_A}}g_{A*}f^!_A\mathcal{O}_Z\xrightarrow{\xi_{\sigma_A}}f^!g_*\mathcal{O}_Z\Big].\end{aligned}$$

Here, the first equality follows from Lemma 3.2(i) and the second equality follows from (4.7.1). Thus $E(V\xrightarrow{f}Y\xleftarrow{g}Z)$ holds if and only if $E(V\xrightarrow{f}Y\xleftarrow{g\circ\pi_E}E)$ holds, which proves the proposition. □

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5. Proofs

Proof of Theorem 1.4. By Corollary 4.6 and Proposition 4.7 we can assume that $g : Z \hookrightarrow Y$ is a closed immersion of codimension 1.

Step 1: Reduction to V being normal. Let $\nu : \tilde{V} \to V$ be the normalization. We claim that via the map

$$\mathrm{Hom}(\mathcal{O}_{\tilde{V}}, (f \circ \nu)^! Rg_* \mathcal{O}_Z) \to \mathrm{Hom}(\mathcal{O}_V, f^! Rg_* \mathcal{O}_Z) \quad a \mapsto \mathrm{Tr}_\nu(\nu_*(a) \circ \nu^*),$$

both sides of $E(\tilde{V} \xrightarrow{f \circ \nu} Y \xleftarrow{g} Z)$ are mapped to the corresponding side of $E(V \xrightarrow{f} Y \xleftarrow{g} Z)$. For the left-hand side this follows from the construction of c_f, see (2.6.1). For the right-hand side first observe that if $B \xrightarrow{\nu_B} A$ is a finite surjective morphism between integral schemes, then by Lemma 2.7,

$$\mathrm{Tr}_{\nu_B}(\nu_{B*}(c_{f_A \circ \nu_B}) \circ \nu_B^*) = \deg(B/A) \cdot c_{f_A}.$$

Thus the claim follows from

$$\ell_A = \sum_B \ell_B \cdot \deg(B/A),$$

where A is an irreducible component of $V \times_Y Z$, and the sum runs over all irreducible components of $\tilde{V} \times_Y Z$ mapping to A, see [Ful98, Example A.3.1].

Step 2: Reduction to V being regular. By our assumption on g, the following diagram is tor-independent.

$$\begin{tikzcd} V \arrow[d, "f"'] & V \times_Y Z \arrow[l, "g_1"'] \arrow[d, "f_1"] \arrow[dl, phantom, "\sigma"] \\ Y & Z \arrow[l, "g"] \end{tikzcd}$$

Hence $\xi_\sigma(\mathcal{O}_Z) : g_{1*} f_1^! \mathcal{O}_Z \to f^! g_* \mathcal{O}_Z$ is an isomorphism, by Lemma 3.2(vi). Further, f_1 satisfies condition (2.4.1), by [EGAIV, Proposition 5.6.5]. Since we have to prove an equality in $\mathrm{Hom}(\mathcal{O}_V, f^! g_* \mathcal{O}_Z) = \mathrm{Hom}(\mathcal{O}_V, g_{1*} f_1^! \mathcal{O}_Z)$ we can use Proposition 2.4 to remove a codimension $\geqslant 2$ subset of V. Thus we can assume that V is regular and the irreducible components of $V \times_Y Z$ are disjoint.

Step 3: End of proof. Let us write $V \times_Y Z = \coprod_{i=1}^r A_i$ for the decomposition into connected components. Let $s \in \mathrm{Hom}(\mathcal{O}_V, f^! g_* \mathcal{O}_Z)$ be the element corresponding to $\mathcal{O}_V \xrightarrow{c_f} f^! \mathcal{O}_Y \xrightarrow{f^!(g^*)} f^! g_* \mathcal{O}_Z$. We denote by $(s_{A_i})_i$ the image of s via the map

$$\begin{aligned} \mathrm{Hom}(\mathcal{O}_V, f^! g_* \mathcal{O}_Z) &\xrightarrow{=} \Gamma(V, \mathcal{H}^0(f^! g_* \mathcal{O}_Z)) \xrightarrow{\cong} \Gamma(V \times_Y Z, \mathcal{H}^0(f_1^! \mathcal{O}_Z)) \\ &\xrightarrow{=} \bigoplus_i \Gamma(A_i, \mathcal{H}^0(f_{A_i}^! \mathcal{O}_Z)). \end{aligned}$$

We claim that $s_{A_i} = 0$ if A_i does not dominate Z. Indeed, V is regular, and hence $f_{A_i} : A_i \to Z$ is a ci0 morphism. Using the definition of the fundamental class, Definition 2.2 and (2.1.3), one directly checks that $s_{A_i} = c_{f_{A_i}}$. Thus Proposition 2.3(iv) implies $s_{A_i} = 0$.

Having no contributions from the non-dominant irreducible components, we may replace Y by any open subset U such that $U \cap Z \neq \emptyset$, and we may assume that $f^{-1}(U) \to U$ is finite and flat. Now the statement follows from Lemma 4.3. $\square$

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Proof: Theorem 1.4 ⇒ Theorem 1.1. We can assume that Y is noetherian. Since f is birational, the only irreducible component of $X \times_Y X$ which dominates X is the diagonal Δ. Let σ_Δ be the commutative square shown in the following diagram.

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, phantom, "\sigma_\Delta"] & \Delta \arrow[l, "\simeq"'] \arrow[d, "\simeq"] \\ Y & X \arrow[l, "f"] \end{tikzcd}$$

By Lemma 3.2(iii), ξ_{σ_Δ} is the natural transformation $\mathrm{id} \to f^! Rf_*$, which by adjunction corresponds to the identity on Rf_*. Theorem 1.4 gives $f^* \circ c_f = \xi_{\sigma_\Delta}$. Thus by adjunction the identity on $Rf_*\mathcal{O}_X$ factors as $Rf_*\mathcal{O}_X \to \mathcal{O}_Y \xrightarrow{f^*} Rf_*\mathcal{O}_X$. This proves Theorem 1.1. □

Proof of Theorem 1.2. Assertion (i) is equivalent to $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$, by [Lip94, Theorem 4.1]. Assertion (ii) is equivalent to $H^i(X, \omega_X) = 0$ for all $i > 0$, in view of [San84] (see [Lip94, Theorem 4.3] and the following remark). Therefore it follows from Theorem 1.1 by duality. □

Proof of Theorem 1.3. Let p be the characteristic of k. Again, we may assume that Y is noetherian. Let $W_n\mathcal{O}_X$ denote the sheaf of (p-typical) Witt vectors of length n and $W\mathcal{O}_X = \varprojlim_n W_n\mathcal{O}_X$ the sheaf of (p-typical) Witt vectors. Set $W := W(k)$ and $K_0 := \mathrm{Frac}(W) = W[1/p]$. By [BBE07, Corollary 1.3, Proposition 6.3] it suffices to show

$$H^0(X_s, W\mathcal{O}_{X_s}) \otimes_W K_0 = K_0 \quad \text{and} \quad H^i(X_s, W\mathcal{O}_{X_s}) \otimes_W K_0 = 0 \quad i \geqslant 1.$$

If κ is the residue field of the image point of s in Y, then the natural inclusion $W(\kappa) \hookrightarrow W$ is étale. Thus it suffices to prove the above equalities in the case where s is a closed immersion, i.e., $s \in Y$ is a closed point with residue field k.

Set $A = H^0(X_s, \mathcal{O}_{X_s})$. Then $\operatorname{Spec} A \to \operatorname{Spec} k$ is finite, surjective and geometrically connected, and hence radical. Since k is perfect we obtain that A is an artinian local k-algebra with residue field k. In particular,

$$H^0(X_s, W\mathcal{O}_{X_s}) \otimes K_0 = W(A) \otimes K_0 = K_0,$$

where the second equality follows from $F \circ V = p = V \circ F$ on $W(A)$, where $F : W(A) \to W(A)$, $(a_0, a_1, \ldots) \mapsto (a_0^p, a_1^p, \ldots)$ is the Frobenius morphism on the Witt vectors.

Denote by $f_p : X_p = X \times_{\mathbb{Z}} \mathbb{F}_p \to Y_p$ the base change of f over $\mathbb{F}_p$. If $X_p = X$ then

$$R^i f_{p*}\mathcal{O}_{X_p} = 0 \quad \text{for all } i \geqslant 1 \tag{5.0.2}$$

follows immediately from Theorem 1.1. If $p \neq 0$ in $\mathcal{O}_X$ then we can use the exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\cdot p} \mathcal{O}_X \to \mathcal{O}_{X_p} \to 0$$

to prove (5.0.2).

For all $n \geqslant 1$ we have an exact sequence of sheaves of abelian groups

$$0 \to W_{n-1}\mathcal{O}_{X_p} \xrightarrow{V} W_n\mathcal{O}_{X_p} \to \mathcal{O}_{X_p} \to 0,$$

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where V is the Verschiebung, $V(a_0,\ldots,a_{n-2})=(0,a_0,\ldots,a_{n-2})$ and the map on the right is the restriction $(a_0,\ldots,a_{n-1})\mapsto a_0$. Hence $R^if_*W_n\mathcal{O}_{X_p}=0$, for all $n,i\geqslant 1$. Further, we have exact sequences for all $i\geqslant 1$

$$0\to R^1\varprojlim_n R^{i-1}f_*W_n\mathcal{O}_{X_p}\to R^if_*W\mathcal{O}_{X_p}\to \varprojlim_n R^if_*W_n\mathcal{O}_{X_p}\to 0.$$

Thus also $R^if_*W\mathcal{O}_{X_p}=0$ for all $i\geqslant 1$. (For the case $i=1$ we use that the restriction maps $f_*W_n\mathcal{O}_{X_p}\to f_*W_{n-1}\mathcal{O}_{X_p}$ are surjective, which implies the vanishing of $R^1\varprojlim_n f_*W_n\mathcal{O}_{X_p}$.)

Now denote by $\mathcal{I}$ the ideal sheaf of X_s in X_p. We obtain a long exact sequence

$$\cdots\to(R^if_*W\mathcal{O}_{X_p})\otimes K_0\to(R^if_{s*}W\mathcal{O}_{X_s})\otimes K_0\to(R^{i+1}f_*W\mathcal{I})\otimes K_0\to\cdots.$$

By the above the term on the left vanishes and the term on the right vanishes by [CR12, Proposition 4.6.1], which is a slight modification of [BBE07, Theorem 2.4(i)]. (In [BBE07] there is a general assumption that the schemes considered have to be of finite type over a perfect field. One can check immediately that this assumption is not used in the parts we refer to.) This proves Theorem 1.3. □

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