

UNIVERSAL NOTIONS CHARACTERIZING SPECTRAL DECOMPOSITIONS

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1. Introduction. In this note we characterize certain types of spectral decomposition in terms of “universal” notions valid for any operator on a Banach space. To be precise, let X be a complex Banach space and let T be a bounded linear operator on X . If F is a closed set in the plane \mathbf{C} , let $X(T, F)$ consist of all $y \in X$ satisfying the identity

$$y = (z - T)f(z), \tag{1.1}$$

where $f: \mathbf{C} \setminus F \rightarrow X$ is analytic. It is then easy to see that $X(T, F)$ is a T -invariant linear manifold in X . Moreover, if $y \in X$ then

$$\gamma(y, T) = \bigcap \{F: F \text{ is closed and } y \in X(T, F)\} \tag{1.2}$$

is a compact subset of the spectrum $\sigma(T)$. Our aim is to give necessary and sufficient conditions for a decomposable or strongly decomposable operator in terms of $X(T, F)$ and $\gamma(y, T)$. Recall that T is decomposable if whenever G_1, G_2 are open and cover \mathbf{C} there exist T -invariant closed linear manifolds M_1, M_2 with $X = M_1 + M_2$ and $\sigma(T|_{M_i}) \subset G_i (i = 1, 2)$ (equivalently, $\sigma(T|_{M_i}) \subset \bar{G}_i$ see [4, p. 57]). In this case, $X(T, F)$ is norm closed if F is closed and each y in X has a unique maximally defined local resolvent satisfying (1.1) on $\mathbf{C} \setminus F_y$; F_y is called the *local spectrum* $\sigma(y, T)$ and coincides with $\gamma(y, T)$. Hence T has the single valued extension property (SVEP); i.e., zero is the only analytic function $f: V \rightarrow X$ satisfying $(z - T)f(z) = 0$ on V . If T is decomposable and the restriction $T|_{X(T, F)}$ is also decomposable for each closed F , then T is called *strongly decomposable*. We point out that Albrecht [2] has shown by example that not every decomposable operator is strongly decomposable, while Eschmeier [6] has given a simpler construction to show that this phenomenon occurs even in Hilbert space.

In Section 2 we prove our criteria for those types mentioned above. Section 3 gives characterizations for a proper subclass of strongly decomposable operators which we call “decomposable relative to the identity” (see also [5], [10].)

2. Decomposable operators. We shall need the following known criterion [8].

PROPOSITION 1. *An operator $T \in L(X)$ is decomposable if and only if for each open cover $\{G, H\}$ of \mathbf{C} , where G is a disc and H is the complement of a disc, there exist invariant subspaces Y, Z such that $X = Y + Z$, $\sigma(T|_Y) \subset \bar{G}$, $\sigma(T|_Z) \subset \bar{H}$.*

THEOREM 1. *For $T \in L(X)$ the following assertions are equivalent.*

- (i) *T is decomposable;*
- (ii) *for each open cover $\{G, H\}$ of \mathbf{C} there exists a linear transformation $P: X \rightarrow X$ such that*

$$\gamma(Py, T) \subset G \quad \text{and} \quad \gamma(y - Py, T) \subset H \quad \text{for } y \in X, \tag{2.1}$$

for each closed $F \subset G \setminus \bar{H}$ and $y \in \overline{X(T, F)}$

$$Py = y, \tag{2.2}$$

and for each closed $K \subset H \setminus \bar{G}$ and $y \in \overline{X(T, K)}$

$$Py = 0; \tag{2.3}$$

(iii) for each open cover $\{G, H\}$ of \mathbf{C} , where G is a disc and H is the complement of a disc, there exists a linear transformation $P : X \rightarrow X$ satisfying (2.1)–(2.3)

Proof. Since (ii) \Rightarrow (iii) is obvious, we prove (i) \Rightarrow (ii) and (iii) \Rightarrow (i).

(i) \Rightarrow (ii). Let $G \cup H = \mathbf{C}$ denote an open cover. Since T is decomposable we can write

$$X = X(T, \bar{G}) + X(T, \bar{H}) \tag{2.4}$$

because a T -invariant subspace M is contained in $X(T, F)$ whenever $\sigma(T | M) \subset F$. Next put

$$Y = X(T, \bar{G}) \cap X(T, \bar{H}),$$

so that Y is a subspace of X . By a Hamel basis argument [9, Th. I.11.2] there is a linear manifold $W \subset X(T, \bar{H})$ such that

$$X(T, \bar{H}) = W \oplus Y, \tag{2.5}$$

and we may suppose $X(T, K) \subset W$ for closed $K \subset H \setminus \bar{G}$. We now prove that

$$X = X(T | \bar{G}) \oplus W. \tag{2.6}$$

For if $y \in X$ then by (2.4) $y = y_1 + y_2$, where $y_1 \in X(T, \bar{G})$ and $y_2 \in X(T, \bar{H})$. But then (2.5) implies $y_2 = u + w$, where $u \in Y$ and $w \in W$. Thus $u \in X(T, \bar{G})$ and hence

$$X = X(T, \bar{G}) + W. \tag{2.7}$$

To see that (2.7) is direct, suppose $0 = y + w$ with $y \in X(T, \bar{G})$, $w \in W$. Then $y = -w \in X(T, \bar{H})$ and so $y \in Y$. By (2.5) $y = w = 0$.

Let P be the projection of X onto $X(T, \bar{G})$ along W . Hence for $y \in X(T, \bar{G})$ we have $Py = y$, and for $y \in W$ also $Py = 0$. In other words

$$PX = X(T, \bar{G}) \quad \text{and} \quad (I - P)X = W.$$

Now let $x \in X$. Then

$$\begin{aligned} \gamma(Px, T) &= \sigma(Px, T) \subset \bar{G}, \\ \gamma(x - Px, T) &= \sigma(x - Px, T) \subset \bar{H}, \end{aligned} \tag{2.8}$$

and so (2.1) is proved. For F closed in $G \setminus \bar{H}$ it follows from the inclusion $X(T, F) \subset X(T, \bar{G})$ and (2.8) that (2.2) holds; also (2.3) follows from (2.8) for K closed in $H \setminus \bar{G}$.

(iii) \Rightarrow (i). Let F be closed. We prove that $X(T, F)$ is absorbent in the following sense. If $\lambda_0 \in F$, $x_0 \in X$ and $(\lambda_0 - T)x_0 \in X(T, F)$ then $x_0 \in X(T, F)$. In fact, let $y_0 = (\lambda_0 - T)x_0$ and let $f : \mathbf{C} \setminus F \rightarrow X$ be analytic such that

$$y_0 = (z - T)f(z) \quad (z \notin F).$$

Then the function defined by $h(z) = (z - \lambda_0)^{-1}[x_0 - f(z)]$ is analytic on $\mathbf{C} \setminus F$ and satisfies $(z - T)h(z) = x_0$; hence $x_0 \in X(T, F)$.

Next we prove that T has the SVEP. Assume that $f: V \rightarrow X$ is analytic on the open connected set V and satisfies

$$(z - T)f(z) = 0 (z \in V).$$

Let δ_1, δ_2 be disjoint closed discs in V . By absorbcency proved above, if $z \in \delta_i$ ($i = 1, 2$) then $f(z) \in X(T, \delta_i)$. Hence

$$f(z) \in \overline{X(T, \delta_1)} \cap \overline{X(T, \delta_2)} \quad (2.9)$$

for all $z \in V$ by analytic continuation. Now choose an open cover of $\{G, H\}$ of \mathbf{C} where G is a disc, H is the complement of a disc and $\delta_1 \subset G \setminus \bar{H}$ and $\delta_2 \subset H \setminus \bar{G}$.

Let $P: X \rightarrow X$ be as in (iii) satisfying (2.1)–(2.3). Then, by (2.9), $f(z) = Pf(z) = 0$ on V .

We now show that $X(T, F)$ is closed whenever F is closed in \mathbf{C} . Let $\lambda \in F$ be arbitrary and define

$$H_\lambda = \{\mu: |\lambda - \mu| > (1/2)\text{dist}(\lambda, F)\}.$$

By hypothesis there is a linear map $P: X \rightarrow X$ such that $Px = y$ for $y \in \overline{X(T, F)}$ and $\sigma(Px, T) = \gamma(Px, T) \subset \bar{H}_\lambda$ for all $x \in X$. Hence $\sigma(y, T) \subset \bar{H}_\lambda$ if $y \in \overline{X(T, F)}$. Since $\lambda \in F$ is arbitrary, we have $\sigma(y, T) \subset F$ for each $y \in \overline{X(T, F)}$; hence $X(T, F)$ is closed.

From the last paragraph we infer $\sigma(T|X(T, F)) \subset F$ for any closed F (e.g., [4, p. 23]). Now let $\{G_1, G_2\}$ be an open cover of \mathbf{C} by a disc and the complement of a disc. The previous paragraphs show $X = X(T, \bar{G}_1) + X(T, \bar{G}_2)$; hence the remark above and Proposition 1 imply that T is decomposable.

REMARK. A characterization of a bounded decomposable operator similar to Theorem 1 (ii) first appeared in [7, Th. 1.2], but the proof here is simpler; part (iii) of Theorem 1 is new.

Because of the examples cited in the introduction ([2], [6]) we evidently need a separate criterion for strongly decomposable operators. To do this we use the following well-known result [3].

PROPOSITION 2. Let $T \in L(X)$ be decomposable. Then T is strongly decomposable if and only if for each open cover $\{G, H\}$ of \mathbf{C} and closed F

$$X(T, F) = X(T, F \cap \bar{G}) + X(T, F \cap \bar{H}). \quad (2.10)$$

LEMMA 1. Let $T \in L(X)$ be decomposable, and let $\{G, H\}$ be an open cover of \mathbf{C} . Define

$$Y = X(T, \bar{G} \cap \bar{H}) (= X(T, \bar{G}) \cap X(T, \bar{H})),$$

$$Z = \text{lin}\{X(T, K): K \text{ closed in } H \setminus \bar{G}\}$$

where “lin” denotes linear span. Then for F closed

$$X(T, F) \cap (Y \oplus Z) = [X(T, F) \cap Y] \oplus [X(T, F) \cap Z]. \quad (2.11)$$

Proof. Since the right-hand side of (2.11) is clearly contained in the left-hand side, let $u \in X(T, F) \cap (Y \oplus Z)$. Then $u = y + z$ with $y \in Y$ and $z \in Z$, and hence by [4, p. 2]

$$\sigma(y, T) \subset \sigma(u, T) \cup \sigma(z, T).$$

But $\sigma(y, T) \cap \sigma(z, T) = \emptyset$ and so $\sigma(y, T) \subset \sigma(u, T) \subset F$. Similarly, $\sigma(z, T) \subset F$ and so (2.11) follows.

THEOREM 2. *Let $T \in L(X)$. Then T is strongly decomposable if and only if for each open cover $\{G, H\}$ of \mathbf{C} and for F closed in \mathbf{C} there is a linear map $P: X \rightarrow X$ such that*

- (i) $\gamma(Px, T) \subset \bar{G}, \gamma(x - Px, T) \subset \bar{H}$ for all $x \in X$;
- (ii) if $F \subset G \setminus \bar{H}$ and $y \in \overline{X(T, F)}$ then $Py = y$;
- (iii) if K is closed in $H \setminus G$ and $y \in \overline{X(T, K)}$ then $Py = 0$;
- (iv) $X(T, F)$ is invariant under P .

Proof. Sufficiency is easy, for (i)–(iii) imply that T is decomposable by Theorem 1. If $y \in X(T, F)$, F closed, then by (i) and (iv) we have

$$Py \in X(T, F \cap \bar{G}), \quad (I - P)y \in X(T, F \cap \bar{H}).$$

This shows that $X(T, F)$ is contained in the right-hand side of (2.10) and hence T satisfies (2.10) since the reverse inclusion is trivial. By Proposition 2, T is strongly decomposable.

Conversely, let T be strongly decomposable, let $\{G, H\}$ be an open cover of \mathbf{C} and let F be closed. Then (2.10) holds. Let Y and Z be defined as in Lemma 1. Then there is a linear manifold $W_0 \subset X(T, F) \cap X(T, H^-)$ such that

$$X(T, F) \cap X(T, \bar{H}) = W_0 \oplus [X(T, F) \cap Y] \oplus [X(T, F) \cap Z]. \tag{2.12}$$

We claim that

$$(Y \oplus Z) \cap W_0 = (0). \tag{2.13}$$

For if $y \in (Y \oplus Z) \cap W_0$ then $y \in X(T, F)$. Hence y lies in the left-hand side of (2.11). By Lemma 1 we have $y = 0$ and hence (2.13) holds. From this we infer the existence of a linear manifold $W_1 \supset W_0$ such that

$$X(T, \bar{H}) = Y \oplus Z \oplus W_1. \tag{2.14}$$

Since $W_0 \subset X(T, F) \cap W_1$ by (2.12), we prove the reverse inclusion by letting $y \in X(T, F) \cap W_1$. In view of (2.12) $y = u + w_0$, where $w_0 \in W_0$ and

$$u \in [X(T, F) \cap Y] \oplus [X(T, F) \cap Z].$$

Hence $u = y - w_0 \in W_1$, and since $u \in Y \oplus Z$ (2.14) implies that $u = 0$. Thus $y \in W_0$, and so

$$W_0 = X(T, F) \cap W_1. \tag{2.15}$$

From (2.12) and (2.15) we obtain

$$X(T, F) \cap X(T, \bar{H}) = [X(T, F) \cap Y] \oplus [X(T, F) \cap Z] \oplus [X(T, F) \cap W_1]. \tag{2.16}$$

Since $Y \subset X(T, \bar{G})$, from (2.10) and (2.16) we obtain

$$X(T, F) = X(T, F \cap \bar{G}) \oplus [X(T, F) \cap Z] \oplus [X(T, F) \cap W_1]. \tag{2.17}$$

Finally since $X = X(T, \bar{G}) + X(T, \bar{H})$, (2.14) yields

$$X = X(T, \bar{G}) \oplus Z \oplus W_1.$$

Hence there is a unique projection P of X onto $X(T, \bar{G})$ along $Z \oplus W_1$, i.e.,

$$Py = y \text{ for } y \in X(T, \bar{G}) \quad \text{and} \quad Py = 0 \text{ for } y \in Z \oplus W_1. \quad (2.18)$$

If $x \in X(T, F)$ then (2.17) and (2.18) imply

$$PX(T, F) = X(T, F \cap \bar{G}) \cap X(T, F),$$

and (iv) is proved. Assertion (i) follows from (2.18), while (ii) and (iii) follow from the construction of Z and W_1 .

3. Decomposition relative to the identity. In this section we consider another class of spectral decomposition which has been treated recently (see [5], [10]). We use $\{T\}'$ to denote the commutant of T .

DEFINITION 1. Let $T \in L(X)$. We say that T is decomposable relative to the identity if for each finite open cover $\{G_i\}$ of \mathbf{C} there exist corresponding systems $\{M_i\}$ of T -invariant subspaces and bounded operators $\{P_i\} \subset \{T\}'$ such that

$$P_i X \subset M_i \quad \text{and} \quad \sigma(T | M_i) \subset G_i \quad (1 \leq i \leq n),$$

$$I = \sum_i P_i.$$

We remark first that the conditions imply that an operator decomposable relative to the identity is decomposable in the sense of the previous section. Our purpose in the present section is to characterize this new type of "decomposability"; we shall also show that these operators form a proper subclass of the strongly decomposable operators.

THEOREM 3. For $T \in L(X)$, the following assertions are equivalent:

- (i) T is decomposable relative to the identity;
- (ii) for every open cover $\{G, H\}$ of \mathbf{C} there exists $P \in \{T\}'$ such that for all $y \in X$

$$\gamma(Py, T) \subset \bar{G} \quad \text{and} \quad \gamma(y - Py, T) \subset \bar{H}; \quad (3.1)$$

- (iii) for every open cover $\{G, H\}$ of \mathbf{C} , each $x \in X$ has a representation

$$x = x_1 + x_2 \quad \text{with} \quad \gamma(x_1, T) \subset \bar{G}, \gamma(x_2, T) \subset \bar{H}$$

and for every pair of closed disjoint sets F_1, F_2 , there exists $P \in \{T\}'$ such that

$$Px = x, \quad \text{if} \quad \gamma(x, T) \subset F_1;$$

$$Px = 0, \quad \text{if} \quad \gamma(x, T) \subset F_2.$$

- (iv) for every open cover $\{G, H\}$ of \mathbf{C} , where G is an open disc and H is the complement of a closed disc, there exists $P \in \{T\}'$ satisfying conditions (3.1);

- (v) for every open cover $\{G, H\}$ of \mathbf{C} , where G is an open disc and H is the complement of a closed disc, all conditions of (iii) are satisfied.

Proof. The proof will be completed with the sequence of implications

$$(i) \Rightarrow (ii) \Rightarrow \{(iii) \text{ or } (iv)\} \Rightarrow (v) \Rightarrow (i).$$

The first of these follows easily from Definition 1. Indeed, given $\mathbf{C} = G \cup H$ there is a $P \in \{T\}'$ with $PX \subset X(T, \bar{G})$ and $(I - P)X \subset X(T, \bar{H})$. Each $x \in X$ can be written $x = Px + (I - P)x$; hence (3.1) follows.

Since (ii) \Rightarrow (iii) \Rightarrow (v) and (ii) \Rightarrow (iv) \Rightarrow (v) are evident, we prove (v) \Rightarrow (i).

First we show that conditions (iii) of Theorem 1 are satisfied. Let $\{G, H\}$ be an open cover of \mathbf{C} , where G is an open disc and H is the complement of a closed disc. Let G_1 be another open disc such that $\bar{G}_1 \subset G$ and $\{G_1, H\}$ is an open cover of \mathbf{C} ; let H_1 be the complement of a closed disc such that $\{G, H_1\}$ is an open cover of \mathbf{C} and $\bar{G}_1 \cap \bar{H}_1 = \emptyset$. By hypothesis there exists $P \in \{T\}'$ such that

$$\begin{aligned} Px &= x & \text{if } \gamma(x, T) \subset \bar{G}_1, \\ Px &= 0 & \text{if } \gamma(x, T) \subset \bar{H}_1. \end{aligned} \tag{3.2}$$

Since $\{G_1, H\}$ is an open cover of \mathbf{C} , for each $x \in X$ there are x_1 and x_2 such that

$$x = x_1 + x_2, \quad \gamma(x_1, T) \subset \bar{G}_1, \quad \gamma(x_2, T) \subset \bar{H}. \tag{3.3}$$

Now (3.2) and (3.3) imply that $(I - P)x_1 = 0$ and hence $(I - P)x = (I - P)x_2$. As $P \in \{T\}'$ we have

$$\gamma(x - Px, T) = \gamma(x_2 - Px_2, T) \subset \gamma(x_2, T) \subset \bar{H}.$$

Similarly, since $\{G, H_1\}$ is an open cover of \mathbf{C} , for each $x \in X$ there are $x'_1, x'_2 \in X$ with

$$x = x'_1 + x'_2, \quad \gamma(x'_1, T) \subset \bar{G}, \quad \gamma(x'_2, T) \subset \bar{H}_1.$$

Then (3.2) and (3.4) imply that $Px'_2 = 0$; hence

$$\gamma(Px_1, T) = \gamma(Px'_1, T) \subset \gamma(x'_1, T) \subset \bar{G}.$$

Now let F be closed in $G \setminus \bar{H}$. Since $\{G_1, H\}$ is an open cover of \mathbf{C} , one has $F \subset G_1 \setminus \bar{H}$. By (3.2) if $x \in X(T, F)$ then $Px = x$. Hence $Px = x$ on $\overline{X(T, F)}$ because P is bounded. In a similar way we have $Px = 0$ for $x \in \overline{X(T, K)}$ if $K \subset H \setminus \bar{G}$ is closed.

This much proves that T is decomposable by Theorem 1 (iii).

Furthermore, for every open cover $\{G, H\}$ of \mathbf{C} condition (ii) of Theorem 1 holds. Let $\{G_i\}$ be a system of open discs and let $\{H_i\}$ be a system of complements of closed discs such that $\{G_i, H_i\}$ is an open cover of \mathbf{C} for each $1 \leq i \leq n$ and

$$\bigcup_{i=1}^n \bar{G}_i \subset \bar{G}, \quad \bigcap_{i=1}^n \bar{H}_i \subset \bar{H}.$$

For each pair (G_i, H_i) there is $P_i \in \{T\}'$ such that for $x \in X$,

$$\sigma(P_i x, T) \subset \bar{G}_i \quad \text{and} \quad \sigma(x - P_i x, T) \subset \bar{H}_i. \tag{3.5}$$

Put $P = I - \prod_i (I - P_i)$. It follows by an easy argument that

$$\sigma(x - Px, T) \subset \bigcap_i \bar{H}_i \subset \bar{H}. \tag{3.6}$$

Since

$$P = (P_1 + P_2 + \dots + P_n) - (P_1 P_2 + \dots + P_{n-1} P_n) + \dots + (-1)^n P_1 P_2 \dots P_n,$$

from (3.5) we have

$$\sigma(Px, T) \subset \bigcup_i \bar{G}_i \subset \bar{G}. \tag{3.7}$$

Because T is decomposable, (3.6) and (3.7) imply that T is decomposable relative to the identity.

The following corollary first appeared in [5, Cor. 17.10]; the proof here uses the previous theorem.

COROLLARY 1. *If T is decomposable relative to the identity, then T is strongly decomposable.*

Proof. Suppose T is decomposable relative to the identity. Then T is decomposable. Let F be closed, let $S = T|_{X(T, F)}$ and let $P \in \{T\}'$ be as in Theorem 3(ii) corresponding to the open cover $\{G, H\}$ of \mathbf{C} . Then $X(T, F)$ is P -invariant. Put $Q = P|_{X(T, F)}$ so that $Q \in \{S\}'$. For $y \in X(T, F)$ we have $\gamma(Qy, S) = \sigma(Py, T) \subset \bar{G}$ and $\gamma(y - Qy, T) \subset \sigma(y - Py, T) \subset \bar{H}$, by Theorem 3. Hence S is decomposable by Theorem 2.

To give an example of a strongly decomposable operator which is not decomposable relative to the identity, we first sketch a construction due to Albrecht [1]. Let Ω be open in \mathbf{C} and let $C^m(\Omega)$ be the algebra of m -times continuously differentiable complex-valued functions on Ω with the topology of uniform convergence on compact sets. Let $C_0^m(\Omega)$ be the subset of $C^m(\Omega)$ with bounded derivatives up through order m . Then if $\phi \in C_0^m(\Omega)$ the following defines a Banach algebra norm:

$$\|\phi\|_m = \sum \frac{1}{k! p!} \sup \|\partial^k \bar{\partial}^p \phi\|$$

where $0 \leq k, p \leq m, k + p \leq m$, the sup is taken over Ω and $\partial, \bar{\partial}$ are the differential operators $\partial/\partial z = (1/2)(\partial/\partial x + i\partial/\partial y)$ and $\partial/\partial \bar{z} = (1/2)(\partial/\partial x - i\partial/\partial y)$ respectively. Let X_0 denote the subspace of bounded functions in $C^0(\Omega)$ and let X_1 denote those $f \in X_0$ such that $\bar{\partial}f \in X_0$ in the sense of distributions. Let T_0, T_1 denote multiplication by the independent variable z on Ω , where we now take $\Omega = \{z: |z| < 1/2\}$. Let $X = X_0 \oplus X_1$ and $T = T_0 \oplus T_1$. By [1], T_0, T_1 and T are generalized scalar operators and $X_i(T_i, F) = \{f_i \in X_i: \text{supp } f_i \subset F\}$. For $h \in X_0$ we define the nilpotent operators A_h and Q_h by

$$\begin{aligned} A_h(f, g) &= (h\bar{\partial}g, 0), \\ Q_h(f, g) &= (hg, 0), \end{aligned}$$

for $(f, g) \in X$. Then A_h and Q_h commute with T . If $S \in L(X)$ leaves the spectral manifold $X(T, F)$ invariant, then by [1, Prop. 3.2] we have

$$S(f, g) = (b_0f, b_1g) + (A_h + Q_k)(f, g), \tag{3.8}$$

where $h, k, b_0 \in X_0$ and $b_1 \in X_1$.

Now let $V = T + A_1$. Then T is quasinilpotent equivalent to V , hence $X(T, F) = X(V, F)$ for all closed F [4, p. 40], and so the restrictions $V|_{X(T, F)}$, and $T|_{X(T, F)}$ are quasinilpotent equivalent. Since T is generalized scalar, V is strongly decomposable by [4, p. 80].

We prove that V is not decomposable relative to the identity with two lemmas.

LEMMA 2. *If $S \in \{V\}'$, then S has representation (3.8), where $b_0 = b_1$ is analytic on Ω .*

Proof. By the remarks above (3.8) holds. Let $U(f, g) = (b_0 f, b_1 g)$. Since $VS = SV$ and $TU = UT$, it follows that $UA_1 = A_1 U$. Then for $(f, g) = (0, 1)$ we have

$$A_1 U(0, 1) = A_1(0, b_1) = (\bar{\partial} b_1, 0),$$

$$UA_1(0, 1) = U(0, 0) = (0, 0),$$

and hence b_1 is analytic. In a similar way with $g = \bar{z}$ we find that $b_1 = b_0$.

LEMMA 3. *V is not decomposable relative to the identity.*

Proof. Suppose the contrary and let $F = \{z: \operatorname{Re} z \leq 0\}$ and $G = \{z: \operatorname{Re} z < 1/4\}$. Let $P \in \{V\}'$ with

$$PX \subset X(T, \bar{G}), \quad P|X(T, F) = I|X(T, F). \quad (3.9)$$

By Lemma 2, for $(f, g) \in X$ we have

$$P(f, g) = (bf, bg) + (A_h + Q_k)(f, g),$$

where b is analytic and $h, k \in X_0$. Hence $P(0, 1) = (k, b)$. By (3.9) $\operatorname{supp} b \subset \bar{G}$, so b vanishes on $\Omega \cap (\mathbb{C} \setminus \bar{G})$ and hence $b = 0$ throughout Ω . Thus P reduces to the nilpotent operator $A_h + Q_k$, and this is impossible by (3.9).

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