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COEFFICIENT ESTIMATES FOR SOME CLASSES OF FUNCTIONS ASSOCIATED WITH q-FUNCTION THEORY

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Abstract

For every $q \in (0, 1)$, we obtain the Herglotz representation theorem and discuss the Bieberbach problem for the class of q-convex functions of order α with $0 \le \alpha < 1$. In addition, we consider the Fekete–Szegö problem and the Hankel determinant problem for the class of q-starlike functions, leading to two conjectures for the class of q-starlike functions of order α with $0 \le \alpha < 1$.

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1. Introduction

Throughout, $\mathbb{C}$ denotes the set of complex numbers and $\mathcal{H}(\mathbb{D})$ the set of all analytic (or holomorphic) functions in the unit disc $\mathbb{D}$. We use the symbol $\mathcal{A}$ for the class of functions $f \in \mathcal{H}(\mathbb{D})$ with the standard normalisation $f(0) = 0 = f'(0) - 1$, that is, the functions $f \in \mathcal{A}$ having the power series representation of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The set $\mathcal{S}$ denotes the class of *univalent* functions in $\mathcal{A}$ and $\mathcal{S}^*$ and C the classes of *starlike* and *convex* functions in $\mathcal{A}$, respectively (see [4, 7]). The principal value of the logarithm for $z \neq 0$ is denoted by $\operatorname{Log} z := \ln|z| + i\operatorname{Arg}(z)$, where $-\pi \le \operatorname{Arg}(z) < \pi$.

In geometric function theory, finding bounds for the coefficients a_n of functions of the form (1.1) is an important problem, connected with geometric properties of the function. For example, the bound for the second coefficient a_2 of functions in the class $\mathcal{S}$ gives growth and distortion properties as well as covering theorems. In 1916, Bieberbach proposed a conjecture that among all functions in $\mathcal{S}$, the Koebe function $z/(1 - z)^2$ has the largest coefficients. The conjecture was first approached for some subclasses of univalent functions such as $\mathcal{S}^*$ and C. One of the important techniques developed to settle the conjecture is the *Herglotz representation theorem* giving an integral representation for analytic functions with positive real part in $\mathbb{D}$. Finally, de Branges [3] settled the Bieberbach conjecture in 1985.

The kth-order Hankel determinant ($k \geq 1$) of $f \in \mathcal{A}$ is defined by

$$H_k(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+k-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+k} \\ \vdots & \vdots & & \vdots \\ a_{n+k-1} & a_{n+k} & \dots & a_{n+2k-2} \end{vmatrix}.$$

We consider the Hankel determinants $H_2(1)$ (also called the Fekete–Szegö functional) and $H_2(2)$. Also in 1916, Bieberbach proved that if $f \in \mathcal{S}$, then $|a_2^2 - a_3| \leq 1$. In 1933, Fekete and Szegö [5] proved that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3 & \text{if } \mu \geq 1, \\ 1 + 2\exp[-2\mu/(1-\mu)] & \text{if } 0 \leq \mu \leq 1, \\ 3 - 4\mu & \text{if } \mu \leq 0. \end{cases}$$

The result is sharp in the sense that for each μ there is a function in the class under consideration for which equality holds. The coefficient functional $a_3 - \mu a_2^2$ has many applications in function theory. For example, $a_3 - a_2^2$ is equal to $S_f(z)/6$, where $S_f(z)$ is the Schwarzian derivative of the locally univalent function f defined by $S_f(z) = (f''(z)/f'(z))' - (1/2)(f''(z)/f'(z))^2$. The *Fekete–Szegö problem* asks for the maximum value of $a_3 - \mu a_2^2$. Koepf [12] solved the Fekete–Szegö problem for close-to-convex functions and showed that the largest real number μ for which $a_3 - \mu a_2^2$ is maximised by the Koebe function is $\mu = 1/3$. Later, in [13] (see also [15]), this result was generalised for functions that are close-to-convex of order β, $\beta \geq 0$. In [17], Pfluger employed a variational method to give another treatment of the Fekete–Szegö inequality which includes a description of the image domains under extremal functions. Later, Pfluger [18] used Jenkin's method to show that for $f \in \mathcal{S}$,

$$|a_3 - \mu a_2^2| \leq 1 + 2|\exp(-2\mu/(1-\mu))|$$

holds for complex μ such that $\operatorname{Re}(1/(1-\mu)) \geq 1$. The inequality is sharp if and only if μ is in a certain pear-shaped subregion of the disc given by

$$\mu = 1 - (u + itv)/(u^2 + v^2), \quad -1 \leq t \leq 1,$$

where $u = 1 - \log(\cos\varphi)$ and $v = \tan\varphi - \varphi$, $0 < \varphi < \pi/2$.

Bieberbach problems for functions belonging to q-analogues of subclasses of univalent functions are discussed in [1, 8, 19]. We discuss the Bieberbach problem for the q-analogue of convex functions of order α with $0 \leq \alpha < 1$. The Hankel determinant and Fekete–Szegö problems do not seem to be treated for q-analogues of subclasses of univalent functions. In this regard, we discuss the Hankel determinant and Fekete–Szegö problems for the q-analogue of starlike functions.

2. Preliminaries and main theorems

For $0 < q < 1$, *the q-difference operator*, denoted by $D_q f$, is defined by

$$(D_q f)(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \neq 0; \quad (D_q f)(0) = f'(0).$$

The class, $\mathcal{S}_q^*(\alpha)$, of *q-starlike functions of order α for* $0 \leq \alpha < 1$ is defined as follows.

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DEFINITION 2.1 [1, Definition 1.1]. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_q^*(\alpha)$, $0 \le \alpha < 1$, if

$$\left|\frac{1}{1-\alpha}\left(\frac{z(D_q f)(z)}{f(z)} - \alpha\right) - \frac{1}{1-q}\right| \le \frac{1}{1-q}, \quad z \in \mathbb{D}.$$

The choice $\alpha = 0$ gives the class $\mathcal{S}_q^*$ of q-starlike functions (see [8, Definition 1.3]). By using the idea of Alexander's theorem [4, Theorem 2.12], Baricz and Swaminathan [2] defined the class C_q of *q-convex functions* in the following way.

DEFINITION 2.2 [2, Definition 3.1]. A function $f \in \mathcal{A}$ is said to be in the class C_q if and only if $z(D_q f)(z) \in \mathcal{S}_q^*$.

The class C_q is nonempty as shown in [2, Theorem 3.2]. Note that as $q \to 1$, the classes $\mathcal{S}_q^*$ and C_q reduce to $\mathcal{S}^*$ and C, respectively. It is natural to define the class $C_q(\alpha)$ of *q-convex functions of order α for* $0 \le \alpha < 1$ as follows.

DEFINITION 2.3. A function $f \in \mathcal{A}$ is said to be in the class $C_q(\alpha)$ for $0 \le \alpha < 1$, if and only if $z(D_q f)(z) \in \mathcal{S}_q^*(\alpha)$.

As $q \to 1$, $C_q(\alpha)$ reduces to the class $C(\alpha)$ of convex functions of order α (see [7]). Thomae [20], a pupil of Heine, introduced a particular case of the q-integral,

$$\int_0^1 f(t)\, d_q t = (1-q) \sum_{n=0}^{\infty} q^n f(q^n),$$

provided the q-series converges. In 1910, Jackson defined the general q-integral [9] (see also [6, 20]) in the following manner:

$$\int_a^b f(t)\, d_q t := \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t,$$

where

$$I_q(f(x)) := \int_0^x f(t)\, d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(xq^n),$$

provided the q-series converges. Observe that

$$D_q I_q f(x) = f(x) \quad \text{and} \quad I_q D_q f(x) = f(x) - f(0),$$

where the second equality holds if f is continuous at $x = 0$.

We now state our main results. We first deal with the Fekete–Szegö problem for the class $\mathcal{S}_q^*$.

THEOREM 2.4. *Let $f \in \mathcal{S}_q^*$ be of the form (1.1) and μ be any complex number. Then*

$$|a_3 - \mu a_2^2| \le \max\left\{\left|2(1-2\mu)\left(\frac{\ln q}{q-1}\right)^2 + 2\left(\frac{\ln q}{q^2-1}\right)\right|, 2\left(\frac{\ln q}{q^2-1}\right)\right\}.$$

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Equality occurs for the functions

$$F_1(z) := z\Big\{\exp\Big[\sum_{n=1}^{\infty} \frac{2\ln q}{q^n - 1} z^n\Big]\Big\} \tag{2.1}$$

and

$$F_2(z) := z\Big\{\exp\Big[\sum_{n=1}^{\infty} \frac{2\ln q}{q^{2n} - 1} z^{2n}\Big]\Big\}. \tag{2.2}$$

Next, we estimate the second-order Hankel determinant for the class $\mathcal{S}_q^*$.

THEOREM 2.5. *Let $f \in \mathcal{S}_q^*$ be of the form (1.1). Then*

$$|H_2(2)| = |a_2 a_4 - a_3^2| \le 4\Big(\frac{\ln q}{q^2 - 1}\Big)^2.$$

Equality occurs for the function $F_2(z)$ defined in (2.2).

REMARK 2.6. As $q \to 1$, Theorem 2.4 reduces to the Fekete–Szegö problem for the class $\mathcal{S}^*$ [11, Theorem 1] and Theorem 2.5 gives the Hankel determinant for the class $\mathcal{S}^*$ [10, Theorem 3.1]. Later, in Section 4, we pose two conjectures on the Fekete–Szegö problem and Hankel determinant for the class $\mathcal{S}_q^*(\alpha)$.

Next, we present the Herglotz representation for functions in the class $C_q(\alpha)$.

THEOREM 2.7. *Let $f \in \mathcal{A}$. Then $f \in C_q(\alpha)$, $0 \le \alpha < 1$, if and only if there exists a probability measure μ supported on the unit circle such that*

$$\frac{z(D_q f)'(z)}{(D_q f)(z)} = \int_{|\sigma|=1} \sigma z F'_{q,\alpha}(\sigma z)\, d\mu(\sigma)$$

where

$$F_{q,\alpha}(z) = \sum_{n=1}^{\infty} \frac{-2}{1 - q^n} \ln\Big(\frac{q}{1 - \alpha(1 - q)}\Big) z^n, \quad z \in \mathbb{D}. \tag{2.3}$$

REMARK 2.8. As $q \to 1$, $F'_{q,\alpha}(z) \to 2(1-\alpha)/(1-z)$ and $z(D_q f)'(z)/(D_q f)(z) \to zf''(z)/f'(z)$. Hence, when $q \to 1$, Theorem 2.7 leads to the Herglotz representation of convex functions of order α (see [7, page 172, Problem 3]).

The Bieberbach problem for the classes $\mathcal{S}_q^*$ and $\mathcal{S}_q^*(\alpha)$ is treated in [8] and [1], respectively. Our next result is on the Bieberbach problem for the class $C_q(\alpha)$, $0 \le \alpha < 1$, which does not seem to have been considered before. One can also consider Hankel determinant and Fekete–Szegö problems for $C_q(\alpha)$.

THEOREM 2.9. *Let*

$$E_q(z) := I_q\{\exp[F_{q,\alpha}(z)]\} = z + \sum_{n=2}^{\infty} \Big(\frac{1-q}{1-q^n}\Big) c_n z^n \tag{2.4}$$

where c_n is the nth coefficient of the function $z\exp[F_{q,\alpha}(z)]$. Then $E_q \in C_q(\alpha)$ for $0 \le \alpha < 1$. Moreover, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C_q(\alpha)$, then $|a_n| \le ((1-q)/(1-q^n))c_n$, with equality holding for all n if and only if f is a rotation of E_q.

Remark 2.10. As $q \to 1$, $F_{q,\alpha}(z) \to -2(1-\alpha)\log(1-z)$ and hence $z\exp[F_{q,\alpha}(z)] \to z/(1-z)^{2(1-\alpha)}$. Therefore, as $q \to 1$, the coefficient $c_n \to \prod_{k=2}^{n}(k-2\alpha)/(n-1)!$, showing that $|a_n|$ is bounded by $\prod_{k=2}^{n}(k-2\alpha)/n!$ for $f \in C(\alpha)$. Thus, when $q \to 1$, Theorem 2.9 leads to the Bieberbach problem for the class $C(\alpha)$ (see [7, page 140, Theorem 2]). It would be interesting to get an explicit form of the extremal function independent of the q-integral in Theorem 2.9.

3. Properties of the class $C_q(\alpha)$, $0 \le \alpha < 1$

This section is concerned with some basic properties of the class $C_q(\alpha)$. We first remark that a function $f \in C_q(\alpha)$ can be written in terms of a function g in $\mathcal{S}_q^*(\alpha)$. The proof is obvious and follows from the definition of $C_q(\alpha)$.

Proposition 3.1. *Let $f \in C_q(\alpha)$, $0 \le \alpha < 1$. Then there exists a unique function $g \in \mathcal{S}_q^*(\alpha)$, $0 \le \alpha < 1$, such that*

$$g(z) = z(D_q f)(z). \tag{3.1}$$

Similarly, given $g \in \mathcal{S}_q^*(\alpha)$, there exists a unique function $f \in C_q(\alpha)$ satisfying (3.1).

The next result gives a characterisation of the functions in the class $C_q(\alpha)$.

Theorem 3.2. *Let $f \in \mathcal{A}$. Then $f \in C_q(\alpha)$, $0 \le \alpha < 1$, if and only if*

$$\left| q\frac{(D_q f)(qz)}{(D_q f)(z)} - \alpha q \right| \le 1 - \alpha, \quad z \in \mathbb{D}.$$

Proof. By Definition 2.3, $f \in C_q(\alpha)$ if and only if $z(D_q f)(z) \in \mathcal{S}_q^*(\alpha)$. The result follows immediately from [1, Theorem 2.2]. □

Corollary 3.3. *The class $C_q(\alpha)$ satisfies the relations*

$$\bigcap_{q<p<1} C_p(\alpha) \subseteq C_q(\alpha) \quad \text{and} \quad \bigcap_{0<q<1} C_q(\alpha) = C(\alpha).$$

Proof. If $f \in C_p(\alpha)$ for all $p \in (q,1)$, then letting $p \to q$ shows $f \in C_q(\alpha)$. Hence the first inclusion in the corollary holds. Similarly, if $f \in C_q(\alpha)$ for all $q \in (0,1)$, then letting $q \to 1$ shows $f \in C(\alpha)$, that is,

$$\bigcap_{0<q<1} C_q(\alpha) \subseteq C(\alpha).$$

It remains to prove the converse inclusion that

$$C(\alpha) \subseteq \bigcap_{0<q<1} C_q(\alpha).$$

For this, let $f \in C(\alpha)$ so that $zf' \in \mathcal{S}^*(\alpha)$. By [1, Corollary 2.3], $\mathcal{S}^*(\alpha) = \bigcap_{0<q<1} \mathcal{S}_q^*(\alpha)$, so that $zf' \in \mathcal{S}_q^*(\alpha)$ for all $q \in (0,1)$. Thus, by Proposition 3.1, there exists a unique $h \in C_q(\alpha)$ satisfying (3.1) with $h(z) = f(z)$. That is, $f \in C_q(\alpha)$ for all $q \in (0,1)$. This completes the proof. □

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Define the two sets

$$B_q = \{g : g \in \mathcal{H}(\mathbb{D}),\ g(0) = q \text{ and } g : \mathbb{D} \to \mathbb{D}\} \quad \text{and} \quad B_q^0 = \{g : g \in B_q \text{ and } 0 \notin g(\mathbb{D})\}.$$

Lemma 3.4 [1, Lemma 2.4]. *For $h \in B_q$, the infinite product*

$$\prod_{n=0}^{\infty} \frac{(1-\alpha)h(zq^n) + \alpha q}{q}$$

converges uniformly on compact subsets of $\mathbb{D}$.

Lemma 3.5. *For $h \in B_q^0$, the infinite product $\prod_{n=0}^{\infty}\{((1-\alpha)h(zq^n) + \alpha q)/q\}$ converges uniformly on compact subsets of $\mathbb{D}$ to a nonzero function in $\mathcal{H}(\mathbb{D})$ with no zeros. Furthermore, the function f satisfying the relation*

$$z(D_q f)(z) = \frac{z}{\prod_{n=0}^{\infty}\{((1-\alpha)h(zq^n) + \alpha q)/q\}} \tag{3.2}$$

belongs to $C_q(\alpha)$ and $h(z) = (1/(1-\alpha))(q(D_q f)(qz)/(D_q f)(z) - \alpha q)$.

Proof. The convergence of the infinite product is given by Lemma 3.4. Since $h \in B_q^0$, $h(z) \neq 0$ in $\mathbb{D}$ and the infinite product does not vanish in $\mathbb{D}$. Thus, $z(D_q f)(z) \in \mathcal{A}$ and

$$q\frac{(D_q f)(qz)}{(D_q f)(z)} = q \lim_{k\to\infty} \prod_{n=0}^{k} \frac{(1-\alpha)h(zq^n) + \alpha q}{(1-\alpha)h(zq^{n+1}) + \alpha q} = (1-\alpha)h(z) + \alpha q.$$

Since $h \in B_q^0$, we see that $f \in C_q(\alpha)$ and the proof of the lemma is complete. □

Let $\mathcal{P}$ be the family of all functions $p \in \mathcal{H}(\mathbb{D})$ for which $\operatorname{Re}\{p(z)\} \geq 0$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad \text{for } z \in \mathbb{D}. \tag{3.3}$$

Lemma 3.6 [8, Lemma 2.4]. *A function $g \in B_q^0$ if and only if it has the representation*

$$g(z) = \exp\{(\ln q)p(z)\} \quad \text{with } p(z) \in \mathcal{P}. \tag{3.4}$$

Theorem 3.7. *The mapping $\rho : C_q(\alpha) \to B_q^0$ defined by*

$$\rho(f)(z) = \frac{1}{1-\alpha}\Big(q\frac{(D_q f)(qz)}{(D_q f)(z)} - \alpha q\Big)$$

is a bijection.

Proof. For $h \in B_q^0$, define a mapping $\sigma :\ B_q^0 \to \mathcal{A}$ by

$$z(D_q\sigma(h))(z) = \frac{z}{\prod_{n=0}^{\infty}\{((1-\alpha)h(zq^n) + \alpha q)/q\}}.$$

From Lemma 3.5, $\sigma(h) \in C_q$ and $(\rho \circ \sigma)(h) = h$. Considering the composite mapping $\sigma \circ \rho$, we compute

$$\begin{aligned} z(D_q(\sigma\circ\rho)(f))(z) &= \frac{z}{\prod_{n=0}^{\infty}\{((1-\alpha)\rho(f)(zq^n) + \alpha q)/q\}} \\ &= \frac{z}{\prod_{n=0}^{\infty}\{q(D_q f)(zq^{n+1})/q(D_q f)(zq^n)\}} = z(D_q f)(z) \end{aligned}$$

whence $(\sigma \circ \rho)(f) = f$. Hence σ is the inverse of ρ and $\rho(f)$ is a bijection. □

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4. Proof of the main theorems

We now prove the main theorems stated in Section 2. The following lemmas are used to obtain the results on the Fekete–Szegö problem and the Hankel determinant.

LEMMA 4.1 [8, Theorem 1.13]. *The mapping* $\rho : \mathcal{S}_q^* \to B_q^0$ *defined by*

$$\rho(f)(z) = \frac{f(qz)}{f(z)}$$

is a bijection.

LEMMA 4.2 [8, Theorem 1.15]. *Let* $f \in \mathcal{A}$. *Then* $f \in \mathcal{S}_q^*$ *if and only if there exists a probability measure* μ *supported on the unit circle such that*

$$\frac{zf'(z)}{f(z)} = 1 + \int_{|\sigma|=1} \sigma z F_q'(\sigma z)\, d\mu(\sigma)$$

where

$$F_q(z) = \sum_{n=1}^{\infty} \frac{2 \ln q}{q^n - 1} z^n, \quad z \in \mathbb{D}. \tag{4.1}$$

LEMMA 4.3 [14, pages 254–256]. *Let the function* $p \in \mathcal{P}$ *be given by the power series (3.3). Then*

$$2p_2 = p_1^2 + x(4 - p_1^2),$$
$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some x *and* z *satisfying* $|x| \le 1$, $|z| \le 1$, *and* $p_1 \in [0, 2]$.

LEMMA 4.4 [16, Lemma 1]. *Let the function* $p \in \mathcal{P}$ *be given by the power series (3.3). Then for any real number* λ,

$$|p_2 - \lambda p_1^2| \le 2 \max\{1, |2\lambda - 1|\}$$

and the result is sharp.

PROOF OF THEOREM 2.4. Let $f \in \mathcal{S}_q^*$. By Lemma 4.1, $g(z) = f(qz)/f(z) \in B_q^0$. By Lemma 3.6, $g(z)$ has the representation (3.4), Define the function

$$\phi(z) = \mathrm{Log} \frac{f(z)}{z} = \sum_{n=1}^{\infty} \phi_n z^n.$$

From (3.4), we find $\phi(qz) - \phi(z) = (\ln q)p(z) - \ln q$ which implies

$$\phi_n = p_n \left(\frac{\ln q}{q^n - 1} \right). \tag{4.2}$$

So, $f(z)$ can be written as

$$f(z) = z \exp\left[\sum_{n=1}^{\infty} \phi_n z^n \right], \tag{4.3}$$

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where ϕ_n is defined in (4.2) and $f(z)$ has the form (1.1). Equating the coefficients of z^n on both sides of (4.3) and using the value of ϕ_n given in (4.2),

$$a_2 = \phi_1 = p_1\Big(\frac{\ln q}{q-1}\Big), \quad a_3 = \phi_2 + \frac{\phi_1^2}{2} = p_2\Big(\frac{\ln q}{q^2-1}\Big) + \frac{p_1^2}{2}\Big(\frac{\ln q}{q-1}\Big)^2. \tag{4.4}$$

Thus,

$$\begin{aligned}|a_3 - \mu a_2^2| &= \Big|p_2\Big(\frac{\ln q}{q^2-1}\Big) + \frac{p_1^2}{2}\Big(\frac{\ln q}{q-1}\Big)^2 - \mu p_1^2\Big(\frac{\ln q}{q-1}\Big)^2\Big| \\ &\le \max\Big\{\Big|2(1-2\mu)\Big(\frac{\ln q}{q-1}\Big)^2 + 2\Big(\frac{\ln q}{q^2-1}\Big)\Big|, 2\Big(\frac{\ln q}{q^2-1}\Big)\Big\},\end{aligned}$$

where the last inequality follows from Lemma 4.4.

It now remains to prove the sharpness part. From the definition of $\mathcal{S}_q^*$, the functions F_1 and F_2 defined in the statement of Theorem 2.4 belong to $\mathcal{S}_q^*$. Further, $F_1 \in \mathcal{S}_q^*$ as a special case of Lemma 4.2 when the measure has a unit mass. The functions F_1 and F_2 prove the sharpness of the result. This completes the proof of the theorem. □

We now pose the following conjecture on the Fekete–Szegö problem for $\mathcal{S}_q^*(\alpha)$.

Conjecture 4.5. Let $f \in \mathcal{S}_q^*(\alpha)$, $0 \le \alpha < 1$, be of the form (1.1) and μ be any complex number. Set $q(\alpha) = q/(1-\alpha(1-q))$. Then

$$|a_3 - \mu a_2^2| \le \max\Big\{\Big|2(1-2\mu)\Big(\frac{\ln q(\alpha)}{q-1}\Big)^2 + 2\Big(\frac{\ln q(\alpha)}{q^2-1}\Big)\Big|, 2\Big(\frac{\ln q(\alpha)}{q^2-1}\Big)\Big\}.$$

Equality occurs for the functions

$$F_1(z) := z\Big\{\exp\Big[\sum_{n=1}^{\infty} \frac{2\ln q(\alpha)}{q^n-1} z^n\Big]\Big\} \tag{4.5}$$

and

$$F_2(z) := z\Big\{\exp\Big[\sum_{n=1}^{\infty} \frac{2\ln q(\alpha)}{q^{2n}-1} z^{2n}\Big]\Big\}. \tag{4.6}$$

Proof of Theorem 2.5. Let $f \in \mathcal{S}_q^*$ have the form (1.1). In (4.4), we already obtained the values of a_2 and a_3. In a similar way, we can find the value of a_4. Indeed,

$$a_4 = \phi_3 + \phi_1\phi_2 + \frac{\phi_1^3}{6} = p_3\Big(\frac{\ln q}{q^3-1}\Big) + p_1p_2\Big(\frac{\ln q}{q-1}\Big)\Big(\frac{\ln q}{q^2-1}\Big) + \frac{p_1^3}{6}\Big(\frac{\ln q}{q-1}\Big)^3.$$

Hence,

$$|a_2a_4 - a_3^2| = \Big|-\frac{p_1^4}{12}\Big(\frac{\ln q}{q-1}\Big)^4 + p_1p_3\Big(\frac{\ln q}{q-1}\Big)\Big(\frac{\ln q}{q^3-1}\Big) - p_2^2\Big(\frac{\ln q}{q^2-1}\Big)^2\Big|.$$

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Suppose now that $p_1 = c$ and $0 \leq c \leq 2$. Using Lemma 4.3, we obtain

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \bigg| -\frac{c^4}{12}\bigg[\Big(\frac{\ln q}{q-1}\Big)^4 - 3\Big(\frac{\ln q}{q-1}\Big)\Big(\frac{\ln q}{q^3-1}\Big) + 3\Big(\frac{\ln q}{q^2-1}\Big)^2\bigg] \\
&\quad + \frac{c^2}{2}(4-c^2)x\bigg[\Big(\frac{\ln q}{q-1}\Big)\Big(\frac{\ln q}{q^3-1}\Big) - \Big(\frac{\ln q}{q^2-1}\Big)^2\bigg] \\
&\quad + \frac{(4-c^2)(1-|x|^2)cz}{2}\Big(\frac{\ln q}{q-1}\Big)\Big(\frac{\ln q}{q^3-1}\Big) \\
&\quad - \bigg[\frac{c^2}{4}(4-c^2)\Big(\frac{\ln q}{q-1}\Big)\Big(\frac{\ln q}{q^3-1}\Big) + \frac{(4-c^2)^2}{4}\Big(\frac{\ln q}{q^2-1}\Big)^2\bigg]x^2\bigg| \\
&\leq \frac{c^4}{12}\bigg|\Big(\frac{\ln q}{q-1}\Big)^4 - 3\Big(\frac{\ln q}{q-1}\Big)\Big(\frac{\ln q}{q^3-1}\Big) + 3\Big(\frac{\ln q}{q^2-1}\Big)^2\bigg| \\
&\quad + \frac{(4-c^2)c}{2}\Big(\frac{\ln q}{q-1}\Big)\Big(\frac{\ln q}{q^3-1}\Big) \\
&\quad + \frac{c^2}{2}(4-c^2)\bigg[\Big(\frac{\ln q}{q-1}\Big)\Big(\frac{\ln q}{q^3-1}\Big) - \Big(\frac{\ln q}{q^2-1}\Big)^2\bigg]\rho \\
&\quad + \Big(\frac{4-c^2}{4}\Big)\bigg[(4-c^2)\Big(\frac{\ln q}{q^2-1}\Big)^2 + c(c-2)\Big(\frac{\ln q}{q-1}\Big)\Big(\frac{\ln q}{q^3-1}\Big)\bigg]\rho^2 \\
&= F(\rho),
\end{aligned}$$

with $\rho = |x| \leq 1$. Furthermore, $F'(\rho) \geq 0$ which implies that F is an increasing function of ρ and that the upper bound for $|a_2a_4 - a_3^2|$ corresponds to $\rho = 1$. Hence,

$$|a_2a_4 - a_3^2| \leq F(1) = G(c) \ \text{(say)}.$$

We can see that

$$\Big(\frac{\ln q}{q-1}\Big)^4 - 3\Big(\frac{\ln q}{q-1}\Big)\Big(\frac{\ln q}{q^3-1}\Big) + 3\Big(\frac{\ln q}{q^2-1}\Big)^2 > 0,$$

for $0 < q < 1$ and a simple calculation gives

$$\begin{aligned}
G(c) &= \frac{c^4}{12}\bigg[\Big(\frac{\ln q}{q-1}\Big)^4 - 12\Big(\frac{\ln q}{q-1}\Big)\Big(\frac{\ln q}{q^3-1}\Big) + 12\Big(\frac{\ln q}{q^2-1}\Big)^2\bigg] \\
&\quad + c^2\bigg[3\Big(\frac{\ln q}{q-1}\Big)\Big(\frac{\ln q}{q^3-1}\Big) - 4\Big(\frac{\ln q}{q^2-1}\Big)^2\bigg] + 4\Big(\frac{\ln q}{q^2-1}\Big)^2.
\end{aligned}$$

We can verify that $G'(c) = 0$ gives either $c = 0$ or $\pm c(q)$, where $c(q) > 0$ can be given explicitly in terms of q, and that $G''(c)$ is negative for $c = 0$ and positive for other values of c. Hence the maximum of $G(c)$ occurs at $c = 0$ and we obtain

$$|a_2a_4 - a_3^2| \leq 4\Big(\frac{\ln q}{q^2-1}\Big)^2.$$

The function F_2 defined in the statement of the theorem shows the sharpness of the result. This completes the proof of the theorem. □

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We pose a conjecture about the Hankel determinant for the class $\mathcal{S}_q^*(\alpha)$.

CONJECTURE 4.6. Let $f \in \mathcal{S}_q^*(\alpha)$, $0 \le \alpha < 1$, be of the form (1.1). Then

$$|a_2a_4 - a_3^2| \le 4\Big(\frac{\ln q(\alpha)}{q^2-1}\Big)^2.$$

where $q(\alpha) = q/(1-\alpha(1-q))$. Equality occurs for the function F_2 defined in (4.6).

REMARK 4.7. We remark that the inequalities in Conjectures 4.5 and 4.6 can be obtained by calculations similar to those in the proofs of Theorems 2.4 and 2.5, respectively. However, the point of the conjectures is to find the extremal functions which we believe to be (4.5) and (4.6).

PROOF OF THEOREM 2.7. Let $f \in C_q(\alpha)$, $0 \le \alpha < 1$. From the definition of $C_q(\alpha)$, $z(D_qf)(z) \in \mathcal{S}_q^*(\alpha)$. By [1, Theorem 1.1],

$$1 + \frac{z(D_qf)'(z)}{(D_qf)(z)} = z\frac{(z(D_qf)(z))'(z)}{z(D_qf)(z)} = 1 + \int_{|\sigma|=1} \sigma z F'_{q,\alpha}(\sigma z)\, d\mu(\sigma)$$

where $F_{q,\alpha}$ is defined in (2.3). This completes the proof. □

PROOF OF THEOREM 2.9. Let $f(z) = z + \sum_{n=2}^{\infty} a_nz^n \in C_q(\alpha)$. From the definition of $C_q(\alpha)$, $z(D_qf)(z) = z + \sum_{n=2}^{\infty}((1-q^n)/(1-q))a_nz^n \in \mathcal{S}_q^*(\alpha)$. By [1, Theorem 1.3],

$$\Big|\frac{1-q^n}{1-q}a_n\Big| \le c_n.$$

Next, we show that equality holds for the function $E_q \in C_q(\alpha)$. As a special case of Theorem 2.7, when the measure has a unit mass, it is clear that $E_q \in C_q(\alpha)$. Let $E_q(z) = z + \sum_{n=2}^{\infty} b_nz^n$. From this representation of E_q and the definition of D_qf,

$$z(D_qE_q)(z) = z + \sum_{n=2}^{\infty} b_n\Big(\frac{1-q^n}{1-q}\Big)z^n. \tag{4.7}$$

On the other hand, $E_q(z) = I_q\{\exp[F_{q,\alpha}(z)]\}$, so $z(D_qE_q)(z) = z\{\exp[F_{q,\alpha}(z)]\}$ and, since c_n is the nth coefficient of the function $z\exp[F_{q,\alpha}(z)]$,

$$z(D_qE_q)(z) = z + \sum_{n=2}^{\infty} c_nz^n. \tag{4.8}$$

By comparing (4.7) and (4.8), we see that $b_n = c_n(1-q)/(1-q^n)$, that is,

$$E_q(z) = z + \sum_{n=2}^{\infty}\Big(\frac{1-q}{1-q^n}\Big)c_nz^n.$$

This completes the proof of the theorem. □

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