T. Mandai and H. Tahara
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STRUCTURE OF SOLUTIONS TO FUCHSIAN SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

TAKESHI MANDAI AND HIDETOSHI TAHARA

Dedicated to Professor Kunihiko KAJITANI on his sixtieth birthday

Abstract. To a certain Volevič system of singular partial differential equations, called a Fuchsian system, all the solutions of the homogeneous equation in a complex domain are constructed and parametrized in a good way, without any assumption on the characteristic exponents.

§1. Introduction

Let $\boldsymbol{C}$ be the set of complex numbers, t be a variable in $\boldsymbol{C}$, and $x = (x_1, \dots, x_n)$ be variables in $\boldsymbol{C}^n$. We use the notation $D_t := \partial/\partial t$, $D_x := (D_{x_1}, \dots, D_{x_n})$, $D_{x_j} := \partial/\partial x_j$, and $\boldsymbol{N} := \{\,\text{nonnegative integers}\,\}$, $\boldsymbol{Z} := \{\,\text{integers}\,\}$.

We consider a certain Volevič system of singular partial differential operators, considered intensively and called a Fuchsian system in [4]. Namely,

$$P = tD_t I_m - A(t, x; D_x)\ , \tag{1.1}$$

where I_m denotes the $m \times m$ unit matrix, and $A(t, x; D_x) = (A_{i,j}(t, x; D_x))_{1 \le i,j \le m}$ is an $m \times m$ matrix of partial differential operators. Assume that the coefficients of $A_{i,j}$ are holomorphic in a neighborhood of $(t, x) = (0, 0)$. We also assume the following two conditions.

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(A-1) There exist $n_j \in \boldsymbol{N}$ $(1 \le j \le m)$ such that $\mathrm{ord}_{D_x} A_{i,j} \le n_i - n_j + 1$, where $\mathrm{ord}_{D_x} A_{i,j}$ denotes the order of a partial differential operator $A_{i,j}$ with respect to D_x.

(A-2) $A(0, x; D_x) =: A_0(x)$ is independent of D_x.

Such a system is called a *Fuchsian system* ([4]). Hereafter, we fix the integers n_j. Note that if $n_i - n_j + 1 < 0$ then $A_{i,j} = 0$, since $A_{i,j}$ are differential operators.

Let ρ_A be the *matrix order* of A, or, the order of A in the sense of Volevič ([3]), defined by

$$\rho_A := \max_{1 \le p \le m;\ 1 \le i_1 < i_2 < \ldots < i_p \le m} \frac{1}{p} \left(\max_{\pi \in \mathfrak{S}_p} \sum_{k=1}^{p} \mathrm{ord}_{D_x} A_{i_k, i_{\pi(k)}} \right) ,$$

where $\mathfrak{S}_p$ is the symmetric group of p numbers. It is well-known (see [3]) that the condition (A-1) is equivalent to each of the following two conditions.

(A-1)′ $\rho_A \le 1$.

(A-1)″ $D_t I_m - A$ is a kowalevskian system in the sense of Volevič.

M. Miyake showed that a kowalevskian system in the sense of Volevič can be reduced to a first order system. Although our system also can be reduced to a first order system by his method, the condition (A-2) may be violated. We do not use the conditions (A-1)′, (A-1)″ in this article.

For such systems, the second author ([4],[5]) showed fundamental theorems that are extensions of the Cauchy-Kowalevsky theorem and the Holmgren theorem, which shall be stated later (Theorems 1.1, 1.2).

We set $\mathcal{C}(x; \lambda) = \mathcal{C}[P](x; \lambda) := \det(\lambda I_m - A_0(x))$. This polynomial of λ is called the *indicial polynomial* of P, and a root λ of $\mathcal{C}(x; \lambda) = 0$ is called a *characteristic exponent* or a *characteristic index* of P at x.

In order to consider solutions in germ sense, set

$$\begin{aligned}
\mathcal{O}(\Omega) &:= \{ \text{holomorphic functions on } \Omega \} , \\
B_R &:= \{ x \in \boldsymbol{C}^n \ :\ |x| < R \} , \qquad \Delta_T := \{ t \in \boldsymbol{C} \ :\ |t| < T \} \quad (T > 0) , \\
\mathcal{O}_0 &:= \bigcup_{R>0} \mathcal{O}(B_R) , \qquad \mathcal{O}_{(0,0)} := \bigcup_{R>0, T>0} \mathcal{O}(\Delta_T \times B_R) , \\
S_{\infty,T} &:= \mathcal{R}(\Delta_T \setminus \{0\}) \quad (\text{the universal covering of } \Delta_T \setminus \{0\}) ,
\end{aligned}$$

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$$S_{\theta,T} := \{ t \in S_{\infty,T} \,:\, |\arg t| \leq \theta \} \ ,$$
$$\widetilde{\mathcal{O}} := \bigcup_{T>0,R>0} \mathcal{O}(S_{\infty,T} \times B_R) \ .$$

As in the case of single Fuchsian partial differential equations, we have the following fundamental theorems, which correspond to the Cauchy-Kowalevsky theorem and the Holmgren theorem.

THEOREM 1.1. ([4, Theorem 1.2.10]) *If $\mathcal{C}(0;j) \neq 0$ ($j \in \boldsymbol{N}$), then for every $\overrightarrow{f} \in (\mathcal{O}_{(0,0)})^m$, there exists a unique solution $\overrightarrow{u} \in (\mathcal{O}_{(0,0)})^m$ of*

$$(CP) \qquad P\overrightarrow{u} = \overrightarrow{f}(t,x) \ .$$

THEOREM 1.2. ([5, Theorem 2]) *Let Ω be an open neighborhood of $0 \in \boldsymbol{R}^n$. Let $L \in \boldsymbol{R}$ satisfy that if $\mathcal{C}(x;\lambda) = 0$ ($x \in \Omega$) then $\mathrm{Re}\lambda < L$. If $\overrightarrow{u}(t) = \overrightarrow{u}(t,x) \in C^1((0,T],\mathcal{D}'(\Omega))^m$ is a solution of $P\overrightarrow{u} = \overrightarrow{0}$ in the real domain $(0,T) \times \Omega$, and if $\overrightarrow{u}$ satisfies $t^{-L}\overrightarrow{u} \in C^0([0,T],\mathcal{D}'(\Omega))^m$, then there exists a neighborhood $U \subset \boldsymbol{R}^{n+1}$ of $(0,0)$ such that $\overrightarrow{u} = \overrightarrow{0}$ in $U \cap [(0,T) \times \Omega]$. Here, $\mathcal{D}'(\Omega)$ denotes the space of Schwartz distributions on Ω.*

If we impose the condition that the characteristic exponents of P do not differ by integers, then the structure of the kernel $\mathrm{Ker}_{(\widetilde{\mathcal{O}})^m} P$ of the map $P : (\widetilde{\mathcal{O}})^m \to (\widetilde{\mathcal{O}})^m$ has been studied in [4] (Theorem 1.3.6, etc.). The purpose of this article is to give a solution map, that is, a linear isomorphism

$$(1.2) \qquad (\mathcal{O}_0)^m \xrightarrow{\sim} \mathrm{Ker}_{(\widetilde{\mathcal{O}})^m} P := \{ \overrightarrow{u} \in (\widetilde{\mathcal{O}})^m \,:\, P\overrightarrow{u} = \overrightarrow{0} \} \ ,$$

rather explicitly, with no assumptions on the characteristic exponents (Theorem 2.3).

In the case of single Fuchsian partial differential equations, the first author ([2]) gave a good solution map, which can be considered as an extension of the classical Frobenius method to ordinary differential equations. The construction given in this article is based on the same idea.

Remark 1.3. By modifying Example 1.0.9 in [4], we see the following. Let P' be a single Fuchsian partial differential operator with weight 0 ([1], [4], [2], etc.); that is, $P' = (tD_t)^m + \sum_{j=1}^m P'_j(t,x;D_x)(tD_t)^{m-j}$, $\mathrm{ord}_{D_x} P'_j \leq$

j, and $P'_j(0,x;D_x) =: a_j(x)$ is a function of x. Then, by $u_j = (tD_t)^{j-1}u$ $(1 \le j \le m)$, the equation $Pu = f$ is reduced to

$$\left(tD_t I_m - \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -P'_m & -P'_{m-1} & -P'_{m-2} & \dots & -P'_1 \end{pmatrix} \right) \overrightarrow{u} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{pmatrix} .$$

Since this system satisfies (A-1) with $n_j = j$ and (A-2), it is a Fuchsian system. Further, this system has the same indicial polynomial $\mathcal{C}(x;\lambda)$ as P', where the indicial polynomial of P' is defined by $\mathcal{C}[P'](x;\lambda) := \lambda^m + \sum_{j=1}^m a_j(x)\lambda^{m-j} = [t^{-\lambda}P'(t^\lambda)]|_{t=0}$.

§2. Construction of the solution map

We assume that the coefficients of $A_{i,j}$ are holomorphic in $\Delta_{T_0} \times B_{R_{00}}$ $(T_0, R_{00} > 0)$. Let μ_l $(l = 1, \dots, d)$ be all the distinct roots of $\mathcal{C}(0;\lambda) = 0$, and let r_l be the multiplicity of μ_l. As is well-known, there exists an $m \times m$ matrix $Q(x)$ with $\mathcal{O}_0$ entries which satisfies the following conditions.

- $Q(x)^{-1}$ has also the entries in $\mathcal{O}_0$,
- $Q(x)^{-1}A_0(x)Q(x) = A_1(x) \otimes \dots \otimes A_d(x)$

$$:= \begin{pmatrix} A_1(x) & O & \dots & O \\ O & A_2(x) & O & \vdots \\ \vdots & O & \ddots & \vdots \\ O & \dots & O & A_d(x) \end{pmatrix},$$

- A_l is an $r_l \times r_l$ matrix with $\mathcal{O}_0$ entries $(l = 1, \dots, d)$,
- $\det(\lambda I_{r_l} - A_l(0)) = (\lambda - \mu_l)^{r_l}$ $(l = 1, \dots, d)$.

When we decompose an m vector $\overrightarrow{u} \in \boldsymbol{C}^m$ to d blocks corresponding to the above decomposition of $Q(x)^{-1}A_0(x)Q(x)$, we denote the l-th block of $\overrightarrow{u}$ by $\overrightarrow{u}^{\flat(l)} \in \boldsymbol{C}^{r_l}$. Conversely, for an r_l vector $\overrightarrow{v} \in \boldsymbol{C}^{r_l}$, we denote by $\overrightarrow{v}^{\sharp(l)} \in \boldsymbol{C}^m$ the m vector with the entries $\overrightarrow{v}$ in the l-th block and the entries 0 in other blocks.

Set

$$\Lambda_P := \{ \mu_l - j \in \boldsymbol{C} \, : \, 1 \le l \le d, j \in \boldsymbol{N} \} . \tag{2.1}$$

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Take $\epsilon \geq 0$ as $\mathrm{Re}\mu_l - \epsilon \notin \boldsymbol{Z}$ for all l. Take $L_l \in \boldsymbol{Z}$ as $L_l + \epsilon < \mathrm{Re}\mu_l < L_l + \epsilon + 1$. Just in the same way as or more easily than Lemmata 3.1, 3.2 in [2], we have the following two lemmata.

Lemma 2.1. *For each l with $1 \leq l \leq d$, there exists a domain D_l in $\boldsymbol{C}$ enclosed by a simple closed curve Γ_l such that the following holds.*

(a) $\mu_l \in D_l \quad (1 \leq l \leq d)$,

(b) $\overline{D_l} \cap \overline{D_{l'}} = \emptyset \quad (l \neq l')$, *where $\overline{D}$ denotes the closure of D.*

(c) *For every l, we have $\overline{D_l} \cap \Lambda_P = \{\, \mu_l \,\}$.*

(d) *For every l, we have $\overline{D_l} \subset \{\, \lambda \in \boldsymbol{C} \,:\, L_l + \epsilon < \mathrm{Re}\lambda < L_l + \epsilon + 1 \,\}$.*

Lemma 2.2. *There exists $R_0 \in (0, R_{00})$ such that*

(e) *For every $x \in B_{R_0}$, $\lambda \in \bigcup_{l=1}^d \Gamma_l$, and $j \in \boldsymbol{N}$, we have $\mathcal{C}(x; \lambda + j) \neq 0$.*

Set

$$\widetilde{\mathcal{O}_{\theta,R}} := \bigcap_{0<R'<R} \; \bigcup_{T'>0} \mathcal{O}(S_{\theta,T'} \times B_{R'}) \quad \text{for} \quad \theta \in (0, \infty] \quad \text{and} \quad R > 0 \ .$$

If Ω is not an open set, then $\mathcal{O}(\Omega)$ denotes the set of functions holomorphic in a neighborhood of Ω. The following is the main result.

Theorem 2.3. (1) *For every $R \in (0, R_0)$, every $T > 0$, and every $\overrightarrow{F} \in \mathcal{O}(\Delta_T \times B_R \times (\bigcup_{l=1}^d \Gamma_l))^m$, the equation*

$$P(t^\lambda \overrightarrow{V}) = t^\lambda \overrightarrow{F}(t, x; \lambda) \tag{2.2}$$

has a unique solution $\overrightarrow{V} = \overrightarrow{V}[\overrightarrow{F}](t, x; \lambda) \in \mathcal{O}(\{t = 0\} \times B_R \times (\bigcup_{l=1}^d \Gamma_l))$. Further, if $\overrightarrow{F} \in \mathcal{O}(\Delta_T \times B_R \times (\bigcup_{l=1}^d \overline{D_l}))^m$, then for $1 \leq l \leq d$

$$\overrightarrow{u}[l, \overrightarrow{F}](t, x) := \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_l} t^\lambda \overrightarrow{V}[\overrightarrow{F}](t, x; \lambda)\, d\lambda \ \left(\in (\widetilde{\mathcal{O}_{\infty,R}})^m\right) \tag{2.3}$$

is a solution of $P\overrightarrow{u} = \overrightarrow{0}$.

(2) *For $R \in (0, R_0)$ and $\overrightarrow{\varphi_l} \in \mathcal{O}(B_R)^{r_l}$, set $\overrightarrow{u_l}[\overrightarrow{\varphi_l}] := \overrightarrow{u}[l, Q\overrightarrow{\varphi_l}^{\sharp(l)}]$ by considering $\overrightarrow{F}(t, x; \lambda) := Q(x)\overrightarrow{\varphi_l}^{\sharp(l)}(x)$. Then, the map*

$$\mathcal{O}(B_R)^m \ni \begin{pmatrix} \overrightarrow{\varphi_1} \\ \vdots \\ \overrightarrow{\varphi_d} \end{pmatrix} \overset{\sim}{\longmapsto} \sum_{l=1}^d \overrightarrow{u_l}[\overrightarrow{\varphi_l}] \in \mathrm{Ker}_{(\widetilde{\mathcal{O}_{\infty,R}})^m} P \tag{2.4}$$

is a linear isomorphism. Especially, $(\mathcal{O}_0)^m \xrightarrow{\sim} \mathrm{Ker}_{(\widetilde{\mathcal{O}})^m} P$ by this map.

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We actually prove that the map (2.4) is surjective onto $\mathrm{Ker}_{(\widetilde{\mathcal{O}_{\theta,R}})^m} P$ for every $\theta \in (0,\infty]$. This implies that all solutions in $(\widetilde{\mathcal{O}_{\theta,R}})^m$ extend automatically to $(\widetilde{\mathcal{O}_{\infty,R}})^m$.

We end this section by giving an asymptotic expansion of the solution $\overrightarrow{u_l}[\overrightarrow{\varphi_l}]$. Let $\overrightarrow{F} \in \mathcal{O}(\Delta_T \times B_R \times (\bigcup_{l=1}^d \overline{D_l}))^m$.

Expand $\overrightarrow{F} = \sum_{j=0}^\infty t^j \overrightarrow{F_j}(x;\lambda)$, $\overrightarrow{V}[\overrightarrow{F}] = \sum_{j=0}^\infty t^j \overrightarrow{V_j}(x;\lambda)$, and $A(t,x;D_x) = A_0(x) + \sum_{l=1}^\infty t^l B_l(x;D_x)$. Then, from $P(t^\lambda \overrightarrow{V}) = t^\lambda \overrightarrow{F}$, we have

$$(2.5)\qquad (\lambda I_m - A_0(x))\overrightarrow{V_0}(x;\lambda) = \overrightarrow{F_0}(x;\lambda)\ ,$$

$$(2.6)\ ((\lambda+j)I_m - A_0(x))\overrightarrow{V_j}(x;\lambda) = \overrightarrow{F_j}(x;\lambda) + \sum_{l=1}^{j} B_l(x;D_x)\overrightarrow{V_{j-l}}(x;\lambda) \quad (j \geq 1)\ .$$

If $\lambda \in \bigcup_{l=1}^d \Gamma_l$, then $\det((\lambda+j)I_m - A_0(x)) = \mathcal{C}(x;\lambda+j) \neq 0$ ($x \in B_{R_0}$, $j \in \boldsymbol{N}$) by Lemma 2.2. Hence, we can determine $\overrightarrow{V_j} \in \mathcal{O}(B_R \times (\bigcup_{l=1}^d \Gamma_l))^m$ uniquely, and

$$\overrightarrow{V_j} \in \frac{1}{\mathcal{C}(x;\lambda+j)\prod_{\nu=0}^{j-1}\mathcal{C}(x;\lambda+\nu)^{m_{j,\nu}}} \times \mathcal{O}(B_R \times \left(\bigcup_{l=1}^d \overline{D_l}\right))^m\ ,$$

where $m_{j,\nu} \in \boldsymbol{N}$.

Let $\overrightarrow{u_l}[\overrightarrow{\varphi_l}] = \sum_{j=0}^\infty t^j \overrightarrow{u_{l,j}}[\overrightarrow{\varphi_l}](t,x)$ be the expansion of $\overrightarrow{u_l}[\overrightarrow{\varphi_l}]$ according to the expansion of $\overrightarrow{V}[Q\overrightarrow{\varphi_l}^{\sharp(l)}]$. Then, the leading term is

$$\overrightarrow{u_{l,0}}[\overrightarrow{\varphi_l}](t,x) = t^{A_0(x)}Q(x)\overrightarrow{\varphi_l}^{\sharp(l)}(x) = Q(x)\{t^{A_l(x)}\overrightarrow{\varphi_l}(x)\}^{\sharp(l)}\ .$$

§3. Proof of (1) of Theorem 2.3

In this section, we prove (1) of Theorem 2.3.

For every $R \in (0,R_0)$, and for every $\overrightarrow{F} \in \mathcal{O}(\Delta_T \times B_R \times (\bigcup_{l=1}^d \Gamma_l))^m$, we solve the equation (2.2)

$$(tD_t I_m - A(t,x;D_x))(t^\zeta \overrightarrow{V}) = t^\zeta \overrightarrow{F}(t,x;\zeta).$$

(We replace λ by ζ for later convenience.) This equation is equivalent to

$$(3.1)\qquad P^{\#}(\overrightarrow{V}) := (tD_t I_m + \zeta I_m - A(t,x;D_x))\overrightarrow{V} = \overrightarrow{F}(t,x;\zeta).$$

Fix an arbitrary $(x_0, \zeta_0) \in B_R \times (\bigcup_{l=1}^d \Gamma_l)$. As an equation with respect to the variables (t, x, ζ), the system (3.1) is a Fuchsian system and its indicial polynomial is

$$\mathcal{C}[P^{\#}](x, \zeta; \lambda) = \det(\lambda I_m + \zeta I_m - A_0(x)) = \mathcal{C}[P](x; \lambda + \zeta).$$

By Lemma 2.2, we have $\mathcal{C}[P^{\#}](x_0, \zeta_0; j) = \mathcal{C}[P](x_0; \zeta_0 + j) \neq 0$ for all $j \in \boldsymbol{N}$. By considering (x_0, ζ_0) as the origin and by using Theorem 1.1, we have a unique holomorphic solution $\overrightarrow{V}$ of (3.1) in a neighborhood of $(0, x_0, \zeta_0)$. Since $(x_0, \zeta_0) \in B_R \times (\bigcup_{l=1}^d \Gamma_l)$ is arbitrary, and since the solution $\overrightarrow{V}$ is unique, we have a holomorphic solution in a neighborhood of $\{t = 0\} \times B_R \times (\bigcup_{l=1}^d \Gamma_l)$.

The latter part of (1) is trivial.

§4. Function spaces

In this section, we introduce some function spaces, used in [2] in order to “measure” the order of functions as $t \to 0$.

Definition 4.1. ([2, Definition 5.1]) For $\theta \in (0, \infty]$, $T > 0$, and $R > 0$, set

$$W(\theta, T, R) := \{ \phi \in \mathcal{O}(S_{\theta,T} \times B_R) : \text{for every } \theta' \in (0, \theta) \text{ and every } R' \in (0, R), \text{ we have } \sup_{|x| \leq R'} |\phi(t, x)| \to 0 \text{ as } t \to 0 \text{ in } S_{\theta',T} \} , \tag{4.1}$$

$$\begin{aligned} \widetilde{W}(\theta, R) &:= \bigcap_{0<R'<R} \bigcup_{T'>0} W(\theta, T', R') \\ &= \{ \phi \in \widetilde{\mathcal{O}_{\theta,R}} : \text{for every } R' \in (0, R), \text{ there exists } T' > 0 \text{ such that } \phi \in W(\theta, T', R') \} . \end{aligned} \tag{4.2}$$

Further, for $a \in \boldsymbol{R}$, set

$$W^{(a)}(\theta, T, R) : = t^a \times W(\theta, T, R) , \tag{4.3}$$

$$\widetilde{W}^{(a)}(\theta, R) : = t^a \times \widetilde{W}(\theta, R) . \tag{4.4}$$

We have the following fundamental properties of these function spaces.

Lemma 4.2. ([2, Lemma 5.2])

(1) *If $a' < a$, then $W^{(a)}(\theta, T, R) \subset W^{(a')}(\theta, T, R)$ and $\widetilde{W}^{(a)}(\theta, R) \subset \widetilde{W}^{(a')}(\theta, R)$.*

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(2) $t \times W^{(a)}(\theta, T, R) \subset W^{(a+1)}(\theta, T, R)$, $t \times \widetilde{W}^{(a)}(\theta, R) \subset \widetilde{W}^{(a+1)}(\theta, R)$.

(3) $D_t(W^{(a)}(\theta, T, R)) \subset W^{(a-1)}(\theta, T, R)$, $D_t(\widetilde{W}^{(a)}(\theta, R)) \subset \widetilde{W}^{(a-1)}(\theta, R)$.

(4) *If* $B(t, x; D_x)$ *is a differential operator of* x *with the coefficients in* $\mathcal{O}(\Delta_T \times B_R)$, *then*

$$B(t, x; D_x)(W^{(a)}(\theta, T, R)) \subset W^{(a)}(\theta, T, R) \ , \tag{4.5}$$

$$B(t, x; D_x)(\widetilde{W}^{(a)}(\theta, R)) \subset \widetilde{W}^{(a)}(\theta, R) \ . \tag{4.6}$$

PROPOSITION 4.3. ([2, Proposition 5.3]) *For a simple closed curve* Γ *in* $\boldsymbol{C}$ *and a function* $\overrightarrow{V}(t, x; \lambda) \in \mathcal{O}(\Delta_T \times B_R \times \Gamma)^m$, *set*

$$\overrightarrow{u}(t, x) := \int_\Gamma t^\lambda \overrightarrow{V}(t, x; \lambda)\, d\lambda \ \in \mathcal{O}(S_{\infty, T} \times B_R)^m \ .$$

If $a < \min\{\, \mathrm{Re}\lambda \ : \ \lambda \in \Gamma \,\}$, *then we have*

$$\overrightarrow{u} \in W^{(a)}(\infty, T, R)^m \ .$$

If $\overrightarrow{V} \in \mathcal{O}(\{t = 0\} \times B_R \times \Gamma)^m$ *instead, then we have* $\overrightarrow{u} \in \widetilde{W}^{(a)}(\infty, R)^m$.

LEMMA 4.4. *If* $\overrightarrow{\varphi_l} \in \mathcal{O}(B_R)^{r_l}$ $(0 < R < R_0)$, *then we can write*

$$\overrightarrow{u_l}[\overrightarrow{\varphi_l}](t, x) = Q(x)\{t^{A_l(x)}\overrightarrow{\varphi_l}(x)\}^{\sharp(l)} + t \cdot \overrightarrow{r_l}[\overrightarrow{\varphi_l}](t, x) \ ,$$

where $\overrightarrow{r_l}[\overrightarrow{\varphi_l}] \in \widetilde{W}^{(L_l+\epsilon)}(\infty, R)^m$. *Note that* $t^{A_l(x)}\overrightarrow{\varphi_l}(x) \in W^{(L_l+\epsilon)}(\infty, \infty, R)^{r_l}$. *Especially,* $\overrightarrow{u_l}[\overrightarrow{\varphi_l}] \in \widetilde{W}^{(L_l+\epsilon)}(\infty, R)^m$.

Proof. In

$$\overrightarrow{u_l}[\overrightarrow{\varphi_l}](t, x) = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_l} t^\lambda \overrightarrow{V}[Q\overrightarrow{\varphi_l}^{\sharp(l)}](t, x; \lambda)\, d\lambda \ ,$$

we have

$$\begin{aligned} \overrightarrow{V}[Q\overrightarrow{\varphi_l}^{\sharp(l)}](t, x; \lambda) &= (\lambda I_m - A_0(x))^{-1} Q(x) \overrightarrow{\varphi_l}^{\sharp(l)}(x) + t \cdot R_l(t, x; \lambda) \\ &= Q(x)\Big\{(\lambda I_{r_l} - A_l(x))^{-1}\overrightarrow{\varphi_l}(x)\Big\}^{\sharp(l)} + t \cdot R_l(t, x; \lambda) \ , \end{aligned}$$

where $R_l \in \mathcal{O}(\{t = 0\} \times B_R \times (\bigcup_{l=1}^d \Gamma_l))$. This gives the result. $\square$

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§5. Euler systems

In order to prove (2) of Theorem 2.3, we need to study a special system $(tD_t - A)\overrightarrow{u} = \overrightarrow{f}(t)$ of ordinary differential equations with holomorphic parameter x.

The first lemma is easy and so the proof may be omitted.

LEMMA 5.1. *If $\theta \in (0,\infty]$, $R \in (0, R_0)$, $\overrightarrow{u} \in (\widetilde{\mathcal{O}_{\theta,R}})^m$ and $(tD_tI_m - A_0(x))\overrightarrow{u} = \overrightarrow{0}$, then there exist unique $\overrightarrow{\varphi_l} \in \mathcal{O}(B_R)^{r_l}$ $(1 \le l \le d)$ such that*

$$\overrightarrow{u} = \sum_{l=1}^{d} Q(x)\{t^{A_l(x)}\overrightarrow{\varphi_l}(x)\}^{\sharp(l)} = Q(x)\begin{pmatrix} t^{A_1(x)}\overrightarrow{\varphi_1}(x) \\ \vdots \\ t^{A_d(x)}\overrightarrow{\varphi_d}(x) \end{pmatrix}$$

and hence $\overrightarrow{u} \in W^{(L_{\min}+\epsilon)}(\infty,\infty,R)^m$, where $L_{\min} := \min_{1\le l\le d} L_l$.

Further, if $L \in \boldsymbol{Z}$ and if $\overrightarrow{u} \in \widetilde{W}^{(L+\epsilon)}(\theta,R)^m$, then $\overrightarrow{\varphi_l} = \overrightarrow{0}$ for l such that $L_l < L$.

PROPOSITION 5.2. *For every $L \in \boldsymbol{Z}$, $\theta \in (0,\infty]$, $T > 0$, $R \in (0,R_0)$, and for every $\overrightarrow{g} \in W^{(L+\epsilon)}(\theta,T,R)^m$, there exists $\overrightarrow{v} \in W^{(L+\epsilon)}(\theta,T,R)^m$ such that $(tD_tI_m - A_0(x))\overrightarrow{v} = \overrightarrow{g}(t,x)$.*

Proof. Set $\overrightarrow{v} = Q(x)^t(\overrightarrow{v_1},\dots,\overrightarrow{v_d})$ and $\overrightarrow{g} = Q(x)^t(\overrightarrow{g_1},\dots,\overrightarrow{g_d})$.

(i) If $\mathrm{Re}\mu_l > L+\epsilon$, then take $T' \in (0,T)$ and take

$$\overrightarrow{v_l} = t^{A_l(x)}\int_{T'}^{t} \tau^{-A_l(x)}\overrightarrow{g_l}(\tau,x)\,\frac{d\tau}{\tau}\ .$$

By an estimate similar to (6.3) in [2], we can see that $\overrightarrow{v}_l \in W^{(L+\epsilon)}(\theta,T,R)^{r_l}$.

(ii) If $\mathrm{Re}\mu_l < L+\epsilon$, then take

$$\overrightarrow{v_l} = t^{A_l(x)}\int_{0}^{t} \tau^{-A_l(x)}\overrightarrow{g_l}(\tau,x)\,\frac{d\tau}{\tau} = \int_0^1 \sigma^{-A_l(x)}\overrightarrow{g_l}(\sigma t,x)\,\frac{d\sigma}{\sigma}\ .$$

□

§6. Temperedness of all solutions

In this section, we show that all the solutions of $P\overrightarrow{u} = \overrightarrow{0}$ in $(\widetilde{\mathcal{O}})^m$ has at most a polynomial growth as $t \to 0$.

PROPOSITION 6.1. *There exists $a \in \boldsymbol{R}$ such that if $\overrightarrow{u} \in \mathcal{O}(S_{\theta,T} \times B_R)^m$ $(\theta \in (0,\infty],\ T \in (0,T_0),\ R \in (0,R_0))$ satisfies $P\overrightarrow{u} = \overrightarrow{0}$, then $\overrightarrow{u} \in W^{(a)}(\theta,T,R)^m$.*

First, we prepare some notations.

DEFINITION 6.2. (1) For a vector $\overrightarrow{u} = {}^t(u_1, \dots, u_m) \in \boldsymbol{C}^m$, set $||\overrightarrow{u}|| := \sum_{j=1}^m |u_j|$. For a matrix A, set $||A|| := \sup_{||\overrightarrow{u}|| \le 1} ||A\overrightarrow{u}||$.

(2) Set $\mathcal{O}_\rho := \mathcal{O}(B_\rho) \cap C^0(\overline{B_\rho})$ for $\rho > 0$. For $\varphi \in \mathcal{O}_\rho$, we define $||\varphi||_\rho := \max_{|x| \le \rho} |\varphi(x)|$. The norm of a vector $\overrightarrow{\varphi} \in (\mathcal{O}_\rho)^m$ and a matrix A with entries in $\mathcal{O}_\rho$ are defined similarly.

(3) Set $M := ||A_0||_{R_0}$.

(4) Set $p_{i,j} := \max\{ n_j - n_i + 1, 1 \}$ and $p := \max_{i,j}\{ p_{i,j} \} = \max_{i,j}\{ n_i - n_j + 1 \}$.

(5) The (i, j) component of a matrix B is denoted by $B_{i,j}$.

By a change of coordinates from t to t^p, we may assume that P has the following form

$$P = tD_t I_m - A_0(x) - t^p B(t, x; D_x) \ , \tag{6.1}$$

where B has also holomorphic coefficients.

Let $\overrightarrow{u} \in \mathcal{O}(S_{\theta,T} \times B_R)^m$ satisfy $P\overrightarrow{u} = \overrightarrow{0}$. We need to show that for every $\theta' \in (0, \theta)$ and $R' \in (0, R)$, the asymptotics $|t|^{-a}||\overrightarrow{u}(t)||_{R'} \to 0$ holds as $t \to 0$ in $S_{\theta',T}$. In fact, we fix $R'' \in (R', R)$ and show that

$$||\overrightarrow{u}(t)||_{R'} \le \frac{2}{|t|^{M+p-1}} ||\overrightarrow{u}(Tt/|t|)||_{R''} \quad \text{for all } t \in S_{\theta',T} \ ,$$

for sufficiently small $T > 0$.

From now on, we write R instead of R''. We may assume that $R \le 1$. By rotating t, we may also assume $t \in (0, T)$, though we should be careful about how small T should be taken.

LEMMA 6.3. *Set* $e_{i,j}(s, x) := (e^{-sA_0(x)})_{i,j}$*: the* (i, j) *component of* $e^{-sA_0(x)}$*. Then, there exists* $C_0 > 0$ *such that*

$$||e_{i,j}(s)||_R \le C_0 s^{p_{i,j}-1} e^{Ms}$$

for $s \in (0, \infty)$.

Proof. First, we show that if $n_i - n_j + k < 0$ then the (i, j) component $(A_0(x)^k)_{i,j}$ of $A_0(x)^k$ vanishes, by induction on k.

When $k = 1$, this is what we stated just after the condition (A-1).

We assume the claim for k. If $n_i - n_j + k + 1 < 0$, then for each l, we have $n_i - n_l + k < 0$ or $n_l - n_j + 1 < 0$. Hence $(A_0(x)^{k+1})_{i,j} = \sum_l (A_0(x)^k)_{i,l} (A_0(x))_{l,j} = 0$.

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Next, we show that if $0 \le k < p_{i,j} - 1$ then $(A_0(x)^k)_{i,j} = 0$. If $p_{i,j} = 1$, then this is trivial. If $p_{i,j} \ge 2$ then $p_{i,j} = n_j - n_i + 1$. Hence $0 \le k < p_{i,j} - 1$ implies $n_i - n_j + k < 0$ and we have $(A_0(x)^k)_{i,j} = 0$.

Since $e^{-sA_0(x)} = \sum_k (1/k!)(-s)^k A_0(x)^k$, we have $e_{i,j}(s,x) = O(s^{p_{i,j}-1})$ $(s \to 0)$. Since $||e^{-sA_0(x)}||_R \le e^{Ms}$, we have the lemma. □

The following is easy and hence the proof may be omitted.

LEMMA 6.4. *For every $\overrightarrow{f} \in C^0((0,T];\mathcal{O}_R)^m$, there exists a unique solution $\overrightarrow{u} \in C^1((0,T];\mathcal{O}_R)^m$ of $(tD_t - A_0(x))\overrightarrow{u} = \overrightarrow{f}$ and $\overrightarrow{u}|_{t=T} = \overrightarrow{0}$. This solution is expressed as*

$$\overrightarrow{u}(t,x) = -\int_t^T e^{-(\log\tau - \log t)A_0(x)} \overrightarrow{f}(\tau,x)\,\frac{d\tau}{\tau} \quad .$$

Using this lemma, we define an operator $\mathcal{R} = (\mathcal{R}_{i,j})$ from $C^0((0,T];\mathcal{O}_R)^m$ to $C^1((0,T];\mathcal{O}_R)^m$ by $\mathcal{R}[\overrightarrow{f}\,] := \overrightarrow{u}$. Note that

$$\mathcal{R}_{i,j}[\phi](t,x) = -\int_t^T e_{i,j}(\log\tau - \log t, x)\phi(\tau,x)\,\frac{d\tau}{\tau} \quad .$$

By Lemma 6.3 and by the estimate $\log\tau - \log t \le (\tau - t)/t \;\; (0 < t \le \tau)$, we have

$$(6.2) \qquad \begin{aligned} &||\mathcal{R}_{i,j}[\phi](t)||_\rho \\ &\le C_0 \int_t^T \frac{(\tau - t)^{p_{i,j}-1}}{t^{p_{i,j}-1}} \left(\frac{\tau}{t}\right)^M ||\phi(\tau)||_\rho \,\frac{d\tau}{\tau} \quad (0 < t \le T) \end{aligned}$$

for every $\phi \in C^0((0,T];\mathcal{O}_\rho)$, where C_0 is the constant given in Lemma 6.3.

Now, Let $0 < T < \min\{T_0, 1\}$. Set $C := \max_{i,j}\{\,(p_{i,j}-1)!\,C_0\,\}$.

The following is the key lemma in proving Proposition 6.1.

LEMMA 6.5. *Let $B = B(t,x;D_x) = \sum_{|\alpha|=l} b_\alpha(t,x)D_x^\alpha$ and $b_\alpha \in C^0([0,T];\mathcal{O}_R)$. Set $K := \max_{t\in[0,T]} \sum_{|\alpha|=l} ||b_\alpha(t)||_R$. Let a, b and A be nonnegative real numbers.*

(1) If $\phi \in C^0((0,T];\mathcal{O}_R)$ satisfies $||\phi(t)||_\rho \le A/(R-\rho)^a$ for every $t \in (0,T]$ and every $\rho \in (0,R)$, then we have

$$||B\phi(t)||_\rho \le Ke^l(a+1)\ldots(a+l)\frac{A}{(R-\rho)^{a+l}}$$

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for every $t \in (0,T]$ *and every* $\rho \in (0,R)$.
(2) *If* $\phi \in C^0((0,T];\mathcal{O}_R)$ *satisfies* $||\phi(t)||_\rho \le (A/t^{M+p-1})\{(T-t)^b/(R-\rho)^a\}$ *for every* $t \in (0,T]$ *and every* $\rho \in (0,R)$, *then we have*

$$||\mathcal{R}_{i,j}[t^p B\phi](t)||_\rho \le \frac{AKe^l C}{t^{M+p-1}} \, \frac{(a+1)\dots(a+l)}{(b+1)\dots(b+p_{i,j})} \, \frac{(T-t)^{b+p_{i,j}}}{(R-\rho)^{a+l}}$$

for every $t \in (0,T]$ *and every* $\rho \in (0,R)$.

The appearance of $(a+1)\dots(a+l)$ in (1) is the difficulty to get good estimate of $\overrightarrow{u}$. The presence of $(b+1)\dots(b+p_{i,j})$ in the denominator in (2) resolves this difficulty as is seen later.

Proof. (1) By Cauchy's theorem, we have

$$||D_x\phi||_\rho \le \frac{1}{\rho'-\rho}||\phi||_{\rho'} \quad (0<\rho<\rho'\le R) \ ,$$

and hence if $|\alpha| = l$ then

$$||D_x^\alpha \phi||_\rho \le \frac{1}{(\rho'-\rho_{l-1})(\rho_{l-1}-\rho_{l-2})\dots(\rho_2-\rho_1)(\rho_1-\rho)}||\phi||_{\rho'}$$

for every ρ's with $0<\rho=\rho_0<\rho_1<\dots<\rho_{l-1}<\rho_l=\rho'$. Thus, we have

$$||B\phi(t)||_\rho \le AK\frac{1}{(\rho'-\rho_{l-1})\dots(\rho_1-\rho)} \, \frac{1}{(R-\rho')^a} \ . \tag{6.3}$$

Set $\rho_j - \rho_{j-1} = t_j(R-\rho_{j-1})$, $0<t_j<1$ $(j=1,\dots,l)$, then $R-\rho_j = (1-t_j)(R-\rho_{j-1})$ and $\rho_j-\rho_{j-1} = t_j(1-t_{j-1})\dots(1-t_1)(R-\rho)$. Hence,

$$\frac{1}{(\rho'-\rho_{l-1})\dots(\rho_1-\rho)} \, \frac{1}{(R-\rho')^a}$$

$$= \frac{1}{t_1t_2\dots t_l(1-t_1)^{a+l-1}(1-t_2)^{a+l-2}\dots(1-t_l)^a} \, \frac{1}{(R-\rho)^{a+l}} \ .$$

Since

$$\min_{0<t<1}\frac{1}{t(1-t)^p} = (p+1)\Big(1+\frac{1}{p}\Big)^p \le (p+1)e \ ,$$

we can take $\rho_1, \dots, \rho_{l-1}$, and $\rho_l = \rho'$ such that

$$\frac{1}{(\rho'-\rho_{l-1})\dots(\rho_1-\rho)} \, \frac{1}{(R-\rho')^a} \le e^l(a+l)\dots(a+1) \, \frac{1}{(R-\rho)^{a+l}} \ .$$

Thus,

$$||B\phi(t)||_\rho \leq AKe^l(a+l)\dots(a+1)\ \frac{1}{(R-\rho)^{a+l}}\ . \tag{6.4}$$

(2) By the result of (1) and (6.2), we have

$$\begin{aligned} ||\mathcal{R}_{i,j}[t^pB\phi](t)||_\rho \leq \frac{C_0}{t^{M+p_{i,j}-1}} \\ \times \int_t^T (\tau-t)^{p_{i,j}-1}\tau^M\tau^p \frac{(T-\tau)^b}{\tau^{M+p-1}}\frac{d\tau}{\tau}\ \frac{AKe^l(a+1)\dots(a+l)}{(R-\rho)^{a+l}}\ . \end{aligned}$$

Since $t^{M+p_{i,j}-1} \geq t^{M+p-1}$ and since it is easy to show

$$\int_t^T (\tau-t)^{p_{i,j}-1}(T-\tau)^b\, d\tau = \frac{(p_{i,j}-1)!(T-t)^{b+p_{i,j}}}{(b+1)\dots(b+p_{i,j})}\ ,$$

for example by induction on $p_{i,j} \geq 1$, we have the result. □

Now, set $\overrightarrow{\varphi}(x) := \overrightarrow{u}(T,x) \in (\mathcal{O}_R)^m$ and solve

$$\begin{cases} tD_t\overrightarrow{u}^{(0)} - A_0(x)\overrightarrow{u}^{(0)} = \overrightarrow{0}\ , \\ \overrightarrow{u}^{(0)}|_{t=T} = \overrightarrow{\varphi}(x)\ , \end{cases}$$

$$\begin{cases} tD_t\overrightarrow{u}^{(k)} - A_0(x)\overrightarrow{u}^{(k)} = t^pB\overrightarrow{u}^{(k-1)}\ , \\ \overrightarrow{u}^{(k)}|_{t=T} = \overrightarrow{0}\ , \end{cases} \qquad (k \geq 1)\ .$$

That is,

$$\begin{aligned} \overrightarrow{u}^{(0)}(t,x) &= e^{-(\log T - \log t)A_0(x)}\overrightarrow{\varphi}(x)\ , \\ \overrightarrow{u}^{(k)}(t,x) &= R[t^pB\overrightarrow{u}^{(k-1)}](t,x)\ . \end{aligned}$$

We have $\overrightarrow{u} = \sum_{k=0}^\infty \overrightarrow{u}^{(k)}$ if this series converges.

We can easily estimate $\overrightarrow{u}^{(0)}$ as follows.

$$||\overrightarrow{u}^{(0)}(t)||_R \leq \left(\frac{T}{t}\right)^M ||\overrightarrow{\varphi}||_R \leq \frac{T^M}{t^{M+p-1}}||\overrightarrow{\varphi}||_R\ . \tag{6.5}$$

In (6.1), let $B = (B_{i,j})$, $B_{i,j}(t,x;D_x) = \sum_{|\alpha| \leq n_i - n_j + 1} b_{i,j;\alpha}(t,x)D_x^\alpha$, and set $B_{i,j}^{(l)}(t,x;D_x) := \sum_{|\alpha|=l} b_{i,j;\alpha}D_x^\alpha$. Note that if $l > n_i - n_j + 1$ then $B_{i,j}^{(l)} \equiv 0$.

Set $\overrightarrow{u}^{(k)} = {}^t(u_1^{(k)}, \dots, u_m^{(k)})$. For $I := (i[k], \dots, i[1])$, $J := (j[k-1], \dots, j[0])$, and $L := (l[k], \dots, l[1])$, set

$$u_{j,I,J}^{(k)\ L} := (\mathcal{R}_{j,i[k]} t^p B_{i[k],j[k-1]}^{(l[k])}) \dots (\mathcal{R}_{j[2],i[2]} t^p B_{i[2],j[1]}^{(l[2])})(\mathcal{R}_{j[1],i[1]} t^p B_{i[1],j[0]}^{(l[1])}) u_{j[0]}^{(0)} \ .$$

Note that the subscript of $\mathcal{R}$ is in the order (j, i) not (i, j). Since $\overrightarrow{u}^{(k)} = (Rt^pB)\dots(Rt^pB)\overrightarrow{u}^{(0)}$, where k iteration of Rt^pB appears, we have

$$u_j^{(k)} = \sum_{I,J,L} u_{j,I,J}^{(k)\ L} \qquad (1 \le j \le m) \ ,$$

where the summation is taken over only those I, J, L that satisfy

$$1 \le i[s], j[s-1] \le m \quad \text{and} \quad 0 \le l[s] \le n_{i[s]} - n_{j[s-1]} + 1 \qquad (1 \le s \le k) \ .$$

Take K as $\sum_{|\alpha|=l} ||b_{i,j,\alpha}(t)||_R \le K$ for $0 \le l \le p$, $1 \le i, j \le m$, and $0 < t \le T$.

Lemma 6.6.

$$||u_{j[k],I,J}^{(k)\ L}(t)||_\rho \le \frac{K^k C^k e^{l[*k]}}{t^{M+p-1}} T^M ||\overrightarrow{\varphi}||_R \ \frac{l[*k]!}{p_{*k}!} \ \frac{(T-t)^{p_{*k}}}{(R-\rho)^{l[*k]}}$$

for $0 < \rho < R$ *and* $0 < t \le T$, *where* $l[*k] := l[1] + \dots + l[k]$, $p_{*k} := p_{j[1],i[1]} + \dots + p_{j[k],i[k]}$.

Proof. We show by induction on k using Lemma 6.5. When $k = 0$, the estimate is trivial.

We assume that the estimate is valid for k. By Lemma 6.5, we have

$$||u_{j[k+1],I,J}^{(k+1)\ L}(t)||_\rho \le \frac{AKe^{l[k+1]}C}{t^{M+p-1}} \times \frac{(l[*k]+1)\dots(l[*k]+l[k+1])}{(p_{*k}+1)\dots(p_{*k}+p_{j[k+1],i[k+1]})} \ \frac{(T-t)^{p_{*k}+p_{j[k+1],i[k+1]}}}{(R-\rho)^{l[*k]+l[k+1]}} \ ,$$

where $A := K^k C^k e^{l[*k]} T^M ||\overrightarrow{\varphi}||_R \ l[*k]!/p_{*k}!$. This shows that the estimate is valid for $k+1$. ∎

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Corollary 6.7. *There exists H_p such that*

$$||u^{(k)\ L}_{j[k],I,J}(t)||_\rho \le \frac{1}{t^{M+p-1}}\Big\{KCe^pH_p\frac{T}{(R-\rho)^p}\Big\}^k||\overrightarrow{\varphi}||_R$$

for $0<t\le T$ *and* $0<\rho<R$.

Proof. Since $l[*k]\le pk$, $p_{*k}\ge k$, $T^M<1$, $(T-t)^{p_{*k}}\le T^k$, and $(R-\rho)^{l[*k]}\ge(R-\rho)^{pk}$, we have only to show the existence of H_p such that $l[*k]!/p_{*k}!\le H_p^k$.

Since $l[s]\le n_{i[s]}-n_{j[s-1]}+1$, we have $l[*k]\le n_{j[k]}+(n_{i[k]}-n_{j[k]}+1)+(n_{i[k-1]}-n_{j[k-1]}+1)+\ldots+(n_{i[1]}-n_{j[1]}+1)-n_{j[0]}\le p_{*k}+p-1$. Thus,

$$\frac{l[*k]!}{p_{*k}!}\le\frac{(p_{*k}+p-1)!}{p_{*k}!}\le(p_{*k}+p-1)^{p-1}\le(pk+p-1)^{p-1}\ .$$

Since $[(p-1)\log\{p(k+1)-1\}]/k\to 0$ as $k\to\infty$, there exists H_p such that $\{p(k+1)-1\}^{p-1}\le H_p^k\ \ (k=0,1,2,\ldots)$. □

Since the numbers of I and J do not exceed m^k, and that of L does not exceed $(p+1)^k$, if we take $\rho=R'$ and take $T>0$ satisfying

$$KCe^pH_p\frac{T}{(R-R')^p}m^2(p+1)\le\frac{1}{2}\ ,$$

then we have

$$||u^{(k)}_j(t)||_{R'}\le\frac{1}{t^{M+p-1}}(\frac{1}{2})^k||\overrightarrow{\varphi}||_R\ ,$$

which implies

$$||\overrightarrow{u}(t)||_{R'}\le\frac{2}{t^{M+p-1}}||\overrightarrow{\varphi}||_R\ .$$

§7. Proof of the injectivity of (2.4)

Suppose that $R\in(0,R_0)$, $\overrightarrow{\varphi_l}\in\mathcal{O}(B_R)^{r_l}$ $(1\le l\le d)$ and that $\overrightarrow{u}=\sum_{l=1}^d\overrightarrow{u_l}[\overrightarrow{\varphi_l}]=\overrightarrow{0}$. Further, suppose that there exists l such that $\overrightarrow{\varphi_l}\ne\overrightarrow{0}$. We shall show that this leads to a contradiction.

We can take l_0 as $L_{l_0}=\min\{\,L_l\,:\,\overrightarrow{\varphi_l}\ne\overrightarrow{0}\,\}$.

Consider the l_0-th block $(Q^{-1}\overrightarrow{u})^{\flat(l_0)}=\sum_{l=1}^d\{Q^{-1}\overrightarrow{u_l}[\overrightarrow{\varphi_l}]\}^{\flat(l_0)}$ of $Q^{-1}\overrightarrow{u}$. If $l\ne l_0$, then $L_l\ge L_{l_0}$ and hence

$$\{Q^{-1}\overrightarrow{u_l}[\overrightarrow{\varphi_l}]\}^{\flat(l_0)}\in\widetilde{W}^{(L_l+1+\epsilon)}(\infty,R)^{r_{l_0}}\subset\widetilde{W}^{(L_{l_0}+1+\epsilon)}(\infty,R)^{r_{l_0}}$$

by Lemma 4.4. Thus, $\overrightarrow{0}=(Q^{-1}\overrightarrow{u})^{\flat(l_0)}(t,x)=t^{A_{l_0}(x)}\overrightarrow{\varphi_{l_0}}(x)+\widetilde{W}^{(L_{l_0}+1+\epsilon)}(\infty,R)^{r_{l_0}}$, which means $t^{A_{l_0}(x)}\overrightarrow{\varphi_{l_0}}(x)\in\widetilde{W}^{(L_{l_0}+1+\epsilon)}(\infty,R)^{r_{l_0}}$. This implies $\overrightarrow{\varphi_{l_0}}=\overrightarrow{0}$ by Lemma 5.1, which contradicts the definition of l_0.

§8. Proof of the surjectivity of (2.4)

Assume that $\vec{u} \in (\widetilde{\mathcal{O}_{\theta,R}})^m$ and $P\vec{u} = \vec{0}$. We need to construct $\vec{\varphi_l} \in \mathcal{O}(B_R)^{r_l}$ $(1 \le l \le d)$. Instead of constructing $\vec{\varphi_l}$ on B_R, we fix an arbitrary $R' \in (0, R)$ and construct $\vec{\varphi_l} \in \mathcal{O}(B_{R'})^{r_l}$. Take an arbitrary $R_1 \in (R', R)$. Then there exists $T_1 > 0$ such that $\vec{u} \in \mathcal{O}(S_{\theta,T_1} \times B_{R_1})^m$. Set $A(t, x; D_x) = A_0(x) + tB(t, x; D_x)$.

(I) By Proposition 6.1, there exists $L \in \boldsymbol{Z}$ such that $\vec{u} \in W^{(L+\epsilon)}(\theta, T_1, R_1)^m$. From this, we have $tB(\vec{u}) \in W^{(L+1+\epsilon)}(\theta, T_1, R_1)^m$ by Lemma 4.2.

(II) By Proposition 5.2, there exists $\vec{v} \in W^{(L+1+\epsilon)}(\theta, T_1, R_1)^m$ such that $(tD_t I_m - A_0(x))\vec{v} = tB(\vec{u})$.

(III) Since $(tD_t I_m - A_0(x))(\vec{u} - \vec{v}) = \vec{0}$ and since $\vec{u} - \vec{v} \in W^{(L+\epsilon)}(\theta, T_1, R_1)^m$, there exists $\vec{\varphi_l}[1] \in \mathcal{O}(B_{R_1})^{r_l}$ such that $\vec{u} - \vec{v} = \sum_{1\le l\le d;\, L_l \ge L} Q(x)\{t^{A_l(x)}\vec{\varphi_l}[1](x)\}^{\sharp(l)}$ by Lemma 5.1.

(IV) Set $\vec{u}[1] := \vec{u} - \sum_{1\le l\le d;\, L_l\ge L} \vec{u_l}[\vec{\varphi_l}[1]]$. Then we have $P(\vec{u}[1]) = \vec{0}$ and

$$\begin{aligned}
\vec{u}[1] &= \Big(\vec{u} - \sum_l Q(x)\{t^{A_l(x)}\vec{\varphi_l}[1](x)\}^{\sharp(l)}\Big) \\
&\qquad - \sum_l \Big(\vec{u_l}[\vec{\varphi_l}[1]] - Q(x)\{t^{A_l(x)}\vec{\varphi_l}[1](x)\}^{\sharp(l)}\Big) \\
&= \vec{v} - \sum_l t(\vec{r_l}[\vec{\varphi_l}[1]])^{\sharp(l)} \in \widetilde{W}^{(L+1+\epsilon)}(\theta, R_1)^m \ ,
\end{aligned}$$

using Lemma 4.4. Take $R_2 \in (R', R_1)$. There exists $T_2 > 0$ such that $\vec{u}[1] \in W^{(L+1+\epsilon)}(\theta, T_2, R_2)^m$.

Now, we can return to step (I) using $\vec{u}[1]$ instead of $\vec{u}$ and with the order increased by 1. Thus, by repeating this argument, there exist $R' < R_j < \ldots < R_2 < R_1 < R$ and $\vec{\varphi_l}[j] \in \mathcal{O}(B_{R_j})^{r_l}$ such that $\vec{u}[j] := \vec{u} - \sum_{k=1}^{j}\sum_l \vec{u_l}[\vec{\varphi_l}[k]] \in \widetilde{W}^{(L+j+\epsilon)}(\theta, R_j)^m$ also satisfies $P(\vec{u}[j]) = \vec{0}$.

For a sufficiently large M, if $\vec{u} \in \widetilde{W}^{(L+M+\epsilon)}(\theta, R')$ and $P(\vec{u}) = \vec{0}$, then $\vec{u} = \vec{0}$ by Theorem 1.2. Hence, we have $\vec{u}[M] = \vec{0}$, which implies $\vec{u} = \sum_l \vec{u_l}[\sum_{k=1}^{M} \vec{\varphi_l}[k]]$. Thus, we have constructed $\vec{\varphi_l} = \sum_{k=1}^{M} \vec{\varphi_l}[k] \in \mathcal{O}(B_{R'})^{r_l}$. Since (2.4) is injective, that is, $\vec{\varphi_l}$ are uniquely determined by $\vec{u}$, and since $R'(< R)$ is arbitrary, we have the existence of $\vec{\varphi_l} \in \mathcal{O}(B_R)^{r_l}$ $(1 \le l \le d)$, that is, the surjectivity of (2.4) to $\mathrm{Ker}_{(\widetilde{\mathcal{O}_{\theta,R}})^m} P$.

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Takeshi Mandai
Research Center for Physics and Mathematics
Osaka Electro-Communication University
18-8 Hatsu-cho, Neyagawa-shi, Osaka 572-8530
Japan
mandai@isc.osakac.ac.jp

Hidetoshi Tahara
Department of Mathematics
Sophia University
7-1 Kioi-cho, Chiyoda-ku, Tokyo 102-8554
Japan
h-tahara@hoffman.cc.sophia.ac.jp