ON THE TEMPERATURE GRADIENT IN THE UPPER PART OF COLD ICE SHEETS

By K. PHILBERTH *and* B. FEDERER

(Eidg. Institut für Schnee- und Lawinenforschung, Davos, Switzerland)

ABSTRACT. The influence of the surface slope on the temperature profile in the upper part of a cold ice sheet can be described by a function which is independent of the geothermal heat and the heat of friction. This function is calculated for the two-dimensional and the axisymmetric cases. In the two-dimensional case its simplest form is proportional to the horizontal velocity and to the height above bedrock reduced by a constant; another form of this function is approximately proportional to the square of velocity and height.

RÉSUMÉ. *Sur le gradient de température dans les couches supérieures des calottes glaciaires froides.* L'influence de l'inclinaison de la surface sur le profil de température dans la partie supérieure d'une calotte glaciaire froide peut être décrite par une fonction qui est indépendante des chaleurs géothermique et de friction. Cette fonction est calculée pour les cas bidimensionnel et axisymmetrique. Dans le cas bidimensionnel la forme la plus simple de cette fonction est proportionnelle à la vitesse horizontale et à l'altitude sur le lit rocheux diminuée d' une constante; une autre forme est donnée qui est approximativement proportionnelle au carré de ces deux facteurs.

ZUSAMMENFASSUNG. *Über den Temperaturgradienten im oberen Teil von kalten Eisschilden.* Der Einfluss der Oberflächenneigung auf das Temperaturprofil im oberen Teil eines kalten Eisschildes lässt sich als Funktion beschreiben, die unabhängig ist von der Erd- und der Reibungswärme. Diese Funktion ist für den zwei-dimensionalen und für den rotationssymetrischen Fall angegeben. Im zweidimensionalen Fall ist ihre einfachste Form proportional der Horizontalgeschwindigkeit und der um eine Konstante reduzierten Höhe über dem Felsboden; eine andere Form ist angenähert proportional dem Quadrat dieser beiden Grössen.

I. INTRODUCTION

The negative temperature gradient in the upper part of large ice sheets has two sources: The movement of the ice in the x-direction and climatic changes. Both effects result in a temperature decrease with depth of the order of 1 deg. It is possible to draw conclusions about climatic changes if the influence of the movement is determined precisely.

Theoretical and field work on this subject has been published by Robin (1955, 1970), Bogoslovski (1958), Radok (1959), Weertman (1968), Budd (1969), Budd and others ([1970]), Radok and others (1970), Budd and Radok (1971). It seems, however, that the following useful solution for the upper part of ice sheets has been overlooked so far.

II. GENERAL SOLUTION FOR THE TWO-DIMENSIONAL CASE

The homogeneous form of the heat transport equation under steady-state conditions is

$$\kappa \, \partial^2 T/\partial y^2 + \kappa \, \partial^2 T/\partial x^2 - v_y \, \partial T/\partial y - v_x \, \partial T/\partial x = 0 \qquad (1)$$

where κ is the diffusivity of ice, y the height above bedrock, x the distance from the ice divide, v_y and v_x the vertical and horizontal velocities respectively.

In the upper part of the ice sheet, v_x is considered to be independent of y (Philberth, B., 1956; Haefeli, 1961; Weertman, 1968; Dansgaard and Johnsen, 1969[b]; Philberth, 1972[b]), and $\partial v_x/\partial x$ can be written

$$dv_x/dx = v_x'; \quad \partial^2 v_x/\partial x^2 = d^2v_x/dx^2 = v_x''.$$

Using the continuity equation we can write $\partial v_y/\partial y = -v_x'$. Hence $v_y = -v_x' y_r$, where y_r, the "reduced height" is equal to $y - h_0(x)$. The term $h_0(x)$ can be visualized as the height below which the ice has zero velocity and above which it is governed by the block flow law. Usually $h_0(x)$ is taken to be zero (Robin, 1955; Weertman, 1968; Budd, 1969; Radok and others, 1970). In this paper we shall use the relation:

$$y_r = y - h_0(x) = y - h(1 - v_{xm}/v_x) \qquad (2)$$

where $h = h(x)$ is the total thickness of the ice sheet, v_{xm} is the mean value of the horizontal velocity over the total ice sheet and v_x is its value in the upper part of the ice sheet. Corresponding to y_r we shall use the "reduced ice-thickness" $h_r = h_r(x) = h(x) - h_0(x)$. The ratio h_r/h is equal to v_{xm}/v_x. For the station Jarl Joset, central Greenland, this ratio is 0.9 (Philberth, 1972[b]).

Using $v_y = -v_x'y_r$ and neglecting the heat diffusivity in the x-direction, Equation (1) can now be written

$$\kappa \, \partial^2 T/\partial y^2 + v_x'y_r \, \partial T/\partial y - v_x \, \partial T/\partial x = 0. \tag{3}$$

The simplest non-trivial solution of this equation is

$$T = C_1 v_x y_r \tag{4}$$

where C_1, and C_0 and C_2 used below, are constants.

If $v_x v_x''/v_x'^2$ is small either with respect to $2y_r^2 v_x'/\kappa$ or to 2 (for station Jarl Joset the first term is 0.15; the second term is >10, down to a depth of 1 km), the following approximate solution can also be used

$$T = C_2 v_x^2 [y_r^2 + \kappa/v_x']. \tag{5}$$

Solutions with higher powers of v_x need not be considered for practical use.

The combination of Equations (4) and (5) leads to the general solution

$$T = C_0 + C_1 v_x y_r + C_2 v_x^2 [y_r^2 + \kappa/v_x']. \tag{6}$$

III. GENERAL SOLUTION FOR DIVERGENT MOVEMENT

A generalization of Equation (6) for cases of divergent movement can be obtained by introducing a factor ϵ characterizing the divergence of the streamlines

$$T = C_0 + C_1 v_x^\epsilon y_r + C_2 v_x^{2\epsilon} [y_r^2 + \kappa/(\epsilon v_x')]. \tag{7}$$

In the two dimensional case $\epsilon = 1$. If the accumulation a and the total depth of the ice h are independent of x, ϵ is 2 in the axisymmetric case and between 1 and 2 in intermediate cases. If $a = a(x)$ and $h = h(x)$, however, in the axisymmetric case ϵ is equal to $1 + (dv_z/dz)/v_x' = 1 + (v_x/x)/v_x'$, where z is the horizontal direction orthogonal to the streamline. If these last expressions are not constant, a mean value along the streamline has to be used.

IV. APPLICATION OF THE TWO-DIMENSIONAL SOLUTION

Differentiating Equation (6) with respect to x, using $T = T_s$ (surface temperature) and $y = h(x)$, hence $y_r = h_r$, yields

$$dT_s/dx = C_1 a + 2C_2 [a + \kappa/h_r] \int_0^x a \, dx \tag{8}$$

where we introduced the substitution $v_x = h_r^{-1} \int_0^x a \, dx$, with $a = a(x)$ denoting the ice value of the accumulation rate. In Equation (8) the term $(C_2 \kappa v_x) v_x v_x''/v_x'^2$ is neglected for the reasons explained in connection with Equation (5). Equation (8) allows us to determine the constants C_1 and C_2.

Under normal conditions dT_s/dx, the horizontal gradient of the surface temperature, and $a(x)$ increase with increasing x. If they increase at an equal rate, C_- becomes zero. Using Equation (8) with $C_2 = 0$, Equation (6) becomes

$$T = (a^{-1} \, dT_s/dx) v_x y_r = (dT_s/dx) \, a^{-1} y_r h_r^{-1} \int_0^x a \, dx. \tag{9}$$

Near the surface, y_r/h_r approximates y/h.

The solutions (4) and (9) do not take into account the heat of friction and the geothermal heat. On the other hand, Robin (1955), Dansgaard and Johnsen (1969[a]) and Philberth and Federer (1971) calculated vertical temperature profiles taking T_s as independent of x but considering the geothermal and frictional heats. Let T_g be such a profile, which is normalized by the addition of a constant so that it becomes zero at the surface.

The real temperature profile taking into account all three influences, is obtained simply by adding T_g to Equations (6), (7) and (9) respectively. This can be explained in the following way: T_g is the solution of a differential equation, which differs from Equation (1) only by a function of x and y on the right-hand side, expressing the heat of friction. In the case of a linear differential equation, the sum of

the solutions of the homogeneous plus the inhomogeneous forms is also a solution of the inhomogeneous form. At the surface T_g is zero ($T = T_s$), but for $y_r = 0$ the function T and its vertical gradient are negligibly small with respect to T_g and its vertical gradient.

Of practical significance is the fact that in the upper part (region with negative temperature gradient) not only T but also T_g can be calculated in a simple way. In the upper part T_g can be taken as independent of x (e.g. according to Robin, 1955, equation (8) or Philberth and Federer, 1971, table I). This can be verified as follows: The upper part of the ice sheet is influenced by a heat of friction which is much smaller than the heat of friction produced below point x, because it originated at a considerably smaller x and v_x. Therefore its influence is normally much smaller than the (x-independent) geothermal heat and can be neglected or approximated by a function which does not depend on x. The horizontal gradient $\partial T_g/\partial x$ depends on dh/dx and da/dx; but Weertman's (1968) equation (6b) for $dT_s/dx = 0$ (his dT_u/dx) shows $\partial T_g/\partial x$ to be very small in the range where $T - T_s$ is small.

V. AN EXAMPLE: STATION JARL JOSET (LAT. 33° 30′ W., LONG. 71° 20′ N., 2 865 m A.S.L.

General values:

   $\kappa = 40$ m² a⁻¹; temperature lapse rate $\lambda = 9.5$ deg km⁻¹.

Local values:

   $h = 2.500$ km; $h_r = 2.250$ km; $x = 125$ km (Philberth, 1972[b]).

Values between ice divide and Jarl Joset:

   $a = 0.30$ m a⁻¹ ice value (independent of x; Federer and others, 1970), $\epsilon = 1$ (two dimensional case; Philberth, in press), height of surface $= 3.160$ km $- 1.4 \times 10^{-3}x - 7.5 \times 10^{-6}$ km⁻¹ x^2 (Mälzer, 1964; Lliboutry, 1968; Philberth and Federer, 1970).   (10)

Derived relations:

   Multiplication of Equation (10) by λ yields

$$T_s = \text{const.} + 13.3 \times 10^{-3} \text{ deg km}^{-1} x + 71 \times 10^{-6} \text{ deg km}^{-2} x^2,$$

$$v_x = h_r^{-1} \int_0^x a \, dx = ax/h_r;$$

according to Equation (6) we have:

$$T = C_0 + C_1 axy_r/h_r + C_2 a^2 [y_r^2/h_r^2 + \kappa/(ah_r)]x^2.$$

The comparison of T for $y_r = h_r$ with T_s yields:

$$T = C_0 + 13.3 \times 10^{-3} \text{ deg km}^{-1} xy_r/h_r + 67 \times 10^{-6} \text{ deg km}^{-2} [y_r^2/h_r^2 + 0.06]x^2.$$

For $x = 125$ km (Jarl Joset) the result is:

$$T - T_s = [-2.71 + 1.66 y_r/h_r + 1.05 y_r^2/h_r^2] \text{ deg.} \tag{11}$$

Comparison with measured values:

   At the depth of 615 m ($y_r = 1 635$ m) $-29.30°$ C and at 1 005 m ($y_r = 1 245$ m) $-30.00°$ C have been measured by the thermal probe method (Philberth, 1962, 1970); that is a difference of 0.70 deg. For these two depths the Equation (11) yields a difference of 0.52 deg and the function T_g yields a difference of -0.32 deg. Hence the total amount of the theoretical difference for steady-state conditions is 0.20 deg.

   The measured value (0.70 deg) and theoretical value (0.20 deg) differ by 0.50 deg, which can be explained by palaeoclimatic changes. If a temperature jump θ (end of ice age) is assumed to be at 10 000 years B.P. (Dansgaard and others, 1969), the 0.50 deg difference corresponds to $\theta = 5$ deg, if the jump is assumed to be at 12 000 years B.P., it corresponds to $\theta = 6$ deg (Philberth, 1972[a]).

MS. received 3 April 1973 and in revised form 11 July 1973

REFERENCES

Bogoslovskiy, V. N. 1958. The temperature conditions (regime) and movement of the Antarctic glacial shield. *Union Géodésique et Géophysique Internationale. Association Internationale d'Hydrologie Scientifique. Symposium de Chamonix, 16–24 sept. 1958*, p. 287–305.

Budd, W. F. 1969. The dynamics of ice masses. *ANARE Scientific Reports*. Ser. A(IV). Glaciology. Publication No. 108.

Budd, W. F., *and* Radok, U. 1971. Glaciers and other large ice masses. *Reports on Progress in Physics*, Vol. 34, No. 1, p. 1–70.

Budd, W. F., *and others*. [1970.] The extent of basal melting in Antarctica, by W. [F.] Budd, D. Jenssen and U. Radok. *Polarforschung*, Bd. 6, Jahrg. 39, Nr. 1, 1969, p. 293–306.

Dansgaard, W., *and* Johnsen, S. J. 1969[a]. Comment on paper by J. Weertman, "Comparison between measured and theoretical temperature profiles of the Camp Century, Greenland, borehole". *Journal of Geophysical Research*, Vol. 74, No. 4, p. 1109–10.

Dansgaard, W., *and* Johnsen, S. J. 1969[b]. A flow model and a time scale for the ice core from Camp Century, Greenland. *Journal of Glaciology*, Vol. 8, No. 53, p. 215–23.

Dansgaard, W., *and others*. 1969. One thousand centuries of climatic record from Camp Century on the Greenland ice sheet, by W. Dansgaard, S. J. Johnsen, J. Møller and C. C. Langway, Jr. *Science*, Vol. 166, No. 3903, p. 377–81.

Federer, B., *and others*. 1970. Outflow and accumulation of ice in Jarl-Joset station, Greenland, by B. Federer, H. V. Sury, K. Philberth and M. [R.] de Quervain. *Journal of Geophysical Research*, Vol. 75, No. 24, p. 4567–69.

Haefeli, R. 1961. Contribution to the movement and the form of ice sheets in the Arctic and Antarctic. *Journal of Glaciology*, Vol. 3, No. 30, p. 1133–51.

Lliboutry, L. A. 1968. Steady-state temperatures at the bottom of ice sheets and computation of the bottom ice flow law from the surface profile. *Journal of Glaciology*, Vol. 7, No. 51, p. 363–76.

Mälzer, H. 1964. Das Nivellement über das grönländische Inlandeis der Internationalen Glaziologischen Grönland-Expedition 1959. *Meddelelser om Grønland*, Bd. 173, Nr. 7.

Philberth, B. 1956. Beseitigung radioaktiver Abfallsubstanzen. *Atomkern-Energie* (München), 1. Jahrg., Ht. 11–12, p. 396–400.

Philberth, K. 1962. Une méthode pour mesurer les températures a l'intérieur d'un inlandsis. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences* (Paris), Tom. 254, No. 22, p. 3881–83.

Philberth, K. 1970. Thermische Tiefbohrung in Zentralgrönland. *Umschau in Wissenschaft und Technik*, 1970, Ht. 16, p. 515–16.

Philberth, K. 1972[a]. Factors affecting deep ice temperatures. *Nature, Physical Science*, Vol. 237, No. 72, p. 44–45.

Philberth, K. 1972[b]. Über den inneren Wärmehaushalt in mächtigen Eisschilden. *Polarforschung*, Bd. 7, 42. Jahrg., Nr. 1, p. 11–17.

Philberth, K. In press. Die thermische Tiefbohrung in Station Jarl-Joset und ihre theoretische Auswertung. *Meddelelser om Grønland*.

Philberth, K., *and* Federer, B. 1970. A note on the surface profile of the Greenland ice sheet. *Journal of Glaciology*, Vol. 9, No. 55, p. 150–53.

Philberth, K., *and* Federer, B. 1971. On the temperature profile and the age profile in the central part of cold ice sheets. *Journal of Glaciology*, Vol. 10, No. 58, p. 3–14.

Radok, U. 1959. Temperatures in polar ice caps. *Nature*, Vol. 184, No. 4692, p. 1056–57.

Radok, U., *and others*. 1970. Steady-state temperature profiles in ice sheets, by U. Radok, D. Jenssen and W. [F.] Budd. [*Union Géodésique et Géophysique Internationale.* Association Internationale d'Hydrologie Scientifique.] [*International Council of Scientific Unions. Scientific Committee on Antarctic Research. International Association of Scientific Hydrology. Commission of Snow and Ice.*] *International Symposium on Antarctic Glaciological Exploration (ISAGE), Hanover, New Hampshire, U.S.A., 3–7 September 1968*, p. 151–65.

Robin, G. de Q. 1955. Ice movement and temperature distribution in glaciers and ice sheets. *Journal of Glaciology*, Vol. 2, No. 18, p. 523–32.

Robin, G. de Q. 1970. Stability of ice sheets as deduced from deep temperature gradients. [*Union Géodésique et Géophysique Internationale. Association Internationale d'Hydrologie Scientifique.*] [*International Council of Scientific Unions. Scientific Committee on Antarctic Research. International Association of Scientific Hydrology. Commission of Snow and Ice.*] *International Symposium on Antarctic Glaciological Exploration (ISAGE), Hanover, New Hampshire, U.S.A., 3–7 September 1968*, p. 141–51.

Weertman, J. 1968. Comparison between measured and theoretical temperature profiles of the Camp Century, Greenland, borehole. *Journal of Geophysical Research*, Vol. 73, No. 8, p. 2691–700.