PRIME IDEALS AND RADICALS IN SEMIGROUP-GRADED RINGS

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In this paper we study the ideal structure of the direct limit and direct sum (with a special multiplication) of a directed system of rings; this enables us to give descriptions of the prime ideals and radicals of semigroup rings and semigroup-graded rings.

We show that the ideals in the direct limit correspond to certain families of ideals from the original rings, with prime ideals corresponding to "prime" families. We then assume the indexing set is a semigroup Ω with preorder defined by $\alpha \prec \beta$ if β is in the ideal generated by α, and we use the direct sum to construct an Ω-graded ring; this construction generalizes the concept of a strong supplementary semilattice sum of rings. We show the prime ideals in this direct sum correspond to prime ideals in the direct limits taken over complements of prime ideals in Ω when two conditions are satisfied; one consequence is that when these conditions are satisfied, the prime ideals in the semigroup ring $S[\Omega]$ correspond bijectively to pairs (Φ, Q) with Φ a prime ideal of Ω and Q a prime ideal of S. The two conditions are satisfied in many bands and in any commutative semigroup in which the powers of every element become stationary. However, we show that the above correspondence fails when Ω is an infinite free band, by showing that $S[\Omega]$ is prime whenever S is.

When Ω satisfies the above-mentioned conditions, or is an arbitrary band, we give a description of the radical of the direct sum of a system in terms of the radicals of the component rings for a class of radicals which includes the Jacobson radical and the upper nil radical. We do this by relating the semigroup-graded direct sum to a direct sum indexed by the largest semilattice quotient of Ω, and also to the direct product of the component rings.

Introduction

It is often the case that a semigroup S can be decomposed as a union of subsemigroups S_α indexed by another semigroup Ω, with the property that $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for $\alpha, \beta \in \Omega$. When this can be done, the semigroup algebra $F[S]$ can be written additively as a direct sum of the sets $F[S_\alpha]$, yielding an Ω-gradation of $F[S]$. So far we have not used the semigroup structure of the S_α. In many cases, the multiplication of an element of S_α by an element of S_β in S can be carried out by mapping the elements into $S_{\alpha\beta}$ and then multiplying in that semigroup (see for example [2, Chapter 4] or [10, Chapter IV]). This leads to a corresponding formula for the multiplication in $F[S]$. It is this sort of decomposition of Ω-graded rings that we study in this paper, and so our results can be applied to help find prime ideals and radicals of the ring $F[S]$ if these objects are known for the rings $F[S_\alpha]$. We will describe the prime ideals and various radicals in the

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case where Ω is a band or a commutative power-stationary semigroup (see Section 3). The description of prime ideals involves direct limits over the complements of prime ideals in Ω, and so we start with a discussion of ideals in direct limits. For more information on semigroup rings, we refer the reader to [9] (although the results there are mostly for types of semigroups different from those we consider in this paper).

Throughout this paper, Ω is a set and $\prec$ is a preorder on Ω, i.e., a reflexive and transitive relation on Ω. We assume further that $(\Omega, \prec)$ is *directed*, that is, that for any finite set $\alpha_1, \ldots, \alpha_n \in \Omega$, there is a $\beta \in \Omega$ with $\beta \succ \alpha_i$ for each $i = 1, \ldots, n$. From Section 2 on, Ω is a semigroup for which the product $\alpha\beta$ is an upper bound for α, β; throughout most of the paper, the preorder $\prec$ is defined by $\alpha \prec \beta$ if β is in the ideal generated by α.

A *system of rings* $\mathscr{R}$ over $(\Omega, \prec)$ is a collection $(R_\alpha)_{\alpha \in \Omega}$ of rings, together with ring homomorphisms $\phi_{\alpha, \beta} \colon R_\alpha \to R_\beta$ for all $\alpha, \beta \in \Omega$ with $\alpha \prec \beta$, such that $\phi_{\beta, \gamma} \circ \phi_{\alpha, \beta} = \phi_{\alpha, \gamma}$ whenever $\alpha \prec \beta \prec \gamma$ and such that $\phi_{\alpha, \alpha} = \mathrm{id}_{R_\alpha}$ for all α. We make systems of rings over $(\Omega, \prec)$ into a category by defining a morphism $\Theta \colon \mathscr{R} \to \mathscr{R}'$ between the systems of rings $\mathscr{R}, \mathscr{R}'$, to be a collection of ring homomorphisms $\theta_\alpha \colon R_\alpha \to R'_\alpha$ for $\alpha \in \Omega$ such that $\theta_\beta \circ \phi_{\alpha, \beta} = \phi'_{\alpha, \beta} \circ \theta_\alpha$ for all $\alpha, \beta \in \Omega$ with $\alpha \prec \beta$.

A *prime ideal* in a ring is a proper ideal with the property that whenever it contains a product of ideals it contains one of the factors; a *completely prime* ideal is defined in the same way with products of elements in place of products of ideals. It is easy to see that any completely prime ideal is a prime ideal, and that in a commutative ring, the two types of ideals are the same. We define prime and completely prime ideals in semigroups in exactly the same way; here we regard $\varnothing$ as both a prime and completely prime ideal. Note that an ideal of a semigroup is completely prime if and only if its complement is a subsemigroup.

In the first section we discuss the direct limit functor from the category of systems of rings to the category of rings, and we determine the (one-sided and two-sided) ideal structure of the direct limit of the system $\mathscr{R}$. We show that the ideals correspond to families of ideals in the rings R_α that are compatible with the maps $\phi_{\alpha, \beta}$, and that the prime ideals correspond to "prime" families. We also construct some examples.

In the second section, we assume Ω is a semigroup such that $\alpha, \beta \prec \alpha\beta$ for all $\alpha, \beta \in \Omega$, and we use the maps $\phi_{\alpha, \beta}$ in the system to define a multiplication on the direct sum of the rings R_α, thereby constructing a direct sum functor from the category of systems of rings to the category of Ω-graded rings. This direct sum construction generalizes the notion of a strong supplementary semilattice sum of rings introduced in [13], and includes semigroup rings $S[\Omega]$ as a special case. We wish to determine the prime ideal structure of the Ω-graded rings we obtain, but we are only able to do so when Ω satisfies a condition we call (†) and when all prime ideals in Ω are completely prime. When this is the case, we show that the prime ideals in the direct sum correspond to prime ideals in the direct limits taken over the complements of prime ideals in Ω. We also show that prime ideals corresponding to distinct prime ideals of Ω are incomparable, and so the study of a chain of prime ideals can be done over the complement of a fixed prime ideal in Ω; this leads us to a formula for the classical Krull dimension of the direct sum of a system. We also note in Section 2 that to describe ideals in the direct

sum, it suffices to consider the preorder defined by $\alpha \prec \beta$ if β is in the ideal (α), and so this preorder is used throughout that section and the subsequent ones.

In the third section, we show that the results of Section 2 are valid when Ω is a band that is either regular or satisfies the d.c.c. on principal ideals (a band is called "regular" if it satisfies the identity $xyxzx = xyzx$) or is a commutative power-stationary semigroup (one in which every element x satisfies $x^k = x^{k+1}$ for some k); in these cases, there is a bijective correspondence between prime ideals in the direct sum of the system and prime families over the complements of prime ideals in Ω. We also show that these results are *not* valid when Ω is an infinite free band by showing that $S[\Omega]$ is prime whenever S is.

In the final section, we describe the radical of the direct sum R of the system for any directed, hereditary radical containing the prime radical (for example, the Jacobson radical or upper nil radical); we show that $\bigoplus_{\alpha \in \Omega} r_\alpha$ is in the radical of R if and only if for each idempotent $e \in \Omega$, the radical of R_e contains $\sum_{\alpha \prec e} \phi_{\alpha, e}(r_\alpha)$. This is under the same assumptions as for the description of prime ideals, but at the end of the section we show how to obtain this description for arbitrary bands. To obtain our description, we go through two steps: first we show how to pass to the largest semilattice quotient of Ω by factoring out a special ideal K, and then we show that the direct sum of a system over a finite semilattice is isomorphic to the direct product of the component rings. We use these same techniques to show that the classical Krull dimension of R equals the supremum of the classical Krull dimensions of the rings R_e where e is an idempotent, in case condition (†) is satisfied, all prime ideals of Ω are completely prime, and all the maps $\phi_{\alpha, \beta}$ are onto.

Some remarks are in order before we proceed. First, some of the statements and constructions for our direct sum differ from similar constructions in the literature such as a semilattice sum of rings in that we assume $\alpha, \beta \prec \alpha\beta$ rather than the other way round, and so our maps $\phi_{\alpha, \beta}$ are defined when $\alpha \prec \beta$, while in the literature they are generally defined for $\alpha \geq \beta$. Of course this makes no substantive difference, since one may always replace $\prec$ by its opposite: we have made our choice to agree with the standard convention for direct limits (and because it appears to us that many natural examples of posets, such as the positive integers, follow this convention). One consequence of our choice is that when we speak of a semilattice, we mean an "upper semilattice", while in the literature semilattice generally means "lower semilattice". In any case, algebraically a semilattice is simply a commutative semigroup in which each element is idempotent, and when Ω is a semilattice or a band, it is the case both in this paper and in the literature that when studying systems or sums of rings, there is a map from R_α to R_β when $\beta\alpha\beta = \beta$. We have tried to avoid confusion by mostly writing $\beta \in (\alpha)$ instead of $\alpha \prec \beta$ from Section 2 onward. Second, we have adopted the convention that functions are written on the left of their arguments, so composition of functions proceeds from right to left.

Remark 0.1. We will not assume our rings have an identity, and when they do, we will not assume that the identity of a subring is the same as the identity of the whole ring. If we did work in the category of rings with identity (declaring then that all ring homomorphisms preserve the identity), then the direct limit construction in Section 1

would yield a ring with identity, while the direct sum construction in Section 2 would always yield a ring with identity if we assumed the semigroup Ω had an identity.

1. Direct limits and their ideals

In this section we prepare for our description of prime ideals in semigroup-graded rings by describing prime ideals in direct limits. First we recall the construction of the direct limit of a system of rings, then we describe its ideals in terms of families of ideals over the component rings, and finally we determine which families yield prime ideals. We also provide some examples showing among other things that families yielding prime or maximal ideals need not consist of prime or maximal ideals of the component rings.

Let $\mathscr{R}$ be a system of rings over $(\Omega, \prec)$. A *direct limit* of $\mathscr{R}$ is a ring R together with ring maps $\phi_\alpha \colon R_\alpha \to R$ such that $\phi_\beta \circ \phi_{\alpha,\beta} = \phi_\alpha$ for all $\alpha, \beta \in \Omega$ with $\alpha \prec \beta$, and such that the following universal property is satisfied. For any ring S and collection $(f_\alpha)_{\alpha \in \Omega}$ of ring maps $f_\alpha \colon R_\alpha \to S$ such that $f_\beta \circ \phi_{\alpha,\beta} = f_\alpha$ for all $\alpha, \beta \in \Omega$ with $\alpha \prec \beta$, there is a unique ring homomorphism $f \colon R \to S$ with $f \circ \phi_\alpha = f_\alpha$ for all $\alpha \in \Omega$. We denote the direct limit of $\mathscr{R}$ by $\varinjlim \mathscr{R}$ or by $\varinjlim_{\alpha \in \Omega} R_\alpha$.

Proposition 1.1. *If $\mathscr{R}$ is a system of rings, then $\varinjlim \mathscr{R}$ exists and is unique (up to natural isomorphism), and $\varinjlim$ defines a functor from the category of systems of rings over $(\Omega, \prec)$ to the category of rings. Moreover, $\varinjlim_{\alpha \in \Omega} R_\alpha = \bigcup_{\alpha \in \Omega} \operatorname{Im} \phi_\alpha$ and $\ker \phi_\alpha = \bigcup_{\beta \succ \alpha} \ker \phi_{\alpha,\beta}$ for each $\alpha \in \Omega$.*

Proof. Let T be the Abelian group $\bigoplus_{\alpha \in \Omega} R_\alpha$ and identify each R_α with its image in T. For an element $t = \bigoplus_{\alpha \in \Omega} r_\alpha \in T$, we define the *support* of t to be $\operatorname{supp} t = \{\alpha \in \Omega \,|\, r_\alpha \neq 0\}$. Let J be the subgroup of T generated by all $r_\alpha - \phi_{\alpha,\beta}(r_\alpha)$ with $r_\alpha \in R_\alpha$ and $\alpha \prec \beta$. It is not hard to see that J consists of all $t = \bigoplus_{\alpha \in \Omega} r_\alpha$ for which there exists a $\beta \in \Omega$ with $\beta \succ \alpha$ for all $\alpha \in \operatorname{supp} t$, such that $\sum_{\alpha \prec \beta} \phi_{\alpha,\beta}(r_\alpha) = 0$. It is well-known that $R = T/J$, together with the canonical maps $\phi_\alpha \colon R_\alpha \to R$ given by $\phi_\alpha(r_\alpha) = r_\alpha + J$, is the direct limit of the system $\mathscr{R}$ of Abelian groups. Thus to show it is the direct limit of this system of rings, we need to show how to give R a ring structure and to show that, the Abelian group map determined in the definition of direct limit is a map of rings.

We define multiplication as follows. To multiply the cosets of $t = \bigoplus_{\alpha \in \Omega} r_\alpha$ and $t' = \bigoplus_{\beta \in \Omega} r'_\beta$, we take a $\gamma \succ \alpha, \beta$ for all $\alpha \in \operatorname{supp} t$ and all $\beta \in \operatorname{supp} t'$, and we define

$$(t + J)(t' + J) = \sum_{\alpha \in \operatorname{supp} t, \, \beta \in \operatorname{supp} t'} \phi_{\alpha,\gamma}(r_\alpha) \phi_{\beta,\gamma}(r'_\beta) + J.$$

The definition of J shows that this product is independent of the choice of γ. To see that the product is well-defined, we suppose for example that $t' \in J$ and we show the sum in the definition is in J. There is a $\delta \in \Omega$ with $\delta \succ \beta$ for all $\beta \in \operatorname{supp} t'$, such that $\sum_{\beta \prec \delta} \phi_{\beta,\delta}(r'_\beta) = 0$. Since we may always increase such a δ, we may assume that $\delta \succ \gamma$. Thus $\sum_{\alpha, \beta \prec \gamma} \phi_{\alpha,\gamma}(r_\alpha) \phi_{\beta,\gamma}(r'_\beta) - \sum_{\alpha, \beta \prec \delta} \phi_{\alpha,\delta}(r_\alpha) \phi_{\beta,\delta}(r'_\beta) \in J$. But the second term in this differ-

ence equals $(\sum_{\alpha<\delta}\phi_{\alpha,\delta}(r_\alpha))(\sum_{\beta<\delta}\phi_{\beta,\delta}(r'_\beta))=0$, so that $\sum_{\alpha,\beta<\gamma}\phi_{\alpha,\gamma}(r_\alpha)\phi_{\beta,\gamma}(r'_\beta)\in J$, as desired. We leave it to the reader to check that this multiplication is associative and hence makes R into a ring.

Let S be a ring with maps $f_\alpha\colon R_\alpha\to S$ as above. Then there is a unique Abelian group homomorphism $f\colon R\to S$ with $f\circ\phi_\alpha=f_\alpha$ for all $\alpha\in\Omega$. Let $\alpha,\beta\in\Omega$ and let $\gamma\succ\alpha,\beta$. Then

$$f(\phi_\alpha(r_\alpha)\phi_\beta(r'_\beta))=f(\phi_\gamma(\phi_{\alpha,\gamma}(r_\alpha)\phi_{\beta,\gamma}(r'_\beta)))=f_\gamma(\phi_{\alpha,\gamma}(r_\alpha)\phi_{\beta,\gamma}(r'_\beta))$$
$$=f_\alpha(r_\alpha)f_\beta(r'_\beta)=f(\phi_\alpha(r_\alpha))f(\phi_\beta(r'_\beta)).$$

This shows that f is a ring homomorphism.

The final sentence follows from the corresponding fact about direct limits of Abelian groups, and we leave it to the reader. $\qquad\square$

Remark 1.2. If $\gamma\in\Omega$ and $\phi_{\gamma,\beta}$ is onto for all $\beta\succ\gamma$, then it is not hard to see that ϕ_γ is onto, and so $\varinjlim_{\alpha\in\Omega}R_\alpha\cong R_\gamma/\bigcup_{\beta\succ\gamma}\ker\phi_{\gamma,\beta}$.

A *family of* [*left, right, two-sided*] *ideals* over $\mathscr{R}$ is a collection $\mathscr{I}=(I_\alpha)_{\alpha\in\Omega}$ such that each I_α is a [left, right, two-sided] ideal of R_α and $\phi_{\alpha,\beta}(I_\alpha)\subseteq I_\beta$ for all $\alpha,\beta\in\Omega$ with $\alpha\prec\beta$. We call the family *compatible* if $\phi_{\alpha,\beta}^{-1}(I_\beta)=I_\alpha$ for all $\alpha,\beta\in\Omega$ with $\alpha\prec\beta$. Note that a compatible family has the property that $I_\alpha\supseteq\ker\phi_\alpha$ for all $\alpha\in\Omega$, since $I_\alpha\supseteq\ker\phi_{\alpha,\beta}$ whenever $\beta\succ\alpha$. We partially order families over $\mathscr{R}$ by declaring $\mathscr{I}\subseteq\mathscr{I}'$ if $I_\alpha\subseteq I'_\alpha$ for all $\alpha\in\Omega$.

Proposition 1.3. *Let $\mathscr{R}$ be a system of rings with $R=\varinjlim_{\alpha\in\Omega}R_\alpha$.*
(1) *If $\mathscr{I}=(I_\alpha)_{\alpha\in\Omega}$ is a family of* [*left, right, two-sided*] *ideals over $\mathscr{R}$, then $I=\sum_{\alpha\in\Omega}\phi_\alpha(I_\alpha)=\bigcup_{\alpha\in\Omega}\phi_\alpha(I_\alpha)$ is a* [*left, right, two-sided*] *ideal of R. If $\mathscr{I}$ is a family of two-sided ideals, then $R/I\cong\varinjlim_{\alpha\in\Omega}R_\alpha/I_\alpha$.*
(2) *There is an order-preserving bijection between the set of* [*left, right, two-sided*] *ideals I of R and the set of compatible families of* [*left, right, two-sided*] *ideals $\mathscr{I}$ over $\mathscr{R}$ given by $I\mapsto(\phi_\alpha^{-1}(I))_{\alpha\in\Omega}$, with inverse given by $\mathscr{I}\mapsto\bigcup_{\alpha\in\Omega}\phi_\alpha(I_\alpha)$.*

Proof. (1) The first claim is obvious (using the fact that $R=\bigcup_{\alpha\in\Omega}\phi_\alpha(R_\alpha)$), and the second is not difficult.

(2) Let I be the left ideal of R. Then certainly each $\phi_\alpha^{-1}(I)$ is a left ideal of R_α. Now $\phi_{\alpha,\beta}^{-1}(\phi_\beta^{-1}(I))$ is equal to $\phi_\alpha^{-1}(I)$ by an easy set-theoretic argument, and so we define a compatible family of left ideals by setting $I_\alpha=\phi_\alpha^{-1}(I)$. If $i\in I$, then for some $\alpha\in\Omega$, there is an $r_\alpha\in R_\alpha$ with $i=\phi_\alpha(r_\alpha)$. Certainly $r_\alpha\in\phi_\alpha^{-1}(I)$, whence we see that $I=\bigcup_{\alpha\in\Omega}\phi_\alpha(I_\alpha)$.

Conversely, suppose $\mathscr{I}$ is a compatible family of left ideals: we know by (1) that $I=\bigcup_{\alpha\in\Omega}\phi_\alpha(I_\alpha)$ is a left ideal of R. If $r_\alpha\in\phi_\alpha^{-1}(I)$, then $\phi_\alpha(r_\alpha)=\phi_\beta(i_\beta)$ for some $\beta\in\Omega$, $i_\beta\in I_\beta$. Let $\gamma\in\Omega$ satisfy $\gamma\succ\alpha,\beta$: then $\phi_\gamma(\phi_{\alpha,\gamma}(r_\alpha))=\phi_\gamma(\phi_{\beta,\gamma}(i_\beta))\in\phi_\gamma(I_\gamma)$. Since $\ker\phi_\gamma\subseteq I_\gamma$, this implies $r_\alpha\in\phi_{\alpha,\gamma}^{-1}(I_\gamma)=I_\alpha$. It is obvious that the correspondence preserves order. $\qquad\square$

We call a compatible family *maximal* if it is maximal among proper compatible families, that is, maximal among compatible families $\mathscr{I}$ with at least one $I_\alpha\neq R_\alpha$. Theorem 1.3 implies that maximal [left, right, two-sided] ideals of R correspond to

maximal compatible families of [left, right, two-sided] ideals. If each map $\phi_{\alpha,\beta}$ is onto, it follows from Remark 1.2 that each of the ideals in a maximal family is maximal, but this is not true in general, as Example 1.9 shows.

We now determine which families the prime ideals in a direct limit correspond to. It is not hard to see that if I, J are ideals of $\varinjlim_{\alpha \in \Omega} R_\alpha$ and $\mathscr{I}$, $\mathscr{J}$ are the corresponding compatible families over $\mathscr{R}$, then the compatible family corresponding to IJ contains the component-wise product $\mathscr{I}\mathscr{J} = (I_\alpha J_\alpha)_{\alpha \in \Omega}$. We say a proper compatible family $\mathscr{P}$ is a *prime family* if whenever $\mathscr{P} \supseteq \mathscr{I}\mathscr{J}$ for families $\mathscr{I}, \mathscr{J}$ of ideals, we have $\mathscr{P} \supseteq \mathscr{I}$ or $\mathscr{P} \supseteq \mathscr{J}$.

Proposition 1.4. *Let $\mathscr{P}$ be a compatible family of ideals over $\mathscr{R}$ and let P be the corresponding ideal of $\varinjlim_{\alpha \in \Omega} R_\alpha$. Then the following conditions are equivalent.*
(1) *P is a prime ideal.*
(2) *$\mathscr{P}$ is a prime family.*
(3) *If $\mathscr{I}$, $\mathscr{J}$ are compatible families of ideals with $I_\alpha J_\alpha \subseteq P_\alpha$ for all $\alpha \in \Omega$, then for any $\alpha \in \Omega$, either $I_\alpha \subseteq P_\alpha$ or $J_\alpha \subseteq P_\alpha$.*
(4) *For any $\alpha \in \Omega$ and any $r_\alpha, s_\alpha \in R_\alpha$, if $\phi_{\alpha,\beta}(r_\alpha) R_\beta \phi_{\alpha,\beta}(s_\alpha) \subseteq P_\beta$ for all $\beta \succ \alpha$ then either $r_\alpha \in P_\alpha$ or $s_\alpha \in P_\alpha$.*

Proof.   Set $R = \varinjlim_{\alpha \in \Omega} R_\alpha$.

$(1) \Rightarrow (2)$ Let $\mathscr{I}$, $\mathscr{J}$ be families of ideals with $\mathscr{I}\mathscr{J} \subseteq \mathscr{P}$ and let I, J denote the ideals of R corresponding to $\mathscr{I}$, $\mathscr{J}$. It is easy to see then that $IJ \subseteq P$; since P is prime, we have either $I \subseteq P$ or $J \subseteq P$, say $I \subseteq P$. Then $I_\alpha \subseteq \phi_\alpha^{-1}(I) \subseteq \phi_\alpha^{-1}(P) = P_\alpha$ for all $\alpha \in \Omega$; this implies $\mathscr{I} \subseteq \mathscr{P}$.

$(2) \Rightarrow (3)$ This is trivial.

$(3) \Rightarrow (4)$ Suppose $r_\alpha, s_\alpha \in R_\alpha$ satisfy the hypothesis of (4), let I be the ideal of R generated by $\phi_\alpha(r_\alpha)$, and let J be the ideal of R generated by $\phi_\alpha(s_\alpha)$. Let $\mathscr{I}$ and $\mathscr{J}$ be the compatible families of ideals corresponding to I and J, and recall that $\mathscr{R} = (R_\alpha)_{\alpha \in \Omega}$. If $\beta \in \Omega$ and $\gamma \succ \alpha, \beta$, then $\phi_\alpha(r_\alpha) \phi_\beta(r'_\beta) \phi_\alpha(s_\alpha) = \phi_\gamma(\phi_{\alpha,\gamma}(r_\alpha) \phi_{\beta,\gamma}(r'_\beta) \phi_{\alpha,\gamma}(s_\alpha)) \in \phi_\gamma(P_\gamma) \subseteq P$ for all $r'_\beta \in R_\beta$. This proves $\phi_\alpha(r_\alpha) R \phi_\alpha(s_\alpha) \subseteq P$ and so $IRJ \subseteq P$. Since $\phi_\beta^{-1}(IRJ) \supseteq \phi_\beta^{-1}(I) R \phi_\beta^{-1}(J)$ for any β, this proves $\mathscr{I}\mathscr{R}\mathscr{J} \subseteq \mathscr{P}$. Thus by (3), either $\phi_\alpha^{-1}(I) \subseteq P_\alpha$, $R_\alpha = P_\alpha$, or $\phi_\alpha^{-1}(J) \subseteq P_\alpha$, and so either $r_\alpha \in P_\alpha$ or $s_\alpha \in P_\alpha$.

$(4) \Rightarrow (1)$ Let $r, s \in R$ satisfy $rRs \subseteq P$. Since $R = \bigcup \operatorname{Im} \phi_\alpha$ and Ω is directed, there is an $\alpha \in \Omega$ such that $r = \phi_\alpha(r_\alpha)$ and $s = \phi_\alpha(s_\alpha)$ for some $r_\alpha, s_\alpha \in R_\alpha$. If $\beta \succ \alpha$, we see that $r\phi_\beta(R_\beta)s = \phi_\beta(\phi_{\alpha,\beta}(r_\alpha) R_\beta \phi_{\alpha,\beta}(s_\alpha))$, whence $\phi_{\alpha,\beta}(r_\alpha) R_\beta \phi_{\alpha,\beta}(s_\alpha) \subseteq \phi_\beta^{-1}(P) = P_\beta$. By (4), this implies that one of r_α, s_α is in P_α, and so one of r, s is in P. This shows P is prime. $\square$

There is an analogous result for semiprime ideals. For completely prime ideals, there is a much simpler statement, which we leave to the reader: a proper compatible family $\mathscr{I}$ corresponds to a completely prime ideal of $\varinjlim_{\alpha \in \Omega} R_\alpha$ if and only if each I_α is a completely prime ideal or is all of R_α.

Corollary 1.5. *Let $\mathscr{P}$ be a compatible family of ideals over $\mathscr{R}$ such that for any $\alpha \in \Omega$, there is a $\beta \succ \alpha$ with P_β a prime ideal. Then $\mathscr{P}$ is a prime family.*

Proof. This follows from $(4) \Rightarrow (2)$ in Proposition 1.4. $\qquad\qquad\qquad\square$

Thus when all the ideals in a compatible family of ideals are prime, it is a prime family. We will give an example momentarily to show that the converse is not true, but first we give some instances in which it is true.

Lemma 1.6. *Let $\mathscr{P}$ be a prime family of ideals over $\mathscr{R}$ and let $\alpha \in \Omega$. If either $\mathrm{Im}\phi_{\alpha,\beta}$ is contained in the center of R_β for all $\beta \succ \alpha$ or $\phi_{\alpha,\beta}$ is onto for every $\beta \succ \alpha$, then P_α is a prime ideal (or in the former case, we may have $P_\alpha = R_\alpha$).*

Proof. Suppose first that $\mathrm{Im}\phi_{\alpha,\beta}$ is contained in the center of R_β for all $\beta \succ \alpha$. Suppose $r_\alpha, s_\alpha \in R_\alpha$ satisfy $r_\alpha s_\alpha \in P_\alpha$. Then for any $\beta \succ \alpha$, we have $\phi_{\alpha,\beta}(r_\alpha)\phi_{\alpha,\beta}(s_\alpha) \in P_\beta$, and so by the centrality hypothesis, $\phi_{\alpha,\beta}(r_\alpha)R_\beta\phi_{\alpha,\beta}(s_\alpha) \subseteq P_\beta$. Now Proposition 1.4(4) gives either r_α or s_α in P_α. This implies P_α is prime or all of R_α.

Suppose that $\phi_{\alpha,\beta}$ is onto for all $\beta \succ \alpha$, and let $r_\alpha, s_\alpha \in R_\alpha$ satisfy $r_\alpha R_\alpha s_\alpha \subseteq P_\alpha$. Applying $\phi_{\alpha,\beta}$ to this containment yields $\phi_{\alpha,\beta}(r_\alpha)R_\beta\phi_{\alpha,\beta}(r_\alpha) \subseteq P_\beta$. By Proposition 1.4(4), this implies either r_α or s_α is in P_α. Thus P_α is prime (it is proper since $\mathscr{P}$ is proper). $\qquad\square$

Corollary 1.7. *Let $\mathscr{P}$ be a compatible family of ideals over $\mathscr{R}$ and suppose that either each R_α is commutative or every $\phi_{\alpha,\beta}$ is onto. Then $\mathscr{P}$ is a prime family if and only if every ideal in it is prime (in the commutative case, some ideals P_α may equal R_α).* $\qquad\square$

The next two examples show that even when all the maps $\phi_{\alpha,\beta}$ are one-to-one, the ideals in a prime family need not be prime, and that the [left] ideals in a maximal compatible family of [left] ideals need not be maximal. They also show that the Jacobson radical does not behave in a nice way with respect to direct limits. The third example gives an instance of misbehaviour of the prime radical. Note that Ω is a chain in all of these examples.

Example 1.8. Let V be an infinite dimensional vector space over the field F with basis $e_1, e_2, \ldots$ and let V_n be the subspace spanned by $e_1, \ldots, e_n$. Let R be the ring of all linear transformations from V to itself of finite rank and let R_n be the subring consisting of all elements of R whose range is contained in V_n (in fact R_n is a right ideal of R). Let Ω be the set of positive integers with their natural order, let $\phi_{m,n}: R_m \to R_n$ be the inclusion map for all $m \leq n$, and let $\phi_n: R_n \to R$ be the inclusion map. Then since $R = \bigcup_{n=1}^{\infty} R_n$, it is not hard to see that R is the direct limit of the rings R_n.

It is well-known that R is a simple ring (that is, $R^2 = R$ and R has no non-zero proper ideals), whence 0 is a prime ideal. However, no R_n is a prime ring: if we let I_n be the set of maps in R_n that annihilate V_n, then I_n is an ideal of R_n with $I_n R_n = 0$, and in particular, I_n is nilpotent. (Note that $R_n/I_n \cong M_n(F)$, and so I_n is the Jacobson radical of R_n.) Thus the family $\mathscr{P}$ with $P_n = 0$ for all n is prime family in which no ideal is a prime ideal.

Note that the prime, nil, and Jacobson radicals of the direct limit are 0, even though these radicals are nonzero for all the rings in the system of rings.

Example 1.9. The last example featured a maximal compatible family of ideals in which every ideal is non-maximal. We now give a commutative example of this phenomenon, which shows that the same thing can happen for families of one-sided ideals. Let p_n be the nth prime number and let $R_n = \mathbb{Z}[p_1^{-1}, \ldots, p_n^{-1}]$. Then every R_n is a subring of $\mathbb{Q}$, and $R_1 \subseteq R_2 \subseteq \cdots$; if we take the maps between these rings to be the inclusion maps, their direct limit is $R = \bigcup_{n=1}^{\infty} R_n = \mathbb{Q}$. Thus the only compatible family is the one in which every ideal $I_n = 0$; this is a maximal compatible family, but none of the ideals I_n is maximal.

If we modify this example by letting p_n be the nth *odd* prime number, the direct limit is $R = \bigcup_{n=1}^{\infty} R_n = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, \ b \text{ odd}\}$. In this case, the Jacobson radical of each R_n is zero, but the Jacobson radical of R is $2R \neq 0$.

Example 1.10. It is clear that a direct limit of nil rings is nil, and a direct limit of rings that equal their own Jacobson radical equals its own Jacobson radical. We now give an example of a prime ring R containing nilpotent subrings $R_1 \subseteq R_2 \subseteq \cdots \subseteq R$ with $R = \bigcup_{n=1}^{\infty} R_n$. Thus the ring R is the direct limit of rings which equal their own prime radicals, but R does not equal its own prime radical.

Let F be a field and let R be the free F-algebra without identity on the generators $x_1, x_2, \ldots$, subject to the relations that any product of n variables from $x_1, \ldots, x_n$ is 0. The theory of free algebras tells us that a basis for R consists of all words in $x_1, x_2, \ldots$ that do not contain for any n a subword of length n with letters from $x_1, \ldots, x_n$.

Let R_n be the subalgebra of R generated by $x_1, \ldots, x_n$: then $(R_n)^n = 0$ and R is the ascending union of the subrings R_n. To see that R is prime, suppose $a, b \in R$ are nonzero and let k be the maximum length of any basis word which occurs with nonzero coefficient in either a or b. Then if we choose $n > 2k + 1$, we have $ax_n b \neq 0$, as can be seen by examining the words of maximal length in this product: they are distinct and nonzero. Thus $aRb \neq 0$, which proves that R is prime.

2. Direct sums of systems of rings and their prime ideals

A natural example of a directed set is an upper semilattice, that is, a partially ordered set in which any two elements have a least upper bound. We will generalize this by requiring that any two elements have a distinguished upper bound, but not requiring it to be least. We wish to do this in a compatible way for different pairs of elements. Given an upper semilattice, we may make it into a commutative semigroup by defining the product of two elements to be their least upper bound: this yields an associative product. Following this model, our generalization takes the following form: for the rest of this paper, Ω will be a semigroup with a preorder $\prec$ such that for any $\alpha, \beta \in \Omega$, we have $\alpha \prec \alpha\beta$ and $\beta \prec \alpha\beta$. Thus $\alpha\beta$ plays the role of our distinguished upper bound. We will define a direct sum functor from the category of systems of rings over $(\Omega, \prec)$ to the category of Ω-graded rings and study prime ideals and radicals in the resulting rings.

Recall that a ring R is Ω-*graded* if it can be written as a direct sum $R = \bigoplus_{\alpha \in \Omega} R_\alpha$ of additive subgroups R_α such that $R_\alpha R_\beta \subseteq R_{\alpha\beta}$ for all $\alpha, \beta \in \Omega$. We define a functor from the category of systems of rings to the category of Ω-graded rings by defining for any

system $\mathscr{R}$, a ring $R = \bigoplus_{\alpha \in \Omega} R_\alpha$ called the *direct sum* of $\mathscr{R}$ and denoted $\oplus \mathscr{R}$. As an Abelian group, the ring R is the direct sum of the Abelian groups R_α; the multiplication on R is defined as follows. We identify each R_α with its image in R, and for $r_\alpha \in R_\alpha, r'_\beta \in R_\beta$, we define $r_\alpha \cdot r'_\beta = \phi_{\alpha, \alpha\beta}(r_\alpha) \phi_{\beta, \alpha\beta}(r'_\beta)$; for general products, we extend this definition via the distributive law. Note that R_α is not a subring of R unless $\alpha^2 = \alpha$.

To check that this defines a ring, we simply have to check the associative law:

$$(r_\alpha \cdot r'_\beta) \cdot r''_\gamma = (\phi_{\alpha, \alpha\beta}(r_\alpha) \phi_{\beta, \alpha\beta}(r'_\beta)) \cdot r''_\gamma = \phi_{\alpha\beta, \alpha\beta\gamma}(\phi_{\alpha, \alpha\beta}(r_\alpha) \phi_{\beta, \alpha\beta}(r'_\beta)) \phi_{\gamma, \alpha\beta\gamma}(r''_\gamma)$$

$$= \phi_{\alpha, \alpha\beta\gamma}(r_\alpha) \phi_{\beta, \alpha\beta\gamma}(r'_\beta) \phi_{\gamma, \alpha\beta\gamma}(r''_\gamma).$$

A symmetric calculation shows that this equals $r_\alpha \cdot (r'_\beta \cdot r''_\gamma)$. The associative law follows from this.

If $\Theta: \mathscr{R} \to \mathscr{R}'$ is a morphism of systems of rings over $(\Omega, \prec)$, we define a map $\theta: \oplus \mathscr{R} \to \oplus \mathscr{R}'$ by $\theta = \bigoplus_{\alpha \in \Omega} \theta_\alpha$. This is clearly an additive map, and we leave it to the reader to check that it is a ring homomorphism. Thus we have indeed defined a functor, which we call the *direct sum functor*.

In the case where Ω is a semilattice (with least upper bound as product), what we have called the direct sum of the system $\mathscr{R}$ is also known as a *strong supplementary semilattice sum of the rings* R_α, $\alpha \in \Omega$—see [13] or [1]. An Ω-graded ring which is isomorphic (as an Ω-graded ring) to the direct sum of a system $\mathscr{R}$ has been called a *strong Ω-graded ring*, or in the case of rings with identity, a *special Ω-graded ring* (see the remark below).

Throughout the rest of this paper, we will write R for the ring that is the direct sum of the system $\mathscr{R}$, and we will identify elements of each R_α with their images in R.

Example 2.1. If we let $\prec$ be any preorder with the property that $\alpha, \beta \prec \alpha\beta$ (for example, the relation for which $\alpha \prec \beta$ is true for any α, β, or the ideal preorder introduced below), and for our system of rings we take each R_α equal to a single ring S, and we take all our maps $\phi_{\alpha, \beta}$ to be the identity on S, we obtain $R \cong S[\Omega]$, the semigroup ring of Ω over S.

Conversely, suppose $R = S[\Omega]$ and set $R_\alpha = S\alpha$ for each $\alpha \in \Omega$. Define multiplication in R_α by ignoring α, that is, $(s\alpha)(s'\alpha) = ss'\alpha$. If we define a system of rings $\mathscr{R}$ by defining $\phi_{\alpha, \beta}$: $R_\alpha \to R_\beta$ via $\phi_{\alpha, \beta}(s\alpha) = s\beta$ for all $s \in S$, $\alpha, \beta \in \Omega$, then in R, $(s\alpha)(s'\beta) = ss'\alpha\beta = \phi_{\alpha, \alpha\beta}(s\alpha) \phi_{\beta, \alpha\beta}(s'\beta)$, and so $R = \oplus \mathscr{R}$.

Remark 2.2. If we wish to work in the category of rings with 1, that is, if we assume each R_α has a 1 and each $\phi_{\alpha, \beta}$ preserves 1, and we want the direct sum construction to yield a ring with 1, then it suffices, but is not necessary, to assume that the semigroup Ω has an identity e. Note that e will satisfy $e \prec \alpha$ for all $\alpha \in \Omega$. If we assume each R_α has a 1 and each $\phi_{\alpha, \beta}$ preserves 1, then the notion of a direct sum of a system $\mathscr{R}$ is related to the notion of a special Ω-graded ring introduced in [8]. Given an Ω-graded ring $R = \bigoplus_{\alpha \in \Omega} R_\alpha$ in which each R_α is a subring containing an identity 1_α, the ring R is called a *special Ω-graded ring* if $1_\alpha 1_\beta = 1_{\alpha\beta}$ for all $\alpha, \beta \in \Omega$. Of course this will be the case when $R = \oplus \mathscr{R}$, under the hypotheses in the first sentence of this paragraph. Conversely, when

every element of Ω is idempotent and $\prec$ is the preorder defined by $\alpha \prec \beta$ if $\beta\alpha\beta = \beta$ (this is the case studied in [8], and this preorder is the same as the ideal preorder introduced below), we may realize a special Ω-graded ring as (isomorphic to) the direct sum of the system of rings obtained by defining maps $\phi_{\alpha,\beta}:R_\alpha \rightarrow R_\beta$ via $\phi_{\alpha,\beta}(r_\alpha) = 1_\beta r_\alpha 1_\beta$ whenever $\alpha \prec \beta$: see [8, Lemma 2.2].

We are now ready to begin our study of direct sums of systems: our goals include the description of the prime ideals and the description of various radicals. Before we proceed, however, let us note that these ideals are connected to the ring structure and have nothing to do with the preorder $\prec$. There is a minimal preorder $\prec$ on any Ω with the property that $\alpha, \beta \prec \alpha\beta$ for all $\alpha, \beta \in \Omega$, defined by declaring $\alpha \prec \beta$ if β can be written as a product (with one or more factors) involving α as a factor. If $\gamma \in \Omega$, let (γ) denote the ideal of Ω generated by γ: then $\alpha \prec \beta$ if and only if $\beta \in (\alpha)$, i.e., if and only if $(\beta) \subseteq (\alpha)$. *For the rest of this paper, we will assume $\prec$ is the preorder on Ω just defined, and all direct limits will involve this preorder:* if we need to be explicit, we will refer to this as the *ideal preorder*. We will frequently write $\beta \in (\alpha)$ instead of $\beta \succ \alpha$. If we were to use a preorder other than this minimal one, the difference would be that the maps $\phi_{\alpha,\beta}$ would have more restrictions placed on them (for example, more of them would likely be forced to be isomorphisms). In fact, if $\mathscr{R}'$ is a system of rings over an arbitrary $(\Omega, \prec')$ and $\mathscr{R}$ is the same system over $(\Omega, \prec)$ where $\prec$ is the ideal preorder, then the direct sums $\oplus\mathscr{R}$ and $\oplus\mathscr{R}'$ are identical Ω-graded rings, since in carrying out ring operations, we only need maps $\phi_{\alpha,\beta}$ where β is a product with α as one of the factors.

Another way to think of $\prec$ is the following: it is the preorder with the property that a subset of Ω is an ideal if and only if it is upwardly closed relative to $\prec$.

Recall that the preorder $\prec$ on Ω is *compatible* with the product if whenever $\alpha \prec \beta$ and $\gamma \prec \delta$, we have $\alpha\gamma \prec \beta\delta$. This is equivalent to the condition that if $\alpha \prec \beta$, then $\alpha\gamma \prec \beta\gamma$ and $\gamma\alpha \prec \gamma\beta$ for all $\gamma \in \Omega$. Associated to our preorder $\prec$ is an equivalence relation $\mathfrak{J}$ defined by $\alpha\mathfrak{J}\beta$ if $\alpha \prec \beta$ and $\beta \prec \alpha$, i.e., $\alpha\mathfrak{J}\beta$ if and only if $(\alpha) = (\beta)$ (this is Green's $\mathfrak{J}$-relation). When $\prec$ is compatible with the product, $\mathfrak{J}$ is a congruence on Ω.

It is obvious that the ideal preorder $\prec$ is compatible with the product when Ω is commutative, but in a general semigroup, it is not compatible.

Lemma 2.3. *Let Ω be a semigroup. Then the ideal preorder $\prec$ is compatible with the product on Ω if and only if $(\alpha)(\beta) = (\alpha\beta)$ for all $\alpha, \beta \in \Omega$. When this is the case, all prime ideals in Ω are completely prime.*

Proof. First assume that $\prec$ is compatible with the product Ω, and suppose $\alpha, \beta, \gamma \in \Omega$. Then $\alpha\gamma\beta \succ \alpha\beta$, so $\alpha\gamma\beta \in (\alpha\beta)$. This proves $(\alpha)(\beta) \subseteq (\alpha\beta)$, and the other inclusion is obvious.

For the converse, assume that $(\alpha)(\beta) = (\alpha\beta)$ for all $\alpha, \beta \in \Omega$. Suppose $\alpha, \beta, \gamma \in \Omega$ and $\alpha \prec \beta$, i.e., $(\beta) \subseteq (\alpha)$. Then $(\gamma\beta) = (\gamma)(\beta) \subseteq (\gamma)(\alpha) = (\gamma\alpha)$. Thus $\gamma\alpha \prec \gamma\beta$, and likewise $\alpha\gamma \prec \beta\gamma$; this proves compatibility.

The fact that all prime ideals are completely prime follows easily from the equality $(\alpha)(\beta) = (\alpha\beta)$. $\qquad\square$

It is easy to see that if $\mathscr{I}$ is a family of [left, right, two-sided] ideals over $\mathscr{R}$, then $\bigoplus_{\alpha\in\Omega}I_\alpha$ is a [left, right, two-sided] ideal of R. However, there are ideals of R that are not of this form, and in many cases prime ideals are never of this form. Our next aim is to describe the prime ideals of R, which we do under two fairly strong hypotheses on the semigroup Ω. One is that all prime ideals in Ω are completely prime; the other is condition (†) below.

Let Φ be a completely prime ideal of Ω, and define $I(\Phi)$ to be the additive subgroup of R generated by the elements r_α with $\alpha\in\Phi$ and the elements $r_\alpha-\phi_{\alpha,\beta}(r_\alpha)$ with $\alpha,\beta\in\Omega\backslash\Phi$ and $\alpha\prec\beta$, where in each case r_α is an arbitrary element of R_α. Thus $I(\Phi)\supseteq\bigoplus_{\alpha\in\Phi}R_\alpha$ and $r=\bigoplus_{\alpha\in\Omega}r_\alpha\in I(\Phi)$ if and only if for any $\gamma\in\Omega\backslash\Phi$ which is $\succ$ than every element of $(\mathrm{supp}\ r)\backslash\Phi$, we have $\sum_{\alpha\prec\gamma}\phi_{\alpha,\gamma}(r_\alpha)=0$.

Lemma 2.4. *Let $\mathscr{R}$ be a system of rings and let Φ be a completely prime ideal of Ω. Then $I(\Phi)$ is an ideal of $R=\oplus\mathscr{R}$, and $R/I(\Phi)\cong\varinjlim_{\alpha\in\Omega\backslash\Phi}R_\alpha$.*

Proof. Set $S=\varinjlim_{\alpha\in\Omega\backslash\Phi}R_\alpha$ and let $\phi_\alpha: R_\alpha\to S$ be the canonical map. Define a map $\pi: R\to S$ by

$$\pi(r_\alpha)=\begin{cases}\phi_\alpha(r_\alpha) & \alpha\in\Omega\backslash\Phi,\\ 0 & \alpha\in\Phi.\end{cases}$$

This certainly is an additive surjection and the standard facts about direct limits of Abelian groups (see the proof of Proposition 1.1) tell us that $I(\Phi)$ is its kernel. To check that π is multiplicative, take $r_\alpha\in R_\alpha$, $r'_\beta\in R_\beta$, so that their product lies in $R_{\alpha\beta}$. If either α or β is in Φ, then so is $\alpha\beta$, whence $\pi(r_\alpha r'_\beta)=0=\pi(r_\alpha)\pi(r'_\beta)$.

Suppose that both $\alpha,\beta\in\Omega\backslash\Phi$. Then $\alpha\beta\notin\Phi$, and so

$$\pi(r_\alpha r'_\beta)=\pi(\phi_{\alpha,\alpha\beta}(r_\alpha)\phi_{\beta,\alpha\beta}(r'_\beta))=\phi_{\alpha\beta}(\phi_{\alpha,\alpha\beta}(r_\alpha)\phi_{\beta,\alpha\beta}(r'_\beta))=\phi_\alpha(r_\alpha)\phi_\beta(r'_\beta)=\pi(r_\alpha)\pi(r'_\beta). \qquad\square$$

We can give a complete description of the ideals containing an $I(\Phi)$ by using the results in the previous section; we now translate those results to our current setting. We will show that under certain conditions, every prime ideal of R contains an $I(\Phi)$, and so we will be able to find all prime ideals in this case.

Let Φ be a completely prime ideal of Ω and let $\mathscr{I}$ be a compatible family of [left, right, two-sided] ideals over the system $\mathscr{R}|_{\Omega\backslash\Phi}$. We will say the pair $(\Phi,\mathscr{I})$ is a *compatible family of ideals*, and if $\mathscr{I}$ is a prime family over the system $\mathscr{R}|_{\Omega\backslash\Phi}$, we will say the pair $(\Phi,\mathscr{I})$ is a *prime family*. Now $\mathscr{I}$ determines a [left, right, two-sided] ideal $\bigcup_{\alpha\in\Omega\backslash\Phi}\phi_\alpha(I_\alpha)$ of $R/I(\Phi)$, where ϕ_α takes i_α to the coset $i_\alpha+I(\Phi)$. Thus if we define the ideal

$$I(\Phi,\mathscr{I})=I(\Phi)+\bigoplus_{\alpha\in\Omega\backslash\Phi}I_\alpha=I(\Phi)+\bigcup_{\alpha\in\Omega\backslash\Phi}I_\alpha$$

of R, we have $R/I(\Phi,\mathscr{I})\cong\varinjlim_{\alpha\in\Omega\backslash\Phi}R_\alpha/I_\alpha$. By using the directedness of $\prec$ on $\Omega\backslash\Phi$, it is

easy to see that every element $i \in I(\Phi, \mathscr{I})$ can be written as follows: there is a $\gamma \in \Omega \backslash \Phi$ and an $i_\gamma \in I_\gamma$ such that $i = \bigoplus_{\alpha \in \Phi} r_\alpha + \sum_{\alpha \prec \gamma} (r_\alpha - \phi_{\alpha, \gamma}(r_\alpha)) + i_\gamma$, where each $r_\alpha \in R_\alpha$ and only finitely many nonzero r_α occur. Alternatively, we see that $r = \bigoplus_{\alpha \in \Omega} r_\alpha \in I(\Phi, \mathscr{I})$ if and only if for any $\gamma \in \Omega \backslash \Phi$ that is $\succ$ than every element of $(\mathrm{supp}\ r) \backslash \Phi$, we have $\sum_{\alpha \prec \gamma} \phi_{\alpha, \gamma}(r_\alpha) \in I_\gamma$.

Suppose conversely that I is a [left, right, two-sided] ideal of R containing $I(\Phi)$. Then $I/I(\Phi)$ is an ideal of $\lim_{\alpha \in \Omega \backslash \Phi} R_\alpha$, and so it equals $I(\Phi, \mathscr{I})$ for the compatible family $\mathscr{I}$ given by $I_\alpha = \phi_\alpha^{-1}(I)$. Since $\phi_\alpha(r_\alpha) = r_\alpha + I(\Phi)$, this means $I_\alpha = I \cap R_\alpha$. We summarize this in the next proposition.

Proposition 2.5. *Let $\mathscr{R}$ be a system of rings and let Φ be a completely prime ideal of Ω. Then there is an order-preserving bijection between the set of [left, right, two-sided] ideals I of $R = \oplus \mathscr{R}$ containing $I(\Phi)$ and the set of compatible families $(\Phi, \mathscr{I})$ of [left, right, two-sided] ideals given by $I \mapsto (\Phi, (I \cap R_\alpha)_{\alpha \in \Omega \backslash \Phi})$, with inverse given by $(\Phi, \mathscr{I}) \mapsto I(\Phi, \mathscr{I})$. Moreover, if I is an ideal of R containing $I(\Phi)$, then $R/I \cong \lim_{\alpha \in \Omega \backslash \Phi} R_\alpha/(I \cap R_\alpha)$.* $\square$

We now turn to the question of describing prime ideals, which requires us to introduce a new condition on Ω. We say Ω *satisfies condition* (†) if
(†) for any prime ideal Φ of Ω and for any $\alpha, \beta \in \Omega \backslash \Phi$,
there exists $\gamma \in \Omega \backslash \Phi$ such that $\gamma' \alpha \gamma'' = \gamma' \beta \gamma''$ for all $\gamma', \gamma'' \in (\gamma) \backslash \Phi$.
If all prime ideals in Ω are completely prime, this can be re-stated as follows: Ω satisfies condition (†) if and only if for any prime ideal Φ of Ω and for any $\alpha, \beta \in \Omega \backslash \Phi$, there exists $\gamma \in \Omega \backslash \Phi$ such that for any $\delta, \varepsilon \in \Omega$, the two elements $\gamma \delta \alpha \varepsilon \gamma$ and $\gamma \delta \beta \varepsilon \gamma$ of Ω either are equal or both lie in Φ.

We will show in Proposition 3.1 that semigroups which satisfy condition (†) have the property that the powers of each element become stationary, and that the converse is true for commutative semigroups. In Proposition 3.4, we will show any regular or finite band satisfies condition (†); however, an infinite free band does not satisfy (†).

Lemma 2.6. *Let $\mathscr{R}$ be a system of rings, let P be a prime ideal in $R = \oplus \mathscr{R}$, and let $\Phi = \{\alpha \in \Omega \mid R_\beta \subseteq P \text{ for all } \beta \in (\alpha)\}$. Then*
(1) Φ *is a prime ideal of Ω.*
(2) *If Ω satisfies condition (†) and all prime ideals in Ω are completely prime, then* $P \supseteq I(\Phi)$.

Proof. (1) It is clear that Φ is proper since P is proper. If A, B are ideals of Ω with $AB \subseteq \Phi$, then $I = \bigoplus_{\alpha \in A} R_\alpha$ and $J = \bigoplus_{\beta \in B} R_\beta$ are ideals of R with $IJ \subseteq P$. Thus either $I \subseteq P$ or $J \subseteq P$, and so either $A \subseteq \Phi$ or $B \subseteq \Phi$.

(2) Clearly $\bigoplus_{\alpha \in \Phi} R_\alpha \subseteq P$. Let $\alpha, \beta \in \Omega \backslash \Phi$, $\alpha \prec \beta$ and suppose that $r_\alpha \in R_\alpha$: we need to show that $r_\alpha - \phi_{\alpha, \beta}(r_\alpha) \in P$. By condition (†), there exists $\gamma \in \Omega \backslash \Phi$ with $\gamma' \alpha \gamma'' = \gamma' \beta \gamma''$ for all $\gamma', \gamma'' \in (\gamma) \backslash \Phi$. Set $I = \bigoplus_{\delta \in (\gamma)} R_\delta$; this is an ideal of R, and it is not contained in P, since $\gamma \in \Omega \backslash \Phi$. Let $\gamma', \gamma'' \in (\gamma)$, and suppose $r_{\gamma'} \in R_{\gamma'}$ and $s_{\gamma''} \in R_{\gamma''}$. If either γ' or γ'' is in Φ, then $(\gamma' \alpha \gamma''), (\gamma' \beta \gamma'') \subseteq \Phi$, whence $r_{\gamma'} \cdot (r_\alpha - \phi_{\alpha, \beta}(r_\alpha)) \cdot s_{\gamma''} \in P$. Otherwise,

$$r_{\gamma'} \cdot (r_\alpha - \phi_{\alpha, \beta}(r_\alpha)) \cdot s_{\gamma''} =$$

$$\phi_{\gamma', \gamma'\alpha\gamma''}(r_{\gamma'})\phi_{\alpha, \gamma'\alpha\gamma''}(r_\alpha)\phi_{\gamma'', \gamma'\alpha\gamma''}(s_{\gamma''}) - \phi_{\gamma', \gamma'\beta\gamma''}(r_{\gamma'})\phi_{\alpha, \gamma'\beta\gamma''}(r_\alpha)\phi_{\gamma'', \gamma'\beta\gamma''}(s_{\gamma''}) = 0.$$

Thus $I(r_\alpha - \phi_{\alpha, \beta}(r_\alpha))I \subseteq P$; this implies $r_\alpha - \phi_{\alpha, \beta}(r_\alpha) \in P$. This proves $I(\Phi) \subseteq P$. $\quad\square$

Combined with Proposition 2.5, Lemma 2.6 now yields the following description of all prime ideals of R.

Theorem 2.7. *Let $\mathscr{R}$ be a system of rings and suppose that Ω satisfies condition (†) and that every prime ideal of Ω is completely prime. Then there is a bijective correspondence between the set of prime ideals P in $R = \oplus\mathscr{R}$ and the set of prime families $(\Phi, \mathscr{P})$ with Φ a prime ideal of Ω, given by*

$$P \mapsto (\Phi = \{\alpha \in \Omega \mid R_\beta \subseteq P \text{ for all } \beta \in (\alpha)\}, (P \cap R_\alpha)_{\alpha \in \Omega \backslash \Phi}),$$

with inverse given by $(\Phi, \mathscr{P}) \mapsto I(\Phi, \mathscr{P})$.

If $R_\alpha R_\beta = R_{\alpha\beta}$ for all $\alpha, \beta \in \Omega$, which will be the case for example if each ring R_γ satisfies $R_\gamma^2 = R_\gamma$ and the maps $\phi_{\alpha, \beta}$ are all onto, then it is not hard to see that in Theorem 2.7, $\Phi = \{\alpha \in \Omega \mid R_\alpha \subseteq P\}$. In general this is not true, but one can use Theorem 2.7 to prove that if $\alpha \prec \beta \in \Omega \backslash \Phi$, and $R_\beta \subseteq P$, then $R_\alpha \subseteq P$.

These results can also be used to find maximal one-sided ideals. Recall that a maximal left ideal M of R is *modular* if R^2 is not contained in M. Such an M contains a prime ideal P (namely the annihilator of the module R/M), and so M contains an ideal $I(\Phi)$. Thus we have $M = I(\Phi, \mathscr{M})$ for some maximal compatible family $\mathscr{M}$ of left ideals over $\Omega \backslash \Phi$.

We would like to compare prime ideals as well. The next lemma and its corollary tell us comparable prime ideals correspond to prime families defined over the complement of the same prime ideal Φ.

Lemma 2.8. *Let $\mathscr{R}$ be a system of rings and suppose that Φ is a completely prime ideal of Ω and that J is a proper additive subgroup of $R = \oplus\mathscr{R}$ containing $I(\Phi)$. Then $\Phi = \{\alpha \in \Omega \mid R_\beta \subseteq J \text{ for all } \beta \in (\alpha)\}$.*

Proof. Set $\Phi' = \{\alpha \in \Omega \mid R_\beta \subseteq J \text{ for all } \beta \in (\alpha)\}$: then Φ' is clearly an ideal of Ω. If $\alpha \in \Phi$, then since Φ is an ideal, $(\alpha) \subseteq \Phi$, and since $I(\Phi) \subseteq J$, we have $R_\beta \subseteq J$ for all $\beta \in (\alpha)$. Thus $\alpha \in \Phi'$, which shows $\Phi \subseteq \Phi'$.

Since J is proper, there is an $\alpha \in \Omega$ with $R_\alpha \nsubseteq J$, say $r_\alpha \in R_\alpha \backslash J$: note that $\alpha \notin \Phi$. Now if $\beta \in \alpha\Phi'$, then $\beta \in \Phi'$, so $R_\beta \subseteq J$. If $\beta \notin \Phi$, then $r_\alpha - \phi_{\alpha, \beta}(r_\alpha) \in I(\Phi) \subseteq J$, and $\phi_{\alpha, \beta}(r_\alpha) \in R_\beta \subseteq J$, which contradicts $r_\alpha \notin J$. Thus we must have $\alpha\Phi' \subseteq \Phi$. As Φ is a prime ideal, this implies $\Phi' \subseteq \Phi$. $\quad\square$

Corollary 2.9. *Let $\mathscr{R}$ be a system of rings and suppose that Ω satisfies condition (†) and that every prime ideal of Ω is completely prime. Let P, P' be prime ideals of $R = \oplus\mathscr{R}$*

with corresponding prime families $(\Phi, \mathscr{P})$ *and* $(\Phi', \mathscr{P}')$. *Then* $P \subseteq P'$ *if and only if* $\Phi = \Phi'$ *and* $\mathscr{P} \subseteq \mathscr{P}'$.

Proof. If $\Phi = \Phi'$ and $\mathscr{P} \subseteq \mathscr{P}'$, then clearly $P \subseteq P'$.

For the converse, assume that $P \subseteq P'$. By Proposition 2.5, it is enough to prove $\Phi = \Phi'$. By Lemma 2.6, we have $I(\Phi) \subseteq P'$ and $\Phi' = \{\alpha \in \Omega \mid R_\beta \subseteq P' \text{ for all } \beta \in (\alpha)\}$, and moreover, the hypothesis of Lemma 2.8 is satisfied with $J = P'$. Thus by Lemma 2.8, $\Phi = \Phi'$. □

Recall that if the supremeum of the lengths of chains of prime ideals of a ring R is finite, this number is called the "classical Krull dimension" of R. Our results on prime ideals yield information about the classical Krull dimension of $\oplus \mathscr{R}$; to state the result in the greatest generality, we need to introduce the ordinal-valued definition of classical Krull dimension. We let Spec R denote the set of prime ideals of R and we set Spec$^{-1}R = \varnothing$. For any ordinal η, we define

$$\text{Spec}^\eta R = \{P \in \text{Spec} R \mid \text{for any } Q \in \text{Spec} R \text{ with } Q \supset P, \text{ we have } Q \in \text{Spec}^\kappa R \text{ for some } \kappa < \eta\}.$$

If there is an η with Spec$^\eta R = \text{Spec} R$, we say *R has classical Krull dimension*, and the *classical Krull dimension of R* is the least such η. This agrees with the simpler notion given above when the classical Krull dimension is finite. See for example [3, Chapter 12] for more details on classical Krull dimension.

Proposition 2.10. *Let $\mathscr{R}$ be a system of rings and suppose that Ω satisfies condition* (†) *and that every prime ideal of Ω is completely prime. Then $R = \oplus \mathscr{R}$ has classical Krull dimension if and only if $\varinjlim_{\alpha \in \Omega \setminus \Phi} R_\alpha$ has classical Krull dimension for each prime ideal Φ of Ω, and when this is the case, the classical Krull dimension of R is the supremum of the classical Krull dimensions of the rings $\varinjlim_{\alpha \in \Omega \setminus \Phi} R_\alpha$.*

Proof. For each prime ideal Φ of Ω, let Spec$_\Phi R$ denote the set of prime ideals of R containing $I(\Phi)$. By Theorem 2.7 and Corollary 2.9, Spec R is the disjoint union of the various Spec$_\Phi(R)$, and so Spec $R = \text{Spec}^\eta R$ if and only if Spec$_\Phi R \cap \text{Spec}^\eta R = \text{Spec}_\Phi R$ for all Φ. As Spec$_\Phi R$ corresponds in a natural way to Spec$R/I(\Phi)$ and $\varinjlim_{\alpha \in \Omega \setminus \Phi} R_\alpha \cong R/I(\Phi)$, we have Spec $R = \text{Spec}^\eta R$ if and only if Spec $\varinjlim_{\alpha \in \Omega \setminus \Phi} R_\alpha = \text{Spec}^\eta \varinjlim_{\alpha \in \Omega \setminus \Phi} R_\alpha$ for all Φ. The proposition follows from this observation. □

Let us now summarize the results of this section for the special case of semigroup rings, that is, the special case where each map $\phi_{\alpha, \beta}$ is an isomorphism. Just as we use Spec$R$ to denote the set of prime ideals of a ring R, let us use SpecΩ to denote the set of prime ideals of a semigroup Ω.

Theorem 2.11. *Suppose that Ω satisfies condition* (†) *and that every prime ideal of Ω is completely prime, let S be a ring, and let $S[\Omega]$ be the semigroup ring.*

(1) *There is a bijective correspondence between Spec$\Omega \times$ SpecS and SpecS$[\Omega]$, given by* $(\Phi, \Omega) \mapsto \{\sum_{\alpha \in \Omega} s_\alpha \alpha \mid \sum_{\alpha \in \Omega \setminus \Phi} s_\alpha \in Q\}$. *If S and Ω have identities, the inverse map is given by*

$P \mapsto (P \cap \Omega, P \cap S)$. *In addition, the prime ideals corresponding to* (Φ, Q) *and* (Φ', Q') *are comparable if and only if* $\Phi = \Phi'$ *and* Q *and* Q' *are comparable.*

(2) *The ring* $S[\Omega]$ *is not prime if* $|\Omega| \geqq 2$.

(3) S *has classical Krull dimension if and only if* $S[\Omega]$ *does, and when this is the case, the classical Krull dimensions of these rings are equal.*

Proof. These statements follow easily from the results of this section. For example, to verify (2) suppose $0 = I(\Phi, \mathscr{I})$: then clearly we must have $\Phi = \varnothing$. Now if α, β are distinct elements of Ω, their product cannot equal both of them, say $\alpha\beta \neq \alpha$. Then for any nonzero $s \in S$, we have $s\alpha - s\alpha\beta \in I(\Phi, \mathscr{I})$, which contradicts our assumption. $\qquad\square$

3. Bands, power-stationary semigroups, and condition (†)

Condition (†) is a very strong condition, which will not hold in many instances. Every semilattice satisfies it, and we will show below that a commutative semigroup satisfies condition (†) if and only if for every element x there is a positive integer k with $x^k = x^{k+1}$. We then turn to noncommutative semigroups and show that every finite band and every regular band satisfies condition (†), and so we may apply the results of the last section to describe the prime ideals in direct sums of systems over such semigroups, and we may apply the results of the next section to describe certain radicals. However, an infinite free band does not satisfy condition (†).

Let us call an element α of a semigroup *power-stationary* if there is a positive integer k with $\alpha^k = \alpha^{k+1}$ and call a semigroup *power-stationary* if every one of its elements is power-stationary.

Proposition 3.1. *Let* Ω *be a semigroup.*

(1) *If* Ω *satisfies condition* (†), *then* Ω *is power-stationary.*

(2) *If* Ω *is commutative, then* Ω *satisfies condition* (†) *if and only if* Ω *is power-stationary.*

Proof. (1) Suppose condition (†) holds, let $\alpha \in \Omega$, and let Φ be the largest ideal of Ω not containing any power of α (that is, $\Phi = \{\beta \in \Omega \mid \alpha^n \notin (\beta)$ for all positive integers $n\}$). A standard argument shows that this is a prime ideal of Ω. By condition (†), there will be $\gamma \in \Omega \backslash \Phi$ with $\gamma'\alpha\gamma'' = \gamma'\alpha^2\gamma''$ for all $\gamma' \in (\gamma)\backslash\Phi$, $\gamma'' \in (\gamma)\backslash\Phi$. The definition of Φ tells us there is a positive integer n with $\alpha^n \in (\gamma)\backslash\Phi$, and so $\alpha^{2n+1} = \alpha^{2n+2}$.

(2) Suppose Ω is commutative and power-stationary, let Φ be a prime ideal of Ω, and let $\alpha, \beta \in \Omega \backslash \Phi$. By the power-stationary property, there is a positive integer k with $\alpha^k = \alpha^{k+1}$ and $\beta^k = \beta^{k+1}$. Set $\gamma = \alpha^k\beta^k$: this is an element of $\Omega\backslash\Phi$ since Φ is prime. Note that $\alpha\gamma = \beta\gamma = \gamma$. If $\gamma', \gamma'' \in (\gamma)$, then since γ is idempotent, we have $\gamma' = \gamma\gamma'$ and $\gamma'' = \gamma\gamma''$, whence $\gamma'\alpha\gamma'' = \gamma'\gamma\alpha\gamma\gamma'' = \gamma'\gamma''$. This is also equal to $\gamma'\beta\gamma''$, and so condition (†) holds. $\qquad\square$

Example 3.2. Let $\Omega = \{0, e_{11}, e_{12}, e_{21}, e_{22}\}$, where the e_{ij} are the standard matrix units. This is a finite semigroup that satisfies the identity $x^3 = x^2$, and hence is power-

stationary, but does not satisfy condition (†). This may be seen by taking $\Phi = \{0\}$ and $\alpha = e_{11}$, $\beta = e_{12}$ in the definition of (†).

We say a semigroup Ω is a *band* if every element of Ω is idempotent, that is, if Ω satisfies the identity $x^2 = x$. If in addition Ω satisfies the identity $xyxzx = xyzx$, we say Ω is a *regular band*. (See [11, §II.3] for a discussion of varieties of bands; examples of regular bands include commutative bands and normal bands.) Note that a commutative band is the same thing as a semilattice. We wish to show every band that is either regular or satisfies the d.c.c. on principal ideals satisfies condition (†); this requires us to develop some of the properties of bands.

Lemma 3.3. *Let Ω be a band and let $\prec$ be the ideal preorder on Ω. Then:*
(1) *$\prec$ is compatible with the product on Ω.*
(2) *If $\alpha, \beta \in \Omega$, then $\alpha \prec \beta$ if and only if $\beta\alpha\beta = \beta$.*

Proof. (1) Let $\gamma \in \Omega$. Then

$$\alpha\gamma\beta = \alpha\gamma\beta\alpha\gamma\beta = \alpha\gamma\beta\alpha\beta\alpha\gamma\beta \in (\alpha\beta).$$

This proves $(\alpha)(\beta) \subseteq (\alpha\beta)$; as the other inclusion is obvious, Lemma 2.3 implies $\prec$ is compatible with the product on Ω.

(2) Assume that $\alpha \prec \beta$: then $\beta = \gamma\alpha\delta$ for some $\gamma, \delta \in \Omega$ (since for example, $\gamma\alpha = \gamma\alpha\alpha$). Thus we need to show $\gamma\alpha\delta\alpha\gamma\alpha\delta = \gamma\alpha\delta$ is an identity which holds in any band. To do this, it is of course enough to prove it in the free band on α, γ, δ, in which case it is an immediate consequence of the properties of words in a free band (it follows for example from [11, Proposition II.2.7(iv)] or [5, Lemma IV.4.6]).

The converse is obvious. ☐

We can now show that regular and finite bands satisfy condition (†).

Proposition 3.4. *If Ω is a band that is either regular or satisfies the d.c.c. on principal ideals, then Ω satisfies condition (†) and all prime ideals of Ω are completely prime.*

Proof. The fact that all prime ideals are completely prime follows from Lemma 3.3 (1) and Lemma 2.3.

Let Φ be a prime ideal of Ω, and let $\alpha, \beta \in \Omega\backslash\Phi$. First suppose Ω is regular and let $\gamma', \gamma'' \in (\alpha\beta)$, so that $\gamma', \gamma'' \succ \alpha$. Then $\gamma'\alpha\gamma'' = \gamma'\alpha\gamma'\alpha\gamma''\alpha\gamma''$; by regularity, this equals $\gamma'\alpha\gamma'\gamma''\alpha\gamma''$, and by Lemma 3.3, this equals $\gamma'\gamma''$. Replacing α by β leads to the same conclusion, and so we may take $\gamma = \alpha\beta$ in the definition of (†) (this is not in Φ, since Φ must be completely prime).

Suppose next that Ω satisfies the d.c.c. on principal ideals and choose $\gamma \in \Omega\backslash\Phi$ with (γ) minimal among principal ideals generated by elements of $\Omega\backslash\Phi$. Then for any $\delta \in \Omega\backslash\Phi$, we have $(\delta\gamma) \subseteq (\gamma)$, whence equality holds and $\delta\gamma \prec \gamma$. By Lemma 3.3, this implies

$\gamma = \gamma\delta\gamma\gamma = \gamma\delta\gamma$. Now if $\gamma', \gamma'' \in (\gamma)\backslash\Phi$, we have $\gamma'\alpha\gamma'' = \gamma'\gamma\gamma'\alpha\gamma''\gamma\gamma'' = \gamma'\gamma\gamma''$, and the same is true for β in place of α. Thus condition (†) is satisfied.   □

Thus we may apply Theorem 2.7 and conclude that, if Ω is a commutative power-stationary semigroup or a band that is either regular or satisfies the d.c.c. on principal ideals, then there is a bijective correspondence between prime ideals of $R = \oplus\mathcal{R}$ and prime families over complements of prime ideals of Ω; moreover, Corollary 2.9 tells us prime ideals of R corresponding to distinct prime ideals of Ω are incomparable.

Note that a band Ω satisfies the d.c.c. on principal ideals if and only if its greatest semilattice quotient $\Omega/\mathfrak{J}$ satisfies the a.c.c. as a poset [when regarded as an upper semilattice, or equivalently, satisfies the d.c.c. when regarded as a poset with the "natural order"].

Unfortunately, the results above are not valid for arbitrary bands. In fact, we have the following proposition.

Proposition 3.5. *Let Ω be a free band on infinitely many generators.*
(1) *Ω does not satisfy condition (†).*
(2) *If S is a prime ring, then the semigroup ring $S[\Omega]$ is a prime ring.*

Proof. (1) This follows from (2) and Theorem 2.11 (2); it may also be proved directly by using the same idea as in the proof of (2).

(2) Let $r = \sum_{\alpha\in\Omega} r_\alpha\alpha$, $r' = \sum_{\alpha\in\Omega} r'_\alpha\alpha$ be nonzero elements of $R = S[\Omega]$. Since r and r' have finite support, there is a finite subset Y of the free generating set of Ω with the property that every $\alpha\in\mathrm{supp}\,r\cup\mathrm{supp}\,r'$ lies in the subsemigroup of Ω generated by Y. Let x be an element of the free generating set of Ω that is not in Y, and let $\alpha, \beta\in\Omega$ be such that $r_\alpha\neq 0\neq r'_\beta$. Since S is prime, there is an $s\in S$ with $r_\alpha s r'_\beta\neq 0$. We claim that $r(sx)r'\neq 0$.

Since $r(sx)r' = \sum_{(\gamma,\delta)\in\mathrm{supp}\,r\times\mathrm{supp}\,r'}(r_\gamma s r'_\delta)\gamma x\delta$, the claim follows once we know the following fact: if $\alpha, \beta, \gamma, \delta$ are in the subsemigroup generated by Y, then $\alpha x\beta = \gamma x\delta$ if and only if $(\alpha, \beta) = (\gamma, \delta)$. This fact follows from the properties of words in a free band: it is an immediate consequence of [**4**, Lemma 2] or [**5**, Lemma IV.4.6].   □

As the ideal 0 in $S[\Omega]$ cannot be of the form $I(\Phi, \mathscr{P})$ (since $|\Omega| > 1$), this proposition shows we cannot extend the results of Theorems 2.7 and 2.11 to arbitrary bands. In fact, we can say more. Suppose that Ω is an infinitely generated free band and that x is an element of a free generating set. Then it is not hard to see that if P is the ideal of $S[\Omega]$ generated by x, then $S[\Omega]/P \cong S[\Omega]$. If S is prime, this shows $S[\Omega]$ cannot have classical Krull dimension, and this clearly extends to any ring S which has a prime ideal.

Proposition 3.5 has other interesting consequences for an infinite free band Ω. It follows from Theorem 4.9 that $S[\Omega]$ has a nonzero nil ideal for any nonzero ring S, so if S is prime, the semigroup ring $S[\Omega]$ is neither left nor right Goldie. It follows that $S[\Omega]$ cannot satisfy a polynomial identity and cannot be left or right Noetherian if S is any ring having a prime ideal. (These facts are known.)

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Remark 3.6. Since a band is finite if and only if it is finitely generated (see [4, Theorem 1] or [5, Theorem IV.4.9]), Proposition 3.5 is valid for any infinite free band.

One may use [4, Lemma 2] to show that if r is any positive integer and Ω is a free semigroup on infinitely many generators in the variety defined by the identity $x^r = x$, then $S[\Omega]$ is a prime ring if and only if S is a prime ring.

Proposition 3.5 can be extended to show that if Ω is an infinite free band (or an infinitely generated free semigroup of the type described in the last paragraph) and $\mathscr{P}$ is a prime family, and if either every $\phi_{\alpha,\beta}$ is onto or each R_α is commutative, then $\oplus\mathscr{P}$ is a prime ideal in $\oplus\mathscr{R}$.

4. Radicals of direct sums of systems

Our aim in this section is to compute various radicals of the direct sum $R = \oplus\mathscr{R}$ of the system $\mathscr{R}$, when Ω satisfies condition (†) and all prime ideals of Ω are completely prime; our results apply to commutative power-stationary semigroups and regular or finite bands. At the end of the section, we will show how to drop condition (†) and extend our results to the case where Ω is an arbitrary band. In order to compute the radical, we show that R has a factor ring which is isomorphic to the direct sum of a system indexed by the largest semilattice quotient of Ω, and when Ω is a finite semilattice, we also show R is isomorphic to the componentwise direct product of the rings R_α. We then use these connections to find the Jacobson and upper nil radicals of R, as well as the prime radical in some cases. We also give a formula for the classical Krull dimension of R when Ω satisfies condition (†), all prime ideals of Ω are completely prime, and every map $\phi_{\alpha,\beta}$ is onto.

Descriptions of radicals in semilattice-graded rings and band-graded rings have been given in several places, such as [13], [1], [8], and [7], to name just a few. Our results are more general and our methods are fairly transparent.

We begin our study of radicals by relating quotient semigroups of Ω to quotient rings of $\oplus\mathscr{R}$. Suppose $\equiv$ is a congruence on Ω and $\mathscr{R}$ is a system of rings over Ω with the ideal preorder: we wish to define a system of rings $\mathscr{S}$ over $\Omega/\equiv$. If each equivalence class $x \in \Omega/\equiv$ is directed relative to the preorder inherited from Ω, we may define rings $S_x = \varinjlim_{\alpha \in x} R_\alpha$; for $\alpha \in x$, let $\phi_\alpha^x : R_\alpha \to S_x$ denote the canonical map. Using the directedness of the equivalence classes, one may show that if $x, y \in \Omega/\equiv$ and $x \prec y$ relative to the ideal preorder on $\Omega/\equiv$, then there is a unique ring homomorphism $\phi_{x,y} : S_x \to S_y$ defined by the equation $\phi_{x,y} \circ \phi_\alpha^x = \phi_\beta^y \circ \phi_{\alpha,\beta}$ whenever $\alpha \in x$, $\beta \in y$, $\alpha \prec \beta$. It is easy to verify that the rings S_x together with the maps $\phi_{x,y}$ form a system of rings over $\Omega/\equiv$. Now define a map $\sigma : \oplus\mathscr{R} \to \oplus\mathscr{S}$ by setting $\sigma(r_\alpha) = \phi_\alpha^{\bar\alpha}(r_\alpha)$, where $\bar\alpha$ is the equivalence class of α under $\equiv$. It is clear that σ is an additive map; we leave it to the reader to check that it is a surjective ring homomorphism and that its kernel is given by the next lemma.

Lemma 4.1. *Let $\mathscr{R}$ be a system of rings over the semigroup Ω and let $\equiv$ be a congruence on Ω such that each equivalence class is directed. Then $\mathscr{S}$ defined above is a system of rings over $\Omega/\equiv$, and the map $\sigma : \oplus\mathscr{R} \to \oplus\mathscr{S}$ just defined is a surjective ring homomorphism with kernel*

$$K(\mathcal{R}, \equiv) = \left\{ \bigoplus_{\alpha \in \Omega} r_\alpha \,\Big|\, \sum_{\alpha \in x} \phi_\alpha^x(r_\alpha) = 0 \text{ for all } x \in \Omega/\equiv \right\}. \qquad \square$$

Each equivalence class will certainly be directed if each equivalence class is a subsemigroup of Ω. This will be the case if and only if $\equiv$ is a *band congruence*, that is, a congruence such that $\alpha^2 \equiv \alpha$ for all $\alpha \in \Omega$. There is a particular band congruence that we will use: the *least semilattice congruence* η on Ω. This is a congruence with the property that Ω/η is a semilattice and such that for any congruence $\equiv$ with $\Omega/\equiv$ a semilattice, $\alpha\eta\beta$ implies $\alpha \equiv \beta$. Such a congruence exists and can be described concretely as follows (see for example [10, §II.2]): $\alpha\eta\beta$ if and only if α and β belong to exactly the same completely prime ideals of Ω. Our approach to finding radicals will be to show that the ideal $K(\mathcal{R}) = K(\mathcal{R}, \eta)$ is contained in the radical of R, which allows us to pass to $R/K(\mathcal{R})$ and so assume Ω is a semilattice, where things are easier to prove. In fact, it is almost immediate from the results of Section 2 that we can do this for any radical containing the prime radical when Ω satisfies condition (†) and all prime ideals in Ω are completely prime.

Let us suppose Ω satisfies condition (†) and all prime ideals of Ω are completely prime: then the relation η has a more concrete description. By Lemma 3.1, for every $\alpha \in \Omega$, there is a positive integer k with $\alpha^k = \alpha^{k+1}$: set $e_\alpha = \alpha^k$. Then e_α is an idempotent and it is clear that $\alpha\eta e_\alpha$. It follows that for $\alpha, \beta \in \Omega$, we have $\alpha\eta\beta$ if and only if $e_\alpha\eta e_\beta$. Now let $e, f \in \Omega$ be idempotents. If $(e) = (f)$, then clearly e and f belong to exactly the same ideals of Ω, and so $e\eta f$. Conversely, suppose $e\eta f$ and $e \notin (f)$, and let Φ be the largest ideal of Ω containing f but not e. A standard argument shows Φ is prime, which contradicts $e\eta f$. Thus we see $e \in (f)$ and so $(e) \subseteq (f)$. By symmetry we may conclude $(e) = (f)$, and so we see that $\alpha\eta\beta$ if and only if $e_\alpha \mathfrak{J} e_\beta$. Hence $\alpha\eta\beta$ if and only if there are positive integers m, n with $\alpha^m \mathfrak{J} \beta^n$, and when this is the case, this relation holds for all large enough m and n. Suppose that in addition Ω is commutative. Then it is easy to verify that if e, f are idempotents, $(e) = (f)$ if and only if $e = f$, and so in this case $\alpha\eta\beta$ if and only if $\alpha^m = \beta^n$ for some positive integers m, n.

Suppose that Ω satisfies condition (†) and that every prime ideal in Ω is completely prime, and let $E(\Omega)$ denote the set of idempotents in Ω. Then the last paragraph tells us that if $\alpha \in \Omega$, we have $\beta \prec e_\alpha$ for any $\beta\eta\alpha$. Thus the direct limit $\varinjlim_{\alpha \in x} R_\alpha$ is R_e for any idempotent $e \in x$, where the maps ϕ_α^x agree with the maps $\phi_{\alpha, e}$. It follows from Lemma 4.1 that

$$K(\mathcal{R}) = \left\{ \bigoplus_{\alpha \in \Omega} r_\alpha \,\Big|\, \sum_{\alpha\eta e} \phi_{\alpha, e}(r_\alpha) = 0 \text{ for all } e \in E(\Omega) \right\}.$$

Lemma 4.2. *Suppose that Ω satisfies condition (†) and that every prime ideal of Ω is completely prime, and let $\mathcal{R}$ be a system of rings over Ω. Then the ideal $K(\mathcal{R})$ is the intersection of all the ideals $I(\Phi)$ with Φ a prime ideal of Ω, and so is contained in the prime radical of $R = \oplus\mathcal{R}$.*

Proof. We begin by showing $K = K(\mathscr{R})$ is contained in $I(\Phi)$ for all prime ideals Φ of Ω. To do this, it suffices to show that if Φ is a prime ideal, $e \in E = E(\Omega)$, and $r = \bigoplus_{\alpha \in \bar{e}} r_\alpha$ satisfies $\sum_{\alpha \in \bar{e}} \phi_{\alpha, e}(r_\alpha) = 0$ (where $\bar{e}$ is the equivalence class of e under η), then $r \in I(\Phi)$. Suppose $e \notin \Phi$: then since $e \succ \alpha$ for all $\alpha \in \operatorname{supp} r$, the condition $\sum_{\alpha \in \bar{e}} \phi_{\alpha, e}(r_\alpha) = 0$ ensures that $r \in I(\Phi)$. If $e \in \Phi$, then for every $\alpha \in \bar{e}$, there is a k with $\alpha^k \mathfrak{J} e$, i.e., with $\alpha^k \in (e) \subseteq \Phi$. Since Φ is completely prime, this implies $\alpha \in \Phi$, and so $r \in I(\Phi)$.

It now follows from Theorem 2.7 that K is contained in every prime ideal of R.

For each $e \in E$, let Φ_e be the largest ideal of Ω not containing e: we've noted before that Φ_e is a prime ideal. Now let $r = \bigoplus_{\alpha \in \Omega} r_\alpha \neq 0$ lie in the intersection of all the ideals $I(\Phi)$, and choose $\beta \in \operatorname{supp} r$ such that $e = e_\beta$ is minimal relative to $\prec$ among idempotents e_α with $\alpha \in \operatorname{supp} r$. Set $r' = \bigoplus_{\alpha \prec e} r_\alpha$. Since $r \in I(\Phi_e)$, we have $\sum_{\alpha \prec e} \phi_{\alpha, e}(r_\alpha) = 0$. Moreover, by our choice of e, we must in fact have $r' = \bigoplus_{\alpha \eta e} r_\alpha$ and so $r' \in K$. By the first paragraph of the proof, this implies r' is in the intersection of all the ideals $I(\Phi)$, and so $r - r'$ is as well. Since $r - r'$ has smaller support than r, we may assume by induction that $r - r' \in K$. It follows that $r \in K$. $\square$

In addition to the homomorphism σ defined above from R to $\oplus \mathscr{S}$, there is another homomorphism from R which is very useful. Let $e \in E$ be an idempotent in Ω, and define $\psi_e : R \to R_e$ by $\psi_e(\bigoplus_{\alpha \in \Omega} r_\alpha) = \sum_{\alpha \prec e} \phi_{\alpha, e}(r_\alpha)$. It is straightforward to verify that ψ_e is a surjective ring homomorphism with kernel $I(\Phi_e)$, where Φ_e is the largest ideal of Ω not containing e. It follows from the proof of Lemma 4.2 that $K(\mathscr{R}) = \bigcap_{e \in E} \ker \psi_e$.

Now define a map $\psi: R \to \prod_{e \in E} R_e$, where the multiplication in the direct product is componentwise, by $\psi(r) = (\psi_e(r))_{e \in E}$. Since each ψ_e is a ring homomorphism, so is ψ, and $\ker \psi = K$.

Lemma 4.3. *Let $\mathscr{R}$ be a system of rings over a finite semilattice Ω. Then the map ψ just defined is an isomorphism.*

Proof. We know ψ is an injective homomorphism of rings; we only need to show it is onto. Let $\Omega = \{\beta_1, \ldots, \beta_n\}$, where we label the elements so that $\beta_i \prec \beta_j$ implies $i \leq j$, and let $s = (s_{\beta_i})_{i=1}^n \in \prod_{\beta \in \Omega} R_\beta$. Let i be the smallest index with $s_{\beta_i} \neq 0$. If $i = n$, then $s = \psi(s_{\beta_n})$ since β_n is the greatest element of Ω. In general, if we let $s' = \psi(s_{\beta_i})$, then the elements $\beta_j \in \operatorname{supp} s'$ satisfy $\beta_j \succ \beta_i$, so $j \geq i$. Thus $s - s'$ is an element whose least index with a nonzero component is larger than i, and so we may assume by induction that $s - s' = \psi(r')$ for some $r' \in R$. Hence $s = \psi(r' + s_{\beta_i})$, which proves ψ is onto. $\square$

Remark 4.4. We can explicitly compute the inverse of ψ in Lemma 4.3 as follows: let M be the Möbius matrix of Ω, that is, the inverse of the $\Omega \times \Omega$ matrix that has (α, β) entry 1 if $\alpha \prec \beta$ and 0 otherwise. Then Möbius inversion formula tells us that M exists and is an integer matrix (since $\prec$ is a partial order for a semilattice), and that if we pick $s_\beta \in R_\beta$ for each $\beta \in \Omega$ and we define $r_\alpha = \sum_{\beta \prec \alpha} m_{\beta, \alpha} \phi_{\beta, \alpha}(s_\beta)$ for all $\alpha \in \Omega$, then $s_\beta = \sum_{\alpha \prec \beta} \phi_{\alpha, \beta}(r_\alpha)$, i.e., $\bigoplus_{\beta \in \Omega} s_\beta = \psi(\bigoplus_{\alpha \in \Omega} r_\alpha)$.

A similar result, which proves the same isomorphism as in Lemma 4.3, appears in

[**13**, Theorem 3.7]. The referee has pointed out to us that the result in Lemma 4.3 also appears in [**6**].

We are now ready to describe the radical in many cases. Let ρ be a radical: then since ψ_e is onto, $\psi_e(\rho(R)) \subseteq \rho(R_e)$ for all $e \in E$, whence $\rho(R) \subseteq \bigcap_{e \in E} \psi_e^{-1}(\rho(R_e))$. Recall that a radical ρ is said to be *directed* if any ring that is the union of a system of ρ-radical subrings which form an upper semilattice under inclusion is a ρ-radical ring (see [**1**, §3]) and is said to be *hereditary* if any ideal of a ρ-radical ring is ρ-radical as a ring itself. Examples of directed hereditary radicals include the Jacobson radical and the upper nil radical; Example 1.10 shows that the prime radical is not directed. We will show that if ρ is a directed hereditary radical containing the prime radical, then the equality $\rho(R) = \bigcap_{e \in E} \psi_e^{-1}(\rho(R_e))$ holds (it does not always hold for the prime radical). In particular, if each R_e is ρ-semisimple, then $\rho(R) = K(\mathscr{R})$.

We begin with a description of radicals for semilattices. This description was proven individually for several radicals in [**13**, Chapter 3].

Proposition 4.5. *Let $\mathscr{R}$ be a system of rings over a semilattice Ω and let $R = \oplus \mathscr{R}$. Let ρ be a radical and assume that either Ω is finite or ρ is hereditary and directed. Then*

$$\rho(R) = \left\{ \bigoplus_{\alpha \in \Omega} r_\alpha \mid \sum_{\alpha \text{ with } \alpha\beta = \beta} \phi_{\alpha,\beta}(r_\alpha) \in \rho(R_\beta) \text{ for all } \beta \in \Omega \right\}.$$

Moreover, $\rho(R) = \bigoplus_{\alpha \in \Omega} \rho(R_\alpha)$ if and only if $\phi_{\alpha,\beta}(\rho(R_\alpha)) \subseteq \rho(R_\beta)$ whenever $\alpha\beta = \beta$.

Remark. The condition $\alpha\beta = \beta$ in the description of $\rho(R)$ can of course be re-stated as $\alpha \prec \beta$; in terms of the "natural order" on Ω it can be re-stated as $\alpha \geq \beta$.

Proof. The set claimed to equal $\rho(R)$ is $\bigcap_{\beta \in \Omega} \psi_\beta^{-1}(\rho(R_\beta))$: let us denote this ideal by L.

First assume Ω is finite. Since ρ is a radical, we may apply Lemma 4.3 and get $\rho(R) = \psi^{-1}(\rho(\prod_{\beta \in \Omega} R_\beta)) = \psi^{-1}(\prod_{\beta \in \Omega} \rho(R_\beta)) = L$.

In the general case, we noted above that $\rho(R) \subseteq L$ always holds; to get equality, we need to show that L is ρ-radical as a ring. We proceed as in the proof of [**1**, Theorem 3.2]. For any finite subsemigroup Ω' of Ω, let R' be the direct sum of the system obtained by restricting to Ω', and let L' be the ideal of R' corresponding to L. Clearly $L \cap R' \lhd L'$ and by the previous paragraph, $L' = \rho(R')$, so $L \cap R'$ is a radical ring (since ρ is hereditary). The subrings $L \cap R'$ form a semilattice of subrings of L whose union is L (since the subsemigroup generated by supp r is finite for any $r \in L$). Thus by the directedness of the radical ρ, L is a radical ring.

Suppose each $\phi_{\alpha,\beta}(\rho(R_\alpha)) \subseteq \rho(R_\beta)$, take $r \in L$, and let $\alpha \in$ supp r be minimal relative to $\prec$. Then $r_\alpha \in \rho(R_\alpha)$ (take $\beta = \alpha$ in the definition of L), and so by our hypothesis, $\phi_{\alpha,\beta}(r_\alpha) \in R_\beta$ for all $\beta \succ \alpha$. This implies $r_\alpha \in \rho(R)$. Since $r - r_\alpha \in \rho(R) = L$, we may assume by induction on $|$supp $r|$ that $r_\beta \in \rho(R_\beta)$ for all $\beta \in$ supp r.

Conversely, suppose $\rho(R) = \bigoplus_{\alpha \in \Omega} \rho(R_\alpha)$ and let $r_\alpha \in \rho(R_\alpha)$. Then $r_\alpha \in \rho(R)$, and by the first result, we must have $\phi_{\alpha,\beta}(r_\alpha) \in \rho(R_\beta)$ for all $\beta \succ \alpha$. $\qquad\square$

We are now ready to state our main result, describing radicals when condition (†) is satisfied. Recall that a radical ρ is said to *contain the prime radical* if $\rho(S)$ contains the prime radical of S for every ring S, or equivalently, if every nilpotent ring is ρ-radical, and a radical is said to be *supernilpotent* if it is hereditary and it contains the prime radical (cf. [12, §11]); examples include the upper nil radical and the Jacobson radical. Our best results are for directed supernilpotent radicals. In the case of the Jacobson radical of a special band-graded ring, our description is the same as that given in [8, Theorem 3.1].

Theorem 4.6. *Suppose that Ω satisfies condition (†) and that every prime ideal of Ω is completely prime, let $\mathscr{R}$ be a system of rings over Ω, and let $R = \oplus\mathscr{R}$. Let ρ be a radical containing the prime radical and suppose that either ρ is hereditary and directed or Ω is finitely generated. Then*

$$\rho(R) = \left\{ \bigoplus_{\alpha \in \Omega} r_\alpha \,\middle|\, \sum_{\alpha \, with \, e \in (\alpha)} \phi_{\alpha, e}(r_\alpha) \in \rho(R_e) \text{ for all } e \in E(\Omega) \right\}.$$

Moreover, $\rho(R) = \{\bigoplus_{\alpha \in \Omega} r_\alpha \,|\, \sum_{\alpha \eta e} \phi_{\alpha, e}(r_\alpha) \in \rho(R_e) \text{ for all } e \in E(\Omega)\}$ if and only if $\phi_{\alpha, e}(\rho(R_\alpha)) \subseteq \rho(R_e)$ whenever $e \in (\alpha)$, and $\rho(R) = \bigoplus_{\alpha \in \Omega} \rho(R_\alpha)$ if and only if this condition holds and in addition $\{\alpha \in \Omega \,|\, \rho(R_\alpha) \neq R_\alpha\}$ is contained in $E(\Omega)$ and contains at most one element from each $\mathfrak{J}$-equivalence class.

Proof.   Set $L = \bigcap_{e \in E(\Omega)} \psi_e^{-1}(\rho(R_e))$: we wish to show $L = \rho(R)$. Since ρ contains the prime radical, Lemma 4.2 implies $\rho(R) \supseteq K$, and the discussion just after Lemma 4.2 implies $L \supseteq K$. Thus we may pass to R/K and so by Lemma 4.1, we may assume Ω is a semilattice. The discussion surrounding Lemmas 4.1 and 4.2 shows that when we pass to R/K, we replace $e \in (\alpha)$ by $\alpha e = e$ and η by $=$ in the statement of the theorem, and we also replace finitely generated by finite. The theorem thus follows from Proposition 4.5. (To verify the very last claim in the theorem, suppose that $r_\alpha \in R_\alpha \backslash \rho(R_\alpha)$, and suppose $\alpha \eta e \in E(\Omega)$. Then $r_\alpha - \phi_{\alpha, e}(r_\alpha) \in \rho(R)$ by the above, but it is not in $\bigoplus_{\alpha \in \Omega} \rho(R_\alpha)$ unless $\alpha = e$.)   □

This theorem is not valid for the prime radical, even when Ω is a semilattice, as Example 1.10 shows, since in that example, $\oplus\mathscr{R}$ has a prime ideal corresponding to the family $(\varnothing, 0)$, while Theorem 4.6 would imply $\rho(\oplus\mathscr{R}) = \oplus\mathscr{R}$. We do have the following result.

Proposition 4.7.   *Suppose that Ω satisfies condition (†) and that every prime ideal of Ω is completely prime, let $\mathscr{R}$ be a system of rings over Ω, and let $R = \oplus\mathscr{R}$. Let ρ be the prime radical and suppose every $\phi_{\alpha, \beta}$ is onto or each R_α is commutative. Then*

$$\rho(R) = \left\{ \bigoplus_{\alpha \in \Omega} r_\alpha \,\middle|\, \sum_{\alpha \eta e} \phi_{\alpha, e}(r_\alpha) \in \rho(R_e) \text{ for all } e \in E(\Omega) \right\}.$$

Proof.   Set $L = \{\bigoplus_{\alpha \in \Omega} r_\alpha \mid \sum_{\alpha \eta e} \phi_{\alpha, e}(r_\alpha) \in \rho(R_e)$ for all $e \in E(\Omega)\}$. Lemma 4.2 implies that $K \subseteq \rho(R)$ and clearly $K \subseteq L$, and so just as in the proof of Theorem 4.6, we may pass to R/K and assume Ω is a semilattice and replace η by $=$ in the definition of L.

Note that the hypotheses imply that $\phi_{\alpha, \beta}(\rho(R_\alpha)) \subseteq \rho(R_\beta)$ whenever $\alpha \prec \beta$. When each R_α is commutative, so is R, and the prime and nil radicals agree for R and for each R_α, whence the proposition follows from Proposition 4.5. Thus we may assume each $\phi_{\alpha, \beta}$ is onto.

We need to show that L is contained in every prime ideal of R. If $r_\alpha \in L$, then r_α is in every prime ideal in R_α. Now consider a prime family $I(\Phi, \mathscr{P})$. If $\alpha \in \Omega \backslash \Phi$, then the hypothesis that each $\phi_{\alpha, \beta}$ is onto implies that the ideal P_α is prime, whence $r_\alpha \in P_\alpha \subseteq I(\Phi, \mathscr{P})$. If $\alpha \in \Phi$, then $r_\alpha \in I(\Phi, \mathscr{P})$ for all $r_\alpha \in R_\alpha$. Thus r_α lies in all prime ideals of R, and so $r_\alpha \in \rho(R)$.   □

We can also use these ideas to improve the results at the end of Section 2 on classical Krull dimension in a special case.

Proposition 4.8.   *Suppose that Ω satisfies condition (†) and that every prime ideal of Ω is completely prime, let $\mathscr{R}$ be a system of rings over Ω, and suppose that every $\phi_{\alpha, \beta}$ is onto. Then $R = \oplus \mathscr{R}$ has classical Krull dimension if and only if R_e does for all $e \in E(\Omega)$, and when this is the case, the classical Krull dimension of R is the supremum of the classical Krull dimensions of the rings R_e.*

Proof.   If $\alpha \in \Omega \backslash \Phi$ for a prime ideal Φ, then $e = e_\alpha \in \Omega \backslash \Phi$. Thus by Remark 1.2, $\varinjlim_{\alpha \in \Omega \backslash \Phi} R_\alpha$ is isomorphic to a factor of R_e, and so by Proposition 2.10, the classical Krull dimension of R is bounded by the supremum of the classical Krull dimensions of the rings R_e. But each R_e is a homomorphic image of R (since ψ_e is onto), and so the classical Krull dimension of each R_e is bounded by the classical Krull dimension of R.   □

Proposition 4.8 may fail when the maps $\phi_{\alpha, \beta}$ are not onto, even if the rings are commutative and Ω is a semilattice. To see this, let F be a field, let $\Omega = \{1, 2, \dots\}$ with the natural order, let R_n be the commutative polynomial ring $F[x_1, \dots, x_n]$, and let $\phi_{m, n}$ be the inclusion map (for $m \leq n$). Then each R_n has finite classical Krull dimension, but $R = \oplus \mathscr{R}$ does not have classical Krull dimension, since it has $F[x_1, x_2, \dots]$ as a homomorphic image.

We conclude by showing that the results on directed radicals remain true for arbitrary bands, even when condition (†) is not satisfied. Thus the results on the Jacobson and upper nil radical carry over; however, the results on prime ideals, the prime radical, and classical Krull dimension do not, as Proposition 3.5 and the comments after it show.

The method we use is to pass to finitely generated subsemigroups, just as in the proof of Proposition 4.5.

Theorem 4.9.   *Let $\mathscr{R}$ be a system of rings over a band Ω, and let ρ be a directed supernilpotent radical.*

(1) *The ideal $K(\mathscr{R})$ is nil and is contained in $\rho(R)$.*

(2)

$$\rho(R) = \left\{ \bigoplus_{\alpha \in \Omega} r_\alpha \,\Big|\, \sum_{\alpha \ with \ \beta\alpha\beta = \beta} \phi_{\alpha,\beta}(r_\alpha) \in \rho(R_\beta) \ for \ all \ \beta \in \Omega \right\}.$$

(3) $\rho(R) = \{\bigoplus_{\alpha \in \Omega} r_\alpha \,|\, \sum_{\alpha \Im \beta} \phi_{\alpha,\beta}(r_\alpha) \in \rho(R_\beta)$ *for all $\beta \in \Omega\}$ if and only if we have* $\phi_{\alpha,\beta}(\rho(R_\alpha)) \subseteq \rho(R_\beta)$ *whenever $\beta\alpha\beta = \beta$, and $\rho(R) = \bigoplus_{\alpha \in \Omega} \rho(R_\alpha)$ if and only if this condition holds and in addition, any α with $\rho(R_\alpha) \neq R_\alpha$ is not $\Im$-equivalent to any other element of Ω.*

Remark.   The condition $\beta\alpha\beta = \beta$ is the same as the condition $\alpha \prec \beta$, and the condition $\alpha\Im\beta$ is the same as the condition $\beta\alpha\beta = \beta$ and $\alpha\beta\alpha = \alpha$.

Proof.   Clearly (2) and (3) follow from (1) and Proposition 4.5 just as Theorem 4.6 follows from Proposition 4.5.

To prove (1), we proceed as in the third paragraph of the proof of Proposition 4.5. For any finite subsemigroup Ω' of Ω, let R' be the direct sum of the system obtained by restricting to Ω', and let K' be the ideal of R' corresponding to K. It's easy to see that $K \cap R' \lhd K'$ (since the relation η equals the relation $\Im$, it is the same in Ω and Ω') and we know by Proposition 3.4 and Lemma 4.2 that $K' \subseteq \rho(R')$. It follows that $K \cap R'$ is ρ-radical as a ring. Since finitely generated bands are finite—see [4, Theorem 1] or [5, Theorem IV.4.9]—every element of R is contained in some R' and so the subrings $K \cap R'$ form an upper semilattice of subrings of K whose union is K. Thus by the directedness of the radical ρ, K is a ρ-radical ring, i.e., $K \subseteq \rho(R)$. Taking ρ to be the upper nil radical, we conclude that K is a nil ideal. $\square$

We now summarize the results of this section for the special case of semigroup rings.

Theorem 4.10.   *Suppose that Ω satisfies condition (†) and that every prime ideal of Ω is completely prime, let S be a ring, and let ρ be a radical containing the prime radical. If Ω is finitely generated, or ρ is directed and hereditary, or ρ is the prime radical, then $\rho(S[\Omega]) = \rho(S)[\Omega] + K$, where $K = \{\sum_{\alpha \in \Omega} s_\alpha \alpha \,|\, \sum_{\alpha \eta e} s_\alpha = 0$ for all $e \in E(\Omega)\}$. Moreover, $\rho(S[\Omega]) = \rho(S)[\Omega]$ if and only if either $\rho(S) = S$ or Ω is a semilattice.*

When Ω is finite or ρ is directed and hereditary, these results remain valid if Ω is an arbitrary band.

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