18

Dynamics of quantum fields

In this chapter we describe how to quantize linear classical dynamics. The starting point will be a dual phase space $\mathcal{Y}$ equipped with a dynamics – a one-parameter group of linear transformations $\{r_t\}_{t\in\mathbb{R}}$ preserving the structure of $\mathcal{Y}$.

The most typical examples of $\mathcal{Y}$ are the space of solutions of the *Klein–Gordon equation* and of the *Dirac equation*, possibly on a curved space-time and in the presence of external potentials. We can also consider other systems, not necessarily relativistic, e.g. motivated by condensed-matter physics.

We describe how to quantize $\big(\mathcal{Y}, \{r_t\}_{t\in\mathbb{R}}\big)$ obtaining a model of non-interacting quantum field theory. We demand that quantum fields are represented on a Hilbert space and that the dynamics is implemented by a unitary group generated by a *positive Hamiltonian*. In all the cases we consider, the first step of quantization is the construction of the so-called *one-particle space* $\mathcal{Z}$, equipped with a dynamics generated by a positive *one-particle Hamiltonian* h. Then we apply the usual procedure of the *second quantization* to obtain the *Fock space over* $\mathcal{Z}$ equipped with the dynamics given by the *second quantization of* $\mathrm{e}^{\mathrm{i}th}$.

The positivity of the Hamiltonian of the quantum system means that we are at the *zero temperature*. We will also consider briefly the case of *positive temperatures*, which involves the construction of a state satisfying the *KMS condition*.

The abstract procedure outlined above is used in concrete situations in quantum field theory to construct *free* (i.e. *non-interacting*) quantum fields and many-body quantum systems. In this chapter we will not discuss the construction of *interacting quantum fields*, which is much more difficult. In the physical literature, one usually tries to construct interacting fields by perturbing free ones, which is one of the reasons for the importance of free fields. We will describe the diagrammatic aspects of the formal perturbation theory in Chap. 20. Some mathematical tools involved in the rigorous construction of interacting fields are described in Chap. 21 and will be applied to bosonic models in two space-time dimensions in Chap. 22.

The space $\mathcal{Y}$ will always have an additional structure preserved by the dynamics. We distinguish four kinds of such structures leading to four kinds of quantization formalisms:

(1) **Neutral bosonic systems.** The space $\mathcal{Y}$ is symplectic. This formalism is used e.g. for real solutions of the Klein–Gordon equation.

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(2) **Neutral fermionic systems.** The space $\mathcal{Y}$ is Euclidean. This formalism can be used e.g. for Majorana spinors satisfying the Dirac equation.
(3) **Charged bosonic systems.** The space $\mathcal{Y}$ is charged symplectic (equipped with a non-degenerate anti-Hermitian form). This formalism is used e.g. for complex solutions of the Klein–Gordon equation.
(4) **Charged fermionic systems.** The space $\mathcal{Y}$ is unitary. This formalism is used e.g. for Dirac spinors satisfying the Dirac equation.

Remark 18.1 *In the most common physics applications we encounter the neutral bosonic formalism (e.g. for photons) and the charged fermionic formalism (e.g. for electrons). Charged bosons are also quite common, e.g. charged pions or gluons in the standard model. On the other hand, until recently, the neutral fermionic formalism had mostly theoretical interest. However, in the modern version of the standard model involving massive neutrinos, Majorana spinors can be useful.*

Remark 18.2 *To avoid possible confusion, let us discuss the distinction between the notion of a "phase space" and of a "dual phase space".*

Possible states of a classical system are described by elements (points) of a set $\mathcal{V}$, called a phase space. *$\mathcal{V}$ is typically a manifold, often equipped with an additional structure, e.g. it is a symplectic manifold. The time evolution of a classical system is given by a one-parameter group $\{r_t\}_{t\in\mathbb{R}}$ of isomorphisms of $\mathcal{V}$. Classical observables are described by (real- or complex-valued) functions on $\mathcal{V}$.*

If $\mathcal{V}$ is in addition a vector space, we have in particular linear (i.e. "coordinate") functions $\mathcal{V} \ni v \mapsto v \cdot y \in \mathbb{R}$ labeled $y \in \mathcal{V}^{\#} =: \mathcal{Y}$. We will say that $\mathcal{Y}$ is the dual phase space*. After the bosonic resp. fermionic quantization, we obtain a family of quantum observables $\phi(y)$, $y \in \mathcal{Y}$, which are operators satisfying the CCR, resp. the CAR and are called the* bosonic, *resp.* fermionic fields*. They are labeled by elements of the dual phase space.*

As we see from this discussion, in the quantum case it is the dual $\mathcal{Y}$ of the phase space that has a more fundamental role than the phase space $\mathcal{V}$ itself. Therefore, in our work the starting point is typically $\mathcal{Y}$.

The distinction between the phase space and its dual is rather academic in the fermionic case, where they can be naturally identified using the scalar product. In the bosonic case, if the space $\mathcal{V}$ is symplectic (the form ω is non-degenerate), one can identify the phase space and its dual with help of this form.

The Hamiltonian, which generates a symplectic dynamics, is traditionally defined as a function on the phase space. If the phase space is symplectic we can transport the Hamiltonian by ω from $\mathcal{V}$ to $\mathcal{Y}$, so that it becomes a function on $\mathcal{Y}$. In this chapter, in the bosonic case the phase space will be always symplectic and we will treat Hamiltonians as functions on the dual phase space, as explained above.

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One can distinguish three stages of quantization.

(1) **Classical system.** We consider one of the four kinds of the *dual phase space* $\mathcal{Y}$, together with a one-parameter group of its automorphisms, $\mathbb{R} \ni t \mapsto r_t$, which we view as a *classical dynamics*.
(2) **Algebraic quantization.** We choose an appropriate $*$-algebra $\mathfrak{A}$, together with a one-parameter group of $*$-automorphisms $\mathbb{R} \ni t \mapsto \hat{r}_t$. The algebra $\mathfrak{A}$ is sometimes called the *field algebra* of the quantum system. The commutation, resp. anti-commutation relations satisfied by the appropriate distinguished elements of $\mathfrak{A}$ are governed by the (charged) symplectic form, resp. the scalar product on the dual phase space. $\{\hat{r}_t\}_{t\in\mathbb{R}}$ describes the *quantum dynamics in the Heisenberg picture*. The algebra $\mathfrak{A}$ contains operators that are useful in the theoretical description of the system. However, we do not assume that all of its elements are physically observable, even in principle. Therefore, we also distinguish the *algebra of observables*. It is a certain $*$-sub-algebra of $\mathfrak{A}$, invariant with respect to the dynamics, which consists of operators whose measurement is theoretically possible.
(3) **Hilbert space quantization.** We represent the algebra $\mathfrak{A}$ on a certain Hilbert space $\mathcal{H}$. Typically, the representation of the algebra $\mathfrak{A}$ is faithful, so that we can write $\mathfrak{A} \subset B(\mathcal{H})$. We demand that the dynamics is implemented by a one-parameter unitary group on $\mathcal{H}$. In the case of a zero temperature, we want this unitary group to be generated by a positive operator H, called the *Hamiltonian*, so that

$$\hat{r}_t(A) = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}. \tag{18.1}$$

In the case of a positive temperature, the space $\mathcal{H}$ should contain a cyclic vector satisfying the KMS condition with respect to the dynamics. Its generator is called the *Liouvillean* and denoted L.

Note that, among the three stages of quantization described above, the most important are the first and the third. The second stage – the algebraic quantization – can be skipped altogether. In the usual presentation, typical for physics textbooks, it is limited to a formal level – one says that "commuting classical observables" are replaced by "non-commuting quantum observables" satisfying the appropriate commutation, resp. anti-commutation relations. In our presentation we have tried to interpret this statement in terms of well-defined C^*-algebras. This is quite easy in the case of fermions. Unfortunately, in the case of bosons it leads to certain technical difficulties related to the unboundedness of bosonic fields, and involves a considerable amount of arbitrariness in the choice of a C^*-algebra describing bosonic observables. To some extent, the algebraic quantization is merely an exercise of academic interest. Nevertheless, in some situations it sheds light on some conceptual aspects of quantum theory.

One of the confusing conceptual points that we believe our abstract approach can explain is the difference between the dual phase space and the one-particle space. Throughout our work, the former is typically denoted by $\mathcal{Y}$ and the latter by $\mathcal{Z}$. These two spaces are often identified. They have, however, different physical meanings and are equipped with different algebraic structures.

We also discuss abstract properties of two commonly used discrete symmetries of quantum systems: the time reversal and the charge reversals. Their properties can be quite confusing. We believe that the precise language of linear algebra is particularly adapted to explain their properties. Note, for instance, that the charge reversal is anti-linear with respect to the complex structure on the phase space and linear with respect to the complex structure on the one-particle space. On the other hand, the (Wigner) time reversal is anti-linear with respect to both.

We will always assume that the time and charge reversals are involutions on the observables. Only in the neutral bosonic case do they need to be involutive on the fields as well. In the other three cases observables are even in fields; therefore the time and charge reversals can be anti-involutive.

The first two sections of this chapter present the quantization in an abstract way. In the next two sections, we specify it a little more, considering what we call *abstract Klein–Gordon* and *abstract Dirac dynamics*. This presentation allows us to isolate the main features of various constructions used in quantum field theory and many-body quantum physics.

Throughout the chapter, t is the generic name of a real variable denoting the time.

18.1 Neutral systems

This section is devoted to the neutral bosonic and fermionic formalism of quantization.

In the neutral formalism the vector space $\mathcal{Y}$ is real and is equipped with a symplectic form ω in the bosonic case, resp. with a positive definite scalar product ν in the fermionic case. The dynamics describing the *time evolution* is a one-parameter group $\{r_t\}_{t\in\mathbb{R}}$ with values in $Sp(\mathcal{Y})$, resp. $O(\mathcal{Y})$. The problem addressed in this section is to find a CCR, resp. CAR representation $\mathcal{Y} \ni y \mapsto \phi(y)$ on a Hilbert space $\mathcal{H}$ and a self-adjoint operator H on $\mathcal{H}$ such that $\mathrm{e}^{\mathrm{i}tH}$ implements r_t. In the case of a zero temperature, usually one demands that the Hamiltonian H is *positive*.

We will do this by finding a Kähler anti-involution that commutes with the dynamics, and thus leads to a Fock representation in which the dynamics is implementable.

It turns out that this is easy in the fermionic case. The bosonic case is more technical. In particular, one needs to assume that the dynamics is *stable*, which roughly means that the classical Hamiltonian is positive.

One often assumes that the dynamics $\{r_t\}_{t\in\mathbb{R}}$ is a part of a larger group of symmetries G. In other words, our starting point is a homomorphism of a group G into $Sp(\mathcal{Y})$, resp. $O(\mathcal{Y})$. One often asks whether the action of G can be implemented in the Hilbert space $\mathcal{H}$ by unitary operators.

A different kind of a symmetry is the *time reversal*. After quantization, the time reversal is implemented by an anti-unitary operator.

Recall that if a is an operator on a real space $\mathcal{Y}$, then $a_{\mathbb{C}}$, resp. $a_{\overline{\mathbb{C}}}$ denotes its linear, resp. anti-linear extension to $\mathbb{C}\mathcal{Y}$.

18.1.1 Neutral bosonic systems

Let $(\mathcal{Y}, \omega)$ be a symplectic space. Let $\mathbb{R} \ni t \mapsto r_t \in Sp(\mathcal{Y})$ be a one-parameter group.

Algebraic quantization of a symplectic dynamics

It is easy to describe the quantum counterpart of the above classical dynamical system. We take one of the CCR algebras over $(\mathcal{Y}, \omega)$, say $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$, and equip it with the group of Bogoliubov automorphisms $\{\hat{r}_t\}_{t\in\mathbb{R}}$, defined by

$$\hat{r}_t\big(W(y)\big) = W(r_t y), \quad y \in \mathcal{Y}.$$

Stable symplectic dynamics

Typical symplectic dynamics that appear in physics have *positive Hamiltonians*. We will call such dynamics *stable*. We will see that (under some technical conditions) a stable dynamics leads to a uniquely defined Fock representation.

It is easy to make the concept of stability precise if $\dim \mathcal{Y} < \infty$. In this case $\mathcal{Y}$ has a natural topology. Of course, we assume that the dynamics $t \mapsto r_t$ is continuous. Let a be its generator, so that $r_t = \mathrm{e}^{ta}$. Clearly, the form β defined by

$$y_1 \cdot \beta y_2 := y_1 \cdot \omega a y_2, \quad y_1, y_2 \in \mathcal{Y}, \tag{18.2}$$

is symmetric.

Definition 18.3 *We say that the group $t \mapsto r_t \in Sp(\mathcal{Y})$ is* stable *if β is positive definite.*

The definition of a stable dynamics in the case of infinite dimensions is more complicated, because we need to equip $(\mathcal{Y}, \omega)$ with a topology. There are various possibilities for doing this; let us consider the simplest one.

Definition 18.4 *We say that $\big(\mathcal{Y}, \omega, \beta, \{r_t\}_{t\in\mathbb{R}}\big)$ is a* weakly stable dynamics *if the following conditions are true:*

(1) *β is a positive definite symmetric form. We equip $\mathcal{Y}$ with the norm $\|y\|_{\mathrm{en}} := (y \cdot \beta y)^{\frac{1}{2}}$. We denote by $\mathcal{Y}_{\mathrm{en}}$ the completion of $\mathcal{Y}$ w.r.t. this norm.*
(2) *$\mathbb{R} \ni t \mapsto r_t \in Sp(\mathcal{Y})$ is a strongly continuous group of bounded operators. Thus, we can extend r_t to a strongly continuous group on $\mathcal{Y}_{\mathrm{en}}$ and define its generator a, so that $r_t = \mathrm{e}^{ta}$.*
(3) *$\mathrm{Ker}\, a = \{0\}$, or equivalently, $\bigcap\limits_{t\in\mathbb{R}} \mathrm{Ker}(r_t - 1\!\!1) = \{0\}$.*
(4) *$\mathcal{Y} \subset \mathrm{Dom}\, a$ and $y_1 \cdot \beta y_2 = y_1 \cdot \omega a y_2, \quad y_1, y_2 \in \mathcal{Y}$.*

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If, in addition, ω is bounded for the topology given by β, so that it can be extended to the whole $\mathcal{Y}_{\mathrm{en}}$, we will say that the dynamics is strongly stable. *In this case $(\mathcal{Y}_{\mathrm{en}}, \omega)$ is a symplectic space.*

Note that β has two roles: it endows $\mathcal{Y}$ with a topology and it is the Hamiltonian for r_t.

Theorem 18.5 *Let $\left(\mathcal{Y}, \omega, \beta, \{r_t\}_{t\in\mathbb{R}}\right)$ be a weakly stable dynamics. Then*

(1) *r_t are orthogonal transformations on the real Hilbert space $\mathcal{Y}_{\mathrm{en}}$.*
(2) *a is anti-self-adjoint and* $\mathrm{Ker}\, a = \{0\}$.
(3) *The polar decomposition*

$$a =: |a|\mathrm{j} = \mathrm{j}|a|$$

defines a Kähler anti-involution j *and a self-adjoint operator $|a| > 0$ on $\mathcal{Y}_{\mathrm{en}}$.*
(4) *The dynamics is strongly stable iff $|a| \geq C$ for some $C > 0$.*

Recall that, given an operator $|a| > 0$ on $\mathcal{Y}_{\mathrm{en}}$, we can define a scale of Hilbert spaces $|a|^s\mathcal{Y}_{\mathrm{en}}$ (see Subsect. 2.3.4). Then r_t and j are bounded on $\mathcal{Y}_{\mathrm{en}} \cap |a|^s\mathcal{Y}_{\mathrm{en}}$ for the norm of $|a|^s\mathcal{Y}_{\mathrm{en}}$. Let $r_{s,t}$ and j_s denote their extensions. Similarly, a and $|a|$ are closable on $\mathcal{Y}_{\mathrm{en}} \cap |a|^s\mathcal{Y}_{\mathrm{en}}$ for the norm $|a|^s\mathcal{Y}_{\mathrm{en}}$. Let a_s, $|a|_s$ denote their closures. Clearly, for any s, $a_s = |a|_s\mathrm{j}_s = \mathrm{j}_s|a|_s$ is the polar decomposition, j_s is an orthogonal anti-involution and $r_{s,t} = \mathrm{e}^{ta_s}$ is an orthogonal one-parameter group.

Let $\cdot_s$ denote the natural scalar product on $|a|^s\mathcal{Y}_{\mathrm{en}}$. Let us express the scalar product and the symplectic form in terms of β:

$$\begin{aligned} y_1 \cdot_s y_2 &= y_1\beta|a|^{-2s}y_2 = \left(|a|^{-2s}y_1\right)\cdot\beta y_2, \\ y_1\cdot\omega y_2 &= y_1\cdot\beta a^{-1}y_2 = (a^{-1}y_1)\cdot\beta y_2. \end{aligned} \tag{18.3}$$

Note that the symplectic form does not need to be defined everywhere.

Of particular interest for us is the case $s = \frac{1}{2}$, for which we introduce the notation $\mathcal{Y}_{\mathrm{dyn}} := |a|^{\frac{1}{2}}\mathcal{Y}_{\mathrm{en}}$. In what follows we drop the subscript $s = \frac{1}{2}$ from $r_{s,t}$, j_s, $\cdot_s$, a_s and $|a|_s$.

Proposition 18.6 *$\mathcal{Y}_{\mathrm{dyn}}$ equipped with $(\cdot, \omega, \mathrm{j})$ is a complete Kähler space.*

Proof Setting $s = \frac{1}{2}$ in (18.3), we obtain

$$y_1 \cdot y_2 = y_1\cdot\beta|a|^{-1}y_2 = y_1\cdot\omega a|a|^{-1}y_2 = y_1\cdot\omega\mathrm{j}y_2. \qquad \square$$

Fock quantization of symplectic dynamics

Until the end of this subsection we drop the subscript $_{\mathrm{dyn}}$ from $\mathcal{Y}_{\mathrm{dyn}}$. Let $\mathcal{Z}$ be the holomorphic subspace of $\mathbb{C}\mathcal{Y}$ for the Kähler anti-involution j constructed in Thm. 18.5.

Clearly, $|a|$ commutes with j, hence its complexification $|a|_{\mathbb{C}}$ preserves $\mathcal{Z}$. We set $h := |a|_{\mathbb{C}}\big|_{\mathcal{Z}}$. Note that $h > 0$ and $a_{\mathbb{C}} = \mathrm{i}\begin{bmatrix} h & 0 \\ 0 & -\overline{h} \end{bmatrix}$, if we use the identification

$\mathbb{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$. Likewise, $(r_t)_\mathbb{C} = (\mathrm{e}^{t\mathrm{j}|a|})_\mathbb{C}$ preserves $\mathcal{Z}$, and we have

$$(r_t)_\mathbb{C}\big|_\mathcal{Z} = \mathrm{e}^{\mathrm{i}th}.$$

For $y \in \mathcal{Y}$, define the field operators

$$\phi(y) := a^*\left(\frac{\mathbb{1} - \mathrm{ij}}{2}y\right) + a\left(\frac{\mathbb{1} - \mathrm{ij}}{2}y\right).$$

Then,

$$\mathcal{Y} \ni y \mapsto \mathrm{e}^{\mathrm{i}\phi(y)} \in U\big(\Gamma_\mathrm{s}(\mathcal{Z})\big) \tag{18.4}$$

is a Fock CCR representation. Introduce the positive operator $H := \mathrm{d}\Gamma(h)$ on $\Gamma_\mathrm{s}(\mathcal{Z})$. We have

$$\mathrm{e}^{\mathrm{i}tH}\phi(y)\mathrm{e}^{-\mathrm{i}tH} = \phi(r_t y). \tag{18.5}$$

Definition 18.7 *(18.4) is called the* positive energy Fock quantization *of the weakly stable dynamics* $\{r_t\}_{t\in\mathbb{R}}$. *For any* $y \in \mathcal{Y}$, *the corresponding* time t phase space field *is defined as*

$$\phi_t(y) := \phi(r_{-t}y).$$

Quantizing symplectic dynamics with the (classical) Hamiltonian that is not bounded below is in general more difficult. Even if it is possible, the corresponding quantum Hamiltonian will not be bounded from below. There are some situations in physics when non-positive Hamiltonians arise. An example of such situations is the Klein–Gordon field in the space-time describing a rotating black hole, where the phenomenon of *super-radiance* appears; see Gibbons (1975).

Criterion for a weakly stable symplectic dynamics

In practice, our starting point for quantization of a symplectic dynamics can be somewhat different from that described in Def. 18.4. In this subsection we describe a more general framework that leads to a stable dynamics.

Suppose that the symplectic space $\mathcal{Y}$ is equipped with a Hilbertian topology given by a norm $\|\cdot\|$ such that the symplectic form ω is bounded. Let $\{r_t\}_{t\in\mathbb{R}}$ be a strongly continuous symplectic dynamics. Again, we denote its generator by a, so that $r_t = \mathrm{e}^{ta}$. It is easy to see that

$$y_1 \cdot \omega a y_2 = -(a y_1) \cdot \omega y_2, \quad y_1, y_2 \in \mathrm{Dom}\, a.$$

Hence,

$$y_1 \cdot \beta y_2 := y_1 \cdot \omega a y_2, \quad y_1, y_2 \in \mathrm{Dom}\, a.$$

defines a symmetric quadratic form. Let us assume that there exists $c > 0$ such that

$$y \cdot \beta y \geq c\|y\|^2, \quad y \in \mathrm{Dom}\, a. \tag{18.6}$$

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Lemma 18.8 *Consider the Hilbert space* $\mathcal{Y}_{\mathrm{en}}$ *obtained by completing* $\mathrm{Dom}\, a$ *in the norm* $\|y\|_{\mathrm{en}} = (y \cdot \beta y)^{\frac{1}{2}}$. *Then* $\mathcal{Y}_{\mathrm{en}}$ *can be viewed as a dense subspace of* $\mathcal{Y}$. *Moreover,* r_t *preserves* $\mathcal{Y}_{\mathrm{en}}$ *and is a strongly continuous isometric group on* $\mathcal{Y}_{\mathrm{en}}$.

Proof (18.6) guarantees that $\mathcal{Y}_{\mathrm{en}}$ can be considered as a subspace of $\mathcal{Y}$.

Let $y \in \mathrm{Dom}\, a$. Then,

$$\begin{aligned} y{\cdot}\beta y = y{\cdot}\omega a y &= (r_t y){\cdot}\omega r_t a y \\ &= (r_t y){\cdot}\omega a r_t y = (r_t y){\cdot}\beta r_t y. \end{aligned}$$

Thus r_t is isometric in $\|\cdot\|_{\mathrm{en}}$ on $\mathrm{Dom}\, a$ (and hence on $\mathcal{Y}_{\mathrm{en}}$). Moreover,

$$(r_t y - y){\cdot}\beta(r_t y - y) = (r_t y - y){\cdot}\omega(r_t a y - a y) \to 0.$$

Thus r_t is strongly continuous in $\|\cdot\|_{\mathrm{en}}$ on $\mathrm{Dom}\, a$ (and hence on $\mathcal{Y}_{\mathrm{en}}$). □

Let a_{en} denote the generator of the dynamics $\{r_t\}_{t\in\mathbb{R}}$ restricted to $\mathcal{Y}_{\mathrm{en}}$. Clearly, $a_{\mathrm{en}} \subset a$.

The following theorem is easy:

Theorem 18.9 *Under the assumptions of this subsection, the space* $\mathrm{Dom}\, a_{\mathrm{en}}$ *equipped with* ω, β *and* $\{r_t\}_{t\in\mathbb{R}}$ *restricted to* $\mathrm{Dom}\, a_{\mathrm{en}}$ *satisfy the conditions of a weakly stable dynamics of Def. 18.4.*

18.1.2 Neutral fermionic systems

Let $(\mathcal{Y}, \nu)$ be a real Hilbert space. We think of it as the dual phase space of a fermionic system. A strongly continuous one-parameter group $\mathbb{R} \ni t \mapsto r_t \in O(\mathcal{Y})$ will be called an *orthogonal dynamics*. We view it as a classical dynamical system.

Algebraic quantization of an orthogonal dynamics

We choose $\mathrm{CAR}^{C^*}(\mathcal{Y})$ as the field algebra of our system. It is equipped with the one-parameter group of Bogoliubov automorphisms $\{\hat{r}_t\}_{t\in\mathbb{R}}$, defined by

$$\hat{r}_t\big(\phi(y)\big) = \phi(r_t y), \quad y \in \mathcal{Y}.$$

In quantum physics only even fermionic operators are observable. Therefore, it is natural to use the even sub-algebra $\mathrm{CAR}_0^{C^*}(\mathcal{Y})$ as the observable algebra.

Kähler structure for a non-degenerate orthogonal dynamics

Let a be the generator of r_t, so that $r_t = \mathrm{e}^{ta}$ and $a = -a^{\#}$.

Definition 18.10 *We say that the dynamics* $t \mapsto r_t \in O(\mathcal{Y})$ *is* non-degenerate *if*

$$\mathrm{Ker}\, a = \{0\}, \quad \textit{or equivalently} \ \bigcap_{t\in\mathbb{R}} \mathrm{Ker}(r_t - \mathbb{1}) = \{0\}. \tag{18.7}$$

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Theorem 18.11 *The polar decomposition*

$$a =: |a|\mathrm{j} = \mathrm{j}|a|$$

defines an operator $|a| > 0$ *and a Kähler anti-involution* j *on* $\mathcal{Y}$.

Fock quantization of orthogonal dynamics

Let $\mathcal{Z}$ be the holomorphic subspace of $\mathbb{C}\mathcal{Y}$ for the Kähler anti-involution j.

The operator $|a|$ commutes with j. Hence, its complexification $|a|_{\mathbb{C}}$ preserves $\mathcal{Z}$. We set $h := |a|_{\mathbb{C}}\big|_{\mathcal{Z}}$. Note that $h > 0$ and $a_{\mathbb{C}} = \mathrm{i}\begin{bmatrix} h & 0 \\ 0 & -\overline{h} \end{bmatrix}$. Likewise, $(r_t)_{\mathbb{C}} = (\mathrm{e}^{t\mathrm{j}|a|})_{\mathbb{C}}$ preserves $\mathcal{Z}$, and we have

$$(r_t)_{\mathbb{C}}\big|_{\mathcal{Z}} = \mathrm{e}^{\mathrm{i}th}.$$

Consider the Fock representation associated with the Kähler anti-involution j

$$\mathcal{Y} \ni y \mapsto \phi(y) := a^*\left(\frac{\mathbb{1} - \mathrm{ij}}{2}y\right) + a\left(\frac{\mathbb{1} - \mathrm{ij}}{2}y\right) \in B_{\mathrm{h}}\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big), \tag{18.8}$$

and the positive operator $H := \mathrm{d}\Gamma(h)$ on $\Gamma_{\mathrm{a}}(\mathcal{Z})$. We have

$$\mathrm{e}^{\mathrm{i}tH}\phi(y)\mathrm{e}^{-\mathrm{i}tH} = \phi(r_t y). \tag{18.9}$$

Definition 18.12 *(18.8) is called the* positive energy Fock quantization *of the dynamics* $\{r_t\}_{t\in\mathbb{R}}$. *For any* $y \in \mathcal{Y}$, *the corresponding* time t field *is defined as*

$$\phi_t(y) := \phi(r_{-t}y).$$

18.1.3 Time reversal in neutral systems

Let $(\mathcal{Y}, \omega)$ be a symplectic space in the bosonic case, or let $(\mathcal{Y}, \nu)$ be a real Hilbert space in the fermionic case.

Time reversal and its algebraic quantization

Definition 18.13 *A map* $\tau \in L(\mathcal{Y})$ *is a* time reversal *if*

(1) τ *is anti-symplectic and* $\tau^2 = \mathbb{1}$ *in the bosonic case,*
(2) τ *is orthogonal and* $\tau^2 = \mathbb{1}$ *or* $\tau^2 = -\mathbb{1}$ *in the fermionic case.*

Let us fix a time reversal τ. Let us quantize τ on the algebraic level.

Proposition 18.14 (1) *In the bosonic case, there exists a unique anti-linear* $*$*-homomorphism* $\hat{\tau}$ *of the algebra* $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ *such that* $\hat{\tau}\big(W(y)\big) := W(\tau y)$. $\hat{\tau}^2$ *is the identity.*

(2) *In the fermionic case, there exists a unique anti-linear* $*$*-homomorphism* $\hat{\tau}$ *of the algebra* $\mathrm{CAR}^{C^*}(\mathcal{Y})$ *such that* $\hat{\tau}\big(\phi(y)\big) := \phi(\tau y)$. $\hat{\tau}^2$ *is the identity on* $\mathrm{CAR}_0^{C^*}(\mathcal{Y})$ *(the even algebra).*

Definition 18.15 $\hat{\tau}$ *defined in Prop. 18.14 is called the* algebraic time reversal.

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Suppose that $\{r_t\}_{t\in\mathbb{R}}$ is a dynamics, where $r_t \in Sp(\mathcal{Y})$ in the bosonic case and $r_t \in O(\mathcal{Y})$ in the fermionic case.

Definition 18.16 *We say that the dynamics* $\{r_t\}_{t\in\mathbb{R}}$ *is* time reversal invariant *if*

$$\tau r_t = r_{-t}\tau. \tag{18.10}$$

Clearly, on the algebraic level (18.10) implies $\hat{\tau}\hat{r}_t = \hat{r}_{-t}\hat{\tau}$.

Fock quantization of time reversal

Let $\{r_t\}_{t\in\mathbb{R}}$ be a time reversal invariant dynamics. In the bosonic case we assume that the dynamics is weakly stable; in the fermionic case we assume that it is non-degenerate. In both cases we can introduce a, j, h. We have

$$\tau a = -a\tau, \quad \tau \mathrm{j} = -\mathrm{j}\tau, \quad \tau|a| = |a|\tau.$$

Note that the anti-linear extension of τ, denoted $\tau_{\overline{\mathbb{C}}}$, preserves $\mathcal{Z}$.

Definition 18.17 *We write* $\tau_{\mathcal{Z}} := \tau_{\overline{\mathbb{C}}}\big|_{\mathcal{Z}}$.

Clearly, $\tau_{\mathcal{Z}}$ is anti-unitary and $\tau_{\mathcal{Z}} h = h\tau_{\mathcal{Z}}$. Moreover,

(1) $\tau_{\mathcal{Z}}^2 = \mathbb{1}$ in the bosonic case,
(2) $\tau_{\mathcal{Z}}^2 = \mathbb{1}$ or $\tau_{\mathcal{Z}}^2 = -\mathbb{1}$ in the fermionic case.

Consider the positive energy quantization of the dynamics on the Fock space $\Gamma_{\mathrm{s/a}}(\mathcal{Z})$.

Definition 18.18 *The* Fock quantization of time reversal *is defined as the anti-unitary map* $T := \Gamma(\tau_{\mathcal{Z}})$.

We have $THT^{-1} = H$, $T\mathrm{e}^{\mathrm{i}tH}T^{-1} = \mathrm{e}^{-\mathrm{i}tH}$. T implements $\hat{\tau}$ and

$$T\phi(y)T^{-1} = \phi(\tau y), \quad y \in \mathcal{Y}.$$

Recall that I denotes the parity operator defined in (3.10). We have

(1) $T^2 = \mathbb{1}$ in the bosonic case,
(2) $T^2 = \mathbb{1}$ or $T^2 = I$ in the fermionic case.

18.2 Charged systems

In the charged formalism, the classical system is described by a complex vector space $\mathcal{Y}$.

In the bosonic case, it is equipped with an anti-Hermitian form $(\cdot|\omega\cdot)$ – we say that it is a *charged symplectic space*. The dynamics $\{r_t\}_{t\in\mathbb{R}}$ describing the time evolution is assumed to preserve $(\cdot|\omega\cdot)$ – we say that r_t is *charged symplectic*.

In the fermionic case it is equipped with a positive scalar product $(\cdot|\cdot)$. Without decreasing the generality we can assume that it is complete – $\mathcal{Y}$ is a complex Hilbert space. The dynamics $\{r_t\}_{t\in\mathbb{R}}$ preserves $(\cdot|\cdot)$ – it is unitary.

By a positive energy quantization of a charged classical system we mean a charged CCR or CAR representation $\mathcal{Y} \ni y \mapsto \psi^*(y)$ on a Hilbert space $\mathcal{H}$ and a positive self-adjoint operator H on $\mathcal{H}$ such that $\mathrm{e}^{\mathrm{i}tH}$ implements r_t.

The complex structure of $\mathcal{Y}$ is responsible for the action of a $U(1)$ symmetry $\{\mathrm{e}^{\mathrm{i}\theta}\}_{\theta\in[0,2\pi]}$. On the level of the Fock representation it is implemented by $\mathrm{e}^{\mathrm{i}\theta Q}$, where Q is called the *charge operator*.

Recall that charged systems can be viewed as special cases of neutral systems equipped in addition with a certain symmetry. As discussed in Subsect. 1.3.11, a homomorphism $U(1) \ni \theta \mapsto u_\theta \in L(\mathcal{Y})$ on a real space $\mathcal{Y}$ is called a $U(1)$ *symmetry of charge 1* if there exists an anti-involution j_{ch} such that $u_\theta = \cos\theta \mathbb{1} + \sin\theta \mathrm{j}_{\mathrm{ch}}$. Assume that it preserves the symplectic, resp. Euclidean form ω, resp. ν, which is equivalent to saying that j_{ch} is pseudo-Kähler, resp. Kähler. Assume also that the dynamics $\{r_t\}_{t\in\mathbb{R}}$ commutes with this symmetry, which is equivalent to saying that j_{ch} commutes with r_t. If we equip the space with the complex structure given by j_{ch}, then the symmetry u_θ becomes just the multiplication by $\mathrm{e}^{\mathrm{i}\theta}$. It is then natural to replace the real bilinear forms ω, resp. ν by the closely related sesquilinear forms $(\cdot|\omega\cdot)$, resp. $(\cdot|\cdot)$. The invariance of the dynamics w.r.t. the charge symmetry is now expressed by the fact that the dynamics is complex linear. See Subsects. 8.2.5 and 12.1.7 for further details.

In this section we describe in abstract terms the charged formalism. At the end of the section, we discuss the *charge reversal* and the *time reversal* for charged systems.

We will use θ as the generic variable in $U(1) = \mathbb{R}/2\pi\mathbb{Z}$.

18.2.1 Charged bosonic systems

Let $\big(\mathcal{Y}, (\cdot|\omega\cdot)\big)$ be a charged symplectic space. Let $\mathbb{R} \ni t \mapsto r_t \in ChSp(\mathcal{Y})$ be a charged symplectic dynamics.

Algebraic quantization of a charged symplectic dynamics

By taking $\mathrm{Re}(y_1|\omega y_2)$ we can view $\mathcal{Y}_\mathbb{R}$ as a real symplectic space. We choose $\mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y}_\mathbb{R})$ as the field algebra of our system. This algebra is generated (in the sense described in Subsect. 8.3.4) by the Weyl elements $\mathrm{e}^{\mathrm{i}\psi(y)+\mathrm{i}\psi^*(y)}$, $y \in \mathcal{Y}$, satisfying the relations

$$\mathrm{e}^{\mathrm{i}\psi(y_1)+\mathrm{i}\psi^*(y_1)}\mathrm{e}^{\mathrm{i}\psi(y_2)+\mathrm{i}\psi^*(y_2)} = \mathrm{e}^{-\mathrm{i}\mathrm{Re}(y_1|\omega y_2)}\mathrm{e}^{\mathrm{i}\psi(y_1+y_2)+\mathrm{i}\psi^*(y_1+y_2)}.$$

We equip $\mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y}_\mathbb{R})$ with the automorphism groups $\{\widehat{\mathrm{e}^{\mathrm{i}\theta}}\}_{\theta\in U(1)}$ and $\{\hat{r}_t\}_{t\in\mathbb{R}}$ defined by

$$\begin{aligned}\widehat{\mathrm{e}^{\mathrm{i}\theta}}\big(\mathrm{e}^{\mathrm{i}\psi(y)+\mathrm{i}\psi^*(y)}\big) &= \mathrm{e}^{\mathrm{i}\psi(\mathrm{e}^{\mathrm{i}\theta}y)+\mathrm{i}\psi^*(\mathrm{e}^{\mathrm{i}\theta}y)},\\ \hat{r}_t\big(\mathrm{e}^{\mathrm{i}\psi(y)+\mathrm{i}\psi^*(y)}\big) &= \mathrm{e}^{\mathrm{i}\psi(r_t y)+\mathrm{i}\psi^*(r_t y)}, \qquad y \in \mathcal{Y}.\end{aligned}$$

For the observable algebra it is natural to choose the so-called *gauge-invariant regular CCR algebra* $\mathrm{CCR}^{\mathrm{reg}}_{\mathrm{gi}}(\mathcal{Y})$, which is defined as the set of elements of $\mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y}_{\mathbb{R}})$ fixed by $\widehat{\mathrm{e}^{\mathrm{i}\theta}}$. Note that $\mathrm{CCR}^{\mathrm{reg}}_{\mathrm{gi}}(\mathcal{Y})$ is contained in the even algebra $\mathrm{CCR}^{\mathrm{reg}}_{0}(\mathcal{Y}_{\mathbb{R}})$ and is preserved by the dynamics $\hat{r}_t$.

Remark 18.19 *In this subsection, for the field algebra of our system we have preferred to choose* $\mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y}_{\mathbb{R}})$ *instead of* $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}_{\mathbb{R}})$*. This is motivated by the fact that the only element left invariant by the gauge symmetry* $\widehat{\mathrm{e}^{\mathrm{i}\theta}}$ *in* $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}_{\mathbb{R}})$ *is* $\mathbb{1}$*, whereas in the case of* $\mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y}_{\mathbb{R}})$ *we obtain a large gauge-invariant algebra.*

Fock quantization of a charged symplectic dynamics

The concept of stability of dynamics in the charged case is analogous to the neutral case.

Definition 18.20 *We say that* $\big(\mathcal{Y}, (\cdot|\omega\cdot), (\cdot|\beta\cdot), \{r_t\}_{t\in\mathbb{R}}\big)$ *is a* weakly stable dynamics *if the following conditions are true:*

(1) $(\cdot|\beta\cdot)$ *is a positive definite sesquilinear form. We equip* $\mathcal{Y}$ *with the norm* $\|y\|_{\mathrm{en}} := (y|\beta y)^{\frac{1}{2}}$*. We denote by* $\mathcal{Y}_{\mathrm{en}}$ *the completion of* $\mathcal{Y}$ *w.r.t. this norm.*
(2) *We assume that* $\{r_t\}_{t\in\mathbb{R}}$ *is a strongly continuous group of bounded operators on* $\mathcal{Y}$*. Thus we can extend* r_t *to a strongly continuous group on* $\mathcal{Y}_{\mathrm{en}}$ *and define its generator* $\mathrm{i}b$*, so that* $r_t = \mathrm{e}^{\mathrm{i}tb}$*.*
(3) $\mathrm{Ker}\, b = \{0\}$*, or equivalently,* $\bigcap\limits_{t\in\mathbb{R}} \mathrm{Ker}(r_t - \mathbb{1}) = \{0\}$*.*
(4) *We assume that* $\mathcal{Y} \subset \mathrm{Dom}\, b$ *and*

$$(y_1|\beta y_2) := \mathrm{i}(y_1|\omega b y_2), \quad y_1, y_2 \in \mathcal{Y}. \tag{18.11}$$

If in addition ω *is bounded for the topology given by* β*, so that* $(\cdot|\omega\cdot)$ *can be extended to the whole* $\mathcal{Y}_{\mathrm{en}}$*, we will say that the dynamics is* strongly stable*.*

Theorem 18.21 *Let* $\big(\mathcal{Y}, (\cdot|\omega\cdot), (\cdot|\beta\cdot), \{r_t\}_{t\in\mathbb{R}}\big)$ *be a weakly stable dynamics. Then*

(1) r_t *are unitary transformations on the Hilbert space* $\mathcal{Y}_{\mathrm{en}}$*,*
(2) b *is self-adjoint and* $\mathrm{Ker}\, b = \{0\}$*.*

Set $q := \mathrm{sgn}(b)$ and $\mathrm{j} := \mathrm{i}\,\mathrm{sgn}(b)$. Clearly, $|b|$ is positive and $r_t = \mathrm{e}^{t\mathrm{j}|b|}$.

Set $\mathcal{Y}_{\mathrm{dyn}} := |b|^{\frac{1}{2}}\mathcal{Y}_{\mathrm{en}}$. As in Subsect. 18.1.1, we can view r_t, j, b and $|b|$ as defined on $\mathcal{Y}_{\mathrm{dyn}}$. In what follows we drop the subscript dyn from $\mathcal{Y}_{\mathrm{dyn}}$.

Let $\mathbb{1}_{\pm} := \mathbb{1}_{]0,\infty[}(\pm b) = \mathbb{1}_{\{\pm 1\}}(q)$, $\mathcal{Y}_{\pm} := \mathrm{Ran}\,\mathbb{1}_{\pm}$. Let $\mathcal{Z}$ denote the space $\mathcal{Y}$ equipped with the complex structure given by j. (In other words, $\mathcal{Z} := \mathcal{Y}_{+} \oplus \overline{\mathcal{Y}}_{-}$.)

The operators $|b|$, q and b preserve $\mathcal{Y}_{\pm}$. Hence, they can be viewed as complex linear operators on $\mathcal{Z}$ as well, in which case they will be denoted h, $q_{\mathcal{Z}}$ and $b_{\mathcal{Z}}$.

Consider the space $\Gamma_s(\mathcal{Z})$. For $y \in \mathcal{Y}$, let us introduce the *charged fields* on $\mathcal{Y}$, which are closed operators on $\Gamma_s(\mathcal{Z})$ defined by

$$\begin{aligned} \psi^*(y) &= a^*\left(\mathbb{1}_+ y\right) + a\left(\overline{\mathbb{1}_- y}\right), \\ \psi(y) &= a\left(\mathbb{1}_+ y\right) + a^*\left(\overline{\mathbb{1}_- y}\right). \end{aligned} \tag{18.12}$$

We obtain a charged CCR representation

$$\mathcal{Y} \ni y \mapsto \psi^*(y) \in Cl\big(\Gamma_s(\mathcal{Z})\big). \tag{18.13}$$

Define the self-adjoint operators on $\Gamma_s(\mathcal{Z})$

$$H := \mathrm{d}\Gamma(h), \quad Q := \mathrm{d}\Gamma(q_{\mathcal{Z}}).$$

Clearly,

$$\mathrm{e}^{\mathrm{i}tH}\psi(y)\mathrm{e}^{-\mathrm{i}tH} = \psi(\mathrm{e}^{\mathrm{i}tb}y), \quad \mathrm{e}^{\mathrm{i}\theta Q}\psi(y)\mathrm{e}^{-\mathrm{i}\theta Q} = \psi(\mathrm{e}^{\mathrm{i}\theta}y), \quad y \in \mathcal{Y}.$$

Definition 18.22 *(18.13) is called the* positive energy Fock quantization for the dynamics $\{r_t\}_{t\in\mathbb{R}}$. *For any $y \in \mathcal{Y}$, the corresponding* time t field *is defined as*

$$\psi_t(y) := \psi(r_{-t}y).$$

18.2.2 Charged fermionic systems

Let $\big(\mathcal{Y}, (\cdot|\cdot)\big)$ be a complex Hilbert space describing a charged fermionic system. A strongly continuous one-parameter group $\mathbb{R} \ni t \mapsto r_t \in U(\mathcal{Y})$ will be called a *unitary dynamics.*

Algebraic quantization of a unitary dynamics

Clearly, by taking the real scalar product $y_1 \cdot \nu y_2 := \frac{1}{2}\mathrm{Re}(y_1|y_2)$ we can view $\mathcal{Y}_\mathbb{R}$ as a real Hilbert space. We can associate with our system the field algebra $\mathrm{CAR}^{C^*}(\mathcal{Y}_\mathbb{R})$ with distinguished elements $\psi(y)$, $y \in \mathcal{Y}$. We equip it with the automorphism group $\{\widehat{\mathrm{e}^{\mathrm{i}\theta}}\}_{\theta\in U(1)}$ and $\{\hat{r}_t\}_{t\in\mathbb{R}}$ defined by

$$\begin{aligned} \widehat{\mathrm{e}^{\mathrm{i}\theta}}\big(\psi(y)\big) &= \psi(\mathrm{e}^{\mathrm{i}\theta}y), \\ \hat{r}_t\big(\psi(y)\big) &= \psi(r_t y), \quad y \in \mathcal{Y}. \end{aligned}$$

Similarly to the bosonic case, for the observable algebra we choose the so-called *gauge-invariant CAR algebra* $\mathrm{CAR}_{\mathrm{gi}}^{C^*}(\mathcal{Y})$, which is defined as the set of elements of $\mathrm{CAR}^{C^*}(\mathcal{Y}_\mathbb{R})$ fixed by $\widehat{\mathrm{e}^{\mathrm{i}\theta}}$. Note that $\mathrm{CAR}_{\mathrm{gi}}^{C^*}(\mathcal{Y})$ is contained in the even algebra $\mathrm{CAR}_0^{C^*}(\mathcal{Y}_\mathbb{R})$ and is preserved by the dynamics $\hat{r}_t$.

Fock quantization of a unitary dynamics

Let b be the self-adjoint generator of $\{r_t\}_{t\in\mathbb{R}}$, so that $r_t = \mathrm{e}^{\mathrm{i}tb}$.

Definition 18.23 *We say that the dynamics $t \mapsto r_t \in U(\mathcal{Y})$ is* non-degenerate *if*

$$\mathrm{Ker}\, b = \{0\}, \ \textit{or equivalently} \ \bigcap_{t\in\mathbb{R}} \mathrm{Ker}(r_t - \mathbb{1}) = \{0\}. \tag{18.14}$$

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Set $q := \mathrm{sgn}(b)$ and $\mathrm{j} := \mathrm{i}\,\mathrm{sgn}(b)$. Clearly, $|b|$ is positive, and $r_t = \mathrm{e}^{t\mathrm{j}|b|}$. Let $\mathbb{1}_\pm := \mathbb{1}_{]0,\infty[}(\pm b) = \mathbb{1}_{\{\pm 1\}}(q)$, $\mathcal{Y}_\pm := \mathrm{Ran}\,\mathbb{1}_\pm$. Let $\mathcal{Z}$ denote the space $\mathcal{Y}$ equipped with the complex structure given by j. (In other words, $\mathcal{Z} := \mathcal{Y}_+ \oplus \overline{\mathcal{Y}}_-$).

The operators $|b|$, q and b preserve $\mathcal{Y}_\pm$. Hence, they can also be viewed as complex linear operators on $\mathcal{Z}$ as well, in which case they will be denoted h, $q_{\mathcal{Z}}$ and $b_{\mathcal{Z}}$.

Consider the space $\Gamma_{\mathrm{a}}(\mathcal{Z})$. For $y \in \mathcal{Y}$, let us introduce the *charged fields* on $\mathcal{Y}$, which are closed operators on $\Gamma_{\mathrm{a}}(\mathcal{Z})$ defined by

$$\psi^*(y) = a^*(\mathbb{1}_+ y) + a\left(\overline{\mathbb{1}_- y}\right), \tag{18.15}$$

$$\psi(y) = a(\mathbb{1}_+ y) + a^*\left(\overline{\mathbb{1}_- y}\right). \tag{18.16}$$

We obtain a charged CAR representation

$$\mathcal{Y} \ni y \mapsto \psi^*(y) \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big). \tag{18.17}$$

Define the self-adjoint operators on $\Gamma_{\mathrm{a}}(\mathcal{Z})$

$$H := \mathrm{d}\Gamma(h), \quad Q := \mathrm{d}\Gamma(q_{\mathcal{Z}}).$$

Clearly,

$$\mathrm{e}^{\mathrm{i}tH}\psi(y)\mathrm{e}^{-\mathrm{i}tH} = \psi(\mathrm{e}^{\mathrm{i}tb}y), \quad \mathrm{e}^{\mathrm{i}\theta Q}\psi(y)\mathrm{e}^{-\mathrm{i}\theta Q} = \psi(\mathrm{e}^{\mathrm{i}\theta}y), \quad y \in \mathcal{Y}.$$

Definition 18.24 *(18.17) is called the* positive energy Fock quantization of the dynamics $\{r_t\}_{t\in\mathbb{R}}$. *For any $y \in \mathcal{Y}$, the corresponding* time t phase space field *is defined as*

$$\psi_t(y) := \psi(r_{-t}y).$$

18.2.3 Charge reversal

Let $\big(\mathcal{Y}, (\cdot|\omega\cdot)\big)$ be a charged symplectic space in the bosonic case, or let $\big(\mathcal{Y}, (\cdot|\cdot)\big)$ be a complex Hilbert space in the fermionic case.

Charge reversal and its algebraic quantization

Definition 18.25 $\chi \in L(\mathcal{Y}_{\mathbb{R}})$ *is a* charge reversal *if* $\chi^2 = \mathbb{1}$ *or* $\chi^2 = -\mathbb{1}$, *and*

(1) $(\chi y_1|\omega\chi y_2) = \overline{(y_1|\omega y_2)}$ *(χ is anti-charged symplectic) in the bosonic case;*
(2) $(\chi y_1|\chi y_2) = \overline{(y_1|y_2)}$ *(χ is anti-unitary) in the fermionic case.*

Let us fix a charge reversal χ. Consider now its algebraic quantization.

Proposition 18.26 (1) *In the bosonic case, there exists a unique $*$-automorphism $\hat{\chi}$ of* $\mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y}_{\mathbb{R}})$ *such that*

$$\hat{\chi}\big(\mathrm{e}^{\mathrm{i}\psi(y)+\mathrm{i}\psi^*(y)}\big) = \mathrm{e}^{\mathrm{i}\psi(\chi y)+\mathrm{i}\psi^*(\chi y)}.$$

(2) *In the fermionic case, there exists a unique $*$-automorphism $\hat{\chi}$ of* $\mathrm{CAR}^{C^*}(\mathcal{Y}_{\mathbb{R}})$ *such that*

$$\hat{\chi}\big(\psi^*(y)\big) = \psi(\chi y).$$

In both the bosonic and the fermionic case, $\hat{\chi}$ leaves invariant the gauge-invariant algebra and is involutive on it.

Definition 18.27 *$\hat{\chi}$ defined in Prop. 18.26 is called the* algebraic charge reversal*.*

Let us remark that whereas χ is anti-linear, $\hat{\chi}$ is linear.

Suppose that $\{r_t\}_{t\in\mathbb{R}}$ is a charged symplectic or unitary dynamics.

Definition 18.28 *We say that the dynamics is* invariant under the charge reversal χ *if*

$$\chi r_t = r_t \chi. \tag{18.18}$$

Similarly, if we have a group of symmetries $\{r_g\}_{g\in G}$ we say that it is invariant under the charge reversal χ *if $r_g\chi = \chi r_g$, $g \in G$.*

Clearly, on the algebraic level (18.18) implies $\hat{\chi}\hat{r}_t = \hat{r}_t\hat{\chi}$.

Fock quantization of charge reversal

Let $\{r_t\}_{t\in\mathbb{R}}$ be a charge reversal invariant dynamics. In the bosonic case assume that the dynamics is weakly stable. In the fermionic case assume it is non-degenerate. Let b, h, q etc. be constructed as before. In both the bosonic and the fermionic case it follows that

$$\chi|b| = |b|\chi, \quad \chi b = -b\chi, \quad \chi q = -q\chi, \quad \chi \mathrm{j} = \mathrm{j}\chi.$$

Definition 18.29 *We denote $\chi_{\mathcal{Z}}$ the map χ considered on $\mathcal{Z}$.*

Note that $\chi_{\mathcal{Z}}$ is unitary, unlike χ.

Definition 18.30 *The* Fock quantization of the charge reversal *is the unitary $C := \Gamma(\chi_{\mathcal{Z}})$.*

We have $CHC^{-1} = H$, $CQC^{-1} = -Q$. C implements $\hat{\chi}$ and

$$C\psi^*(y)C^{-1} = \psi(\chi y).$$

Note that $C^2 = \mathbb{1}$ or $C^2 = I$, where we recall that I is the parity operator.

Neutral subspace

Assume that $\chi^2 = \mathbb{1}$. Recall that we can define the spaces

$$\mathcal{Y}^{\pm\chi} := \{y \in \mathcal{Y} \ : \ y = \pm\chi y\}.$$

The dynamics and the symmetry group restrict to $\mathcal{Y}^{\chi}$ and $\mathcal{Y}^{-\chi}$.

Definition 18.31 *We will call $\mathcal{Y}^{\chi}$ the* neutral subspace of $\mathcal{Y}$. *(In the fermionic case, we will also call it the* Majorana subspace*.)*

Note that $\mathcal{Y} = \mathcal{Y}^\chi \oplus \mathrm{i}\mathcal{Y}^\chi$, hence the system can be viewed as a couple of neutral systems.

Let us describe the converse construction. Suppose that we have a neutral system $(\mathcal{Y}, \omega)$ or $(\mathcal{Y}, \nu)$ equipped with the dynamics $\{r_t\}_{t\in\mathbb{R}}$. We can extend it to a charged system as follows. We consider the complexified space $\mathbb{C}\mathcal{Y}$ equipped with the natural conjugation denoted by the "bar". We equip it with the anti-Hermitian form, resp. scalar product

$$\begin{aligned}(w_1|\omega w_2) &:= \overline{w}_1\cdot\omega w_2,\\ \text{or}\quad (w_1|w_2) &:= 2\overline{w}_1\cdot\nu w_2, \quad w_1, w_2 \in \mathbb{C}\mathcal{Y}.\end{aligned}$$

We extend the dynamics r_t to $(r_t)_\mathbb{C}$ on $\mathbb{C}\mathcal{Y}$. Clearly, $(r_t)_\mathbb{C}$ is a charged symplectic, resp. unitary dynamics with the charge reversal given by $\chi w := \overline{w}$, $w \in \mathbb{C}\mathcal{Y}$. It satisfies $\chi^2 = \mathbb{1}$. One gets back the original system by the restriction to the neutral subspace.

18.2.4 Time reversal in charged systems

Let $\big(\mathcal{Y}, (\cdot|\omega\cdot)\big)$ be a charged symplectic space in the bosonic case, or let $\big(\mathcal{Y}, (\cdot|\cdot)\big)$ be a complex Hilbert space in the fermionic case.

In the case of charged systems it is natural to consider two kinds of time reversal. The standard choice is an anti-linear symmetry considered by Wigner. The so-called Racah time reversal is actually historically older than the Wigner time reversal. It is linear and from a purely mathematical point of view may seem more natural.

Wigner time reversal and its algebraic quantization

Definition 18.32 *$\tau \in L(\mathcal{Y}_\mathbb{R})$ is a* Wigner time reversal *if $\tau^2 = \mathbb{1}$ or $\tau^2 = -\mathbb{1}$, and*

(1) *$(\tau y_1|\omega\tau y_2) = -\overline{(y_1|\omega y_2)}$ (τ is anti-charged anti-symplectic) in the bosonic case;*
(2) *$(\tau y_1|\tau y_2) = \overline{(y_1|y_2)}$ (τ is anti-unitary) in the fermionic case.*

Let us fix a Wigner time reversal τ.

Proposition 18.33 (1) *There exists a unique anti-linear $*$-automorphism $\hat\tau$ on the algebra $\mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y}_\mathbb{R})$ such that*

$$\hat\tau\Big(\mathrm{e}^{\mathrm{i}\psi(y)+\mathrm{i}\psi^*(y)}\Big) = \mathrm{e}^{-\mathrm{i}\psi(\tau y)-\mathrm{i}\psi^*(\tau y)}.$$

(2) *There exists a unique anti-linear $*$-automorphism $\hat\tau$ of the algebra $\mathrm{CAR}^{C^*}(\mathcal{Y}_\mathbb{R})$ such that*

$$\hat\tau\big(\psi(y)\big) = \psi(\tau y).$$

In both the bosonic and the fermionic case, $\hat\tau$ leaves invariant the gauge-invariant algebra and is involutive on it.

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Definition 18.34 *$\hat{\tau}$ defined in Prop. 18.33 is called the* algebraic Wigner time reversal*.*

Note that both τ and $\hat{\tau}$ are anti-linear.

Suppose that $\{r_t\}_{t\in\mathbb{R}}$ is a charged symplectic or unitary dynamics.

Definition 18.35 *We say that the dynamics is* invariant under the Wigner time reversal τ *if*

$$\tau r_t = r_{-t}\tau. \tag{18.19}$$

Clearly, on the algebraic level (18.19) implies $\hat{\tau}\hat{r}_t = \hat{r}_{-t}\hat{\tau}$.

Fock quantization of Wigner time reversal

Let $\{r_t\}_{t\in\mathbb{R}}$ be a Wigner time reversal dynamics. In the bosonic case assume that the dynamics is weakly stable. In the fermionic case assume it is non-degenerate. Let b, h, q etc. be constructed as before. In both the bosonic and the fermionic case it follows that

$$\tau|b| = |b|\tau, \quad \tau b = b\tau, \quad \tau q = q\tau, \quad \tau\mathrm{j} = -\mathrm{j}\tau.$$

Thus $\tau\mathcal{Y}_+ = \mathcal{Y}_+$, $\tau\mathcal{Y}_- = \mathcal{Y}_-$.

Definition 18.36 *Let $\tau_{\mathcal{Z}}$ denote τ considered on $\mathcal{Z}$.*

Note that $\tau_{\mathcal{Z}}$ is anti-unitary.

Definition 18.37 *The* Fock quantization of the Wigner time reversal *is given by the anti-unitary $T := \Gamma(\tau_{\mathcal{Z}})$.*

We have $THT^{-1} = H$, $T\mathrm{e}^{\mathrm{i}tH}T^{-1} = \mathrm{e}^{-\mathrm{i}tH}$, $TQT^{-1} = Q$, $T\mathrm{e}^{\mathrm{i}\theta Q}T^{-1} = \mathrm{e}^{-\mathrm{i}\theta Q}$. T implements $\hat{\tau}$ and

$$T\psi(y)T^{-1} = \psi(\tau y), \;\; T\psi^*(y)T^{-1} = \psi^*(\tau y), \quad y \in \mathcal{Y}.$$

Moreover, $T^2 = 1\!\!1$ or $T^2 = I$.

Racah time reversal

Definition 18.38 *$\kappa \in L(\mathcal{Y})$ is a* Racah time reversal *if $\kappa^2 = 1\!\!1$ or $\kappa^2 = -1\!\!1$, and*

(1) *$(\kappa y_1|\omega\kappa y_2) = -(y_1|\omega y_2)$ (κ is charged anti-symplectic) in the bosonic case;*
(2) *$(\kappa y_1|\kappa y_2) = (y_1|y_2)$ (κ is unitary) in the fermionic case.*

Let us stress that the Racah time reversal is linear.

Let $\{r_t\}_{t\in\mathbb{R}}$ be a charged symplectic or unitary dynamics.

Definition 18.39 *$\{r_t\}_{t\in\mathbb{R}}$ is* invariant under the Racah time reversal *if $\kappa r_t = r_{-t}\kappa$.*

Suppose that χ is charge reversal and τ is a Wigner time reversal satisfying

$$\tau\chi = \chi\tau \quad \text{or } \tau\chi = -\chi\tau.$$

Then it is easy to see that $\kappa := \tau\chi$ is a Racah time reversal. In particular, $\kappa^2 = \mathbb{1}$ or $\kappa^2 = -\mathbb{1}$.

Note that we are free to multiply either χ or τ by i. Therefore, possibly after a redefinition of χ or τ, we can always assume that

$$\tau\chi = \chi\tau. \tag{18.20}$$

Thus we have three commuting symmetries: χ, τ and κ.

Consider in addition $\{r_t\}_{t\in\mathbb{R}}$, a charged dynamics invariant under Wigner's time reversal τ and a charge reversal χ. Let us recall the various commutation properties:

$$\begin{aligned} &\tau|b| = |b|\tau, \quad \tau\mathrm{j} = -\mathrm{j}\tau, \quad \tau\mathrm{i} = -\mathrm{i}\tau, \quad \tau q = q\tau, \\ &\chi|b| = |b|\chi, \quad \chi\mathrm{j} = \mathrm{j}\chi, \quad \chi\mathrm{i} = -\mathrm{i}\chi, \quad \chi q = -q\chi. \end{aligned}$$

τ, χ, κ and $q\kappa$ are all either involutions or anti-involutions. The following list describes various possible behaviors of these four symmetries:

τ^2	χ^2	κ^2	$(q\kappa)^2$
$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$-\mathbb{1}$
$\mathbb{1}$	$-\mathbb{1}$	$-\mathbb{1}$	$\mathbb{1}$
$-\mathbb{1}$	$-\mathbb{1}$	$\mathbb{1}$	$-\mathbb{1}$
$-\mathbb{1}$	$\mathbb{1}$	$-\mathbb{1}$	$\mathbb{1}$

Note that both κ and $q\kappa$ satisfy the conditions of the Racah time reversal. If $\tau^2 = \chi^2 = \pm\mathbb{1}$, we have $\kappa^2 = \mathbb{1}$, whereas if $\tau^2 = -\chi^2 = \pm\mathbb{1}$, we have $(q\kappa)^2 = \mathbb{1}$. Therefore, one of the operators κ or $q\kappa$ is always an involution.

18.3 Abstract Klein–Gordon equation and its quantization

In Subsects. 18.1.1, resp. 18.2.1 we described how to quantize a symplectic, resp. charged symplectic dynamics. The most important symplectic or charged symplectic dynamics used in quantum field theory is associated with the *wave equation*

$$(\partial_t^2 - \Delta)\zeta = 0$$

or, more generally, to the closely related *Klein–Gordon equation*

$$(\partial_t^2 - \Delta + m^2)\zeta = 0.$$

One of the characteristic features of the wave and Klein–Gordon equation is the second order of the time derivative. In this section we study an abstract version of the wave or Klein–Gordon equation. We forget about the spatial structure of the system, but we keep the second-order temporal derivative. We describe the corresponding symplectic dynamics and its quantization.

In the next chapter we will consider the true wave and Klein–Gordon equation on the space-time and its quantization. We find it instructive and amusing, however, that many of the constructions used in this context can be described in a rather abstract fashion.

18.3.1 Splitting into configuration and momentum space

Suppose that $\mathcal{Y}$ is a symplectic space equipped with a time reversal τ. Recall that it satisfies $\tau^2 = \mathbb{1}$. Thus τ is a conjugation on a real symplectic space. As discussed in Subsect. 1.1.16, we can split the dual phase space into the direct sum of Lagrangian subspaces $\mathcal{Y} = \mathcal{Y}^{\tau} \oplus \mathcal{Y}^{-\tau}$, where $\mathcal{Y}^{\pm\tau} := \{y \in \mathcal{Y} \ : \ y = \pm\tau y\}$.

$\mathcal{Y}^{\tau}$ has the interpretation of the *dual of the configuration space*, whereas $\mathcal{Y}^{-\tau}$ has the interpretation of the *dual of the momentum space*.

Recall from Subsect. 1.1.16 that $(\mathcal{Y}^{\tau}, \mathcal{Y}^{-\tau})$ can be interpreted as a dual pair so that the symplectic form can be written as

$$(\vartheta_1, \varsigma_1)\cdot\omega(\vartheta_2, \varsigma_2) = \vartheta_1 \cdot \varsigma_2 - \varsigma_1 \cdot \vartheta_2 \qquad (\vartheta_i, \varsigma_i) \in \mathcal{Y}^{\tau} \oplus \mathcal{Y}^{-\tau},\ i = 1, 2. \tag{18.21}$$

The time reversal acts as

$$\tau(\vartheta, \varsigma) = (\vartheta, -\varsigma), \quad (\vartheta, \varsigma) \in \mathcal{Y}^{\tau} \oplus \mathcal{Y}^{-\tau}. \tag{18.22}$$

Let $\{r_t\}_{t\in\mathbb{R}}$ be a time reversal invariant dynamics. For $(\vartheta, \varsigma) \in \mathcal{Y}^{\tau} \oplus \mathcal{Y}^{-\tau}$ write $r_t(\vartheta, \varsigma) = (\vartheta(t), \varsigma(t))$. Then there exist $f \in L(\mathcal{Y}^{\tau}, \mathcal{Y}^{-\tau})$, $g \in L(\mathcal{Y}^{-\tau}, \mathcal{Y}^{\tau})$ such that $f = f^{\#}$, $g = g^{\#}$ and

$$\partial_t \varsigma(t) = f\vartheta(t), \quad \partial_t \vartheta(t) = -g\varsigma(t).$$

The Hamiltonian of the dynamics is

$$\frac{1}{2}\vartheta \cdot g\vartheta + \frac{1}{2}\varsigma \cdot f\varsigma. \tag{18.23}$$

18.3.2 Neutral Klein–Gordon equation

Let $\mathcal{X}$ be a real Hilbert space. Let $\epsilon > 0$ be a strictly positive self-adjoint operator on $\mathcal{X}$. (Recall that $\epsilon > 0$ means that $\epsilon \geq 0$ and $\operatorname{Ker}\epsilon = \{0\}$.)

Definition 18.40 *The equation*

$$\partial_t^2 \zeta(t) + \epsilon^2 \zeta(t) = 0, \tag{18.24}$$

where $\zeta(t)$ *is a function from* $\mathbb{R}$ *to* $\mathcal{X}$*, will be called an* abstract neutral Klein–Gordon equation*.*

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Clearly, if $\zeta(t)$ is a solution, $\zeta(-t)$ is also a solution, so (18.24) is invariant under time reversal.

Examples of (18.24) are the wave or Klein–Gordon equations on *static space-times*; see Chap. 19.

Let us reinterpret (18.24) as a first-order equation. To this end, we consider the space of Cauchy data $\mathcal{Y} = \mathcal{X} \oplus \mathcal{X}$, whose elements are denoted (ϑ, ς) or $\begin{bmatrix} \vartheta \\ \varsigma \end{bmatrix}$. We equip it with the symplectic form

$$(\vartheta_1, \varsigma_1) \cdot \omega (\vartheta_2, \varsigma_2) = \vartheta_1 \cdot \varsigma_2 - \vartheta_2 \cdot \varsigma_1, \qquad (\vartheta_i, \varsigma_i) \in \mathcal{X} \oplus \mathcal{X}, \quad i = 1, 2.$$

Setting

$$\varsigma(t) := \zeta(t), \quad \vartheta(t) := \partial_t \zeta(t), \quad a := \begin{bmatrix} 0 & -\epsilon^2 \\ \mathbb{1} & 0 \end{bmatrix},$$

we rewrite (18.24) as

$$\partial_t \begin{bmatrix} \vartheta(t) \\ \varsigma(t) \end{bmatrix} = a \begin{bmatrix} \vartheta(t) \\ \varsigma(t) \end{bmatrix}, \quad \begin{bmatrix} \vartheta(0) \\ \varsigma(0) \end{bmatrix} = \begin{bmatrix} \vartheta \\ \varsigma \end{bmatrix}. \tag{18.25}$$

(Note that we put the time derivative first, since we are considering the dual phase space.) We see that (18.25) is solved by

$$\begin{bmatrix} \vartheta(t) \\ \varsigma(t) \end{bmatrix} = \begin{bmatrix} \cos(\epsilon t) & -\epsilon \sin(\epsilon t) \\ \epsilon^{-1} \sin(\epsilon t) & \cos(\epsilon t) \end{bmatrix} \begin{bmatrix} \varsigma \\ \vartheta \end{bmatrix} = \mathrm{e}^{ta} \begin{bmatrix} \vartheta \\ \varsigma \end{bmatrix}. \tag{18.26}$$

For bounded ϵ, e^{ta} is a symplectic dynamics on $\mathcal{X} \oplus \mathcal{X}$ with the Hamiltonian

$$\frac{1}{2} \vartheta \cdot \vartheta + \frac{1}{2} \varsigma \cdot \epsilon^2 \varsigma. \tag{18.27}$$

For unbounded ϵ, there is a problem, since $\mathcal{X} \oplus \mathcal{X}$ is not preserved by e^{ta}. In this case, one can replace $\mathcal{X} \oplus \mathcal{X}$ with $\mathcal{Y} = \mathcal{X} \oplus \mathrm{Dom}\, \epsilon$, which is a symplectic space preserved by the dynamics. The dynamics is weakly stable. If in addition $\epsilon \geq m > 0$, then it is stable. The energy space $\mathcal{Y}_{\mathrm{en}}$ is equal to $\mathcal{X} \oplus \epsilon^{-1} \mathcal{X}$.

The Kähler anti-involution of Thm. 18.5 takes the form

$$\mathrm{j} = \begin{bmatrix} 0 & -\epsilon \\ \epsilon^{-1} & 0 \end{bmatrix}. \tag{18.28}$$

The associated Hermitian product is

$$\big((\vartheta_1, \varsigma_1) | (\vartheta_2, \varsigma_2) \big) = \vartheta_1 \cdot \epsilon^{-1} \vartheta_2 + \varsigma_1 \cdot \epsilon \varsigma_2 + \mathrm{i} (\vartheta_1 \cdot \varsigma_2 - \vartheta_2 \cdot \varsigma_1). \tag{18.29}$$

The completion of the Kähler space $\mathcal{Y}_{\mathrm{en}}$ for (18.29) is

$$\mathcal{Y}_{\mathrm{dyn}} := \epsilon^{\frac{1}{2}} \mathcal{X} \oplus \epsilon^{-\frac{1}{2}} \mathcal{X}. \tag{18.30}$$

In the standard way we introduce the space $\mathcal{Z} := \frac{1}{2}(\mathbb{1} - \mathrm{ij}) \mathbb{C} \mathcal{Y}_{\mathrm{dyn}}$, which will serve as the one-particle space for quantization.

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Note that the dual phase space and dynamics of an abstract Klein–Gordon equation belong to the class described in Subsect. 18.3.1. In particular, the time reversal is given by (18.22).

It is natural to introduce the following identification:

$$\mathbb{C}(2\epsilon)^{\frac{1}{2}}\mathcal{X} \ni \vartheta \mapsto U\vartheta := \frac{\mathbb{1}-\mathrm{ij}}{2}(\vartheta, 0) = \Big(\frac{1}{2}\vartheta, \frac{\mathrm{i}}{2\epsilon}\vartheta\Big) \in \mathcal{Z} \subset \mathbb{C}\mathcal{Y}_{\mathrm{dyn}}. \tag{18.31}$$

Note that U is unitary.

Recall that the dynamics can be lifted to the space $\mathcal{Z}$ by $\mathrm{e}^{ta}_{\mathcal{Z}} := \mathrm{e}^{ta}_{\mathbb{C}}\big|_{\mathcal{Z}}$. We have $U^*\mathrm{e}^{ta}_{\mathcal{Z}}U = \mathrm{e}^{\mathrm{i}t\epsilon}$.

Likewise, the time reversal can be lifted to $\mathcal{Z}$ by $\tau_{\mathcal{Z}} := \tau_{\overline{\mathbb{C}}}\big|_{\mathcal{Z}}$. Now $U^*\tau_{\mathcal{Z}}U$ coincides with the usual canonical conjugation on $\mathbb{C}(2\epsilon)^{\frac{1}{2}}\mathcal{X}$.

Note that $\mathcal{Y}_{\mathrm{dyn}}$ is a complete Kähler space with a conjugation τ. Recall that we considered the CCR over such spaces in Subsect. 8.2.7. The operator $(2c)^{-1}$ of Subsect. 8.2.7 can be identified with ϵ of this subsection. The map U is the same map as (8.32).

Remark 18.41 *An abstract neutral Klein–Gordon equation describes the most general stable dynamics invariant w.r.t. a time reversal. In fact, recall the Hamiltonian (18.23), discussed in Subsect. 18.1.3 about the time-reversal invariance, and assume that it is strictly positive. Then it is easy to see that (18.23) can be brought to the form (18.27).*

18.3.3 Neutral Klein–Gordon equation in an external potential

We consider now the following modification of (18.24):

$$\begin{aligned} &(\partial_t + d)^2\zeta(t) + \epsilon^2\zeta(t) = 0, \\ \text{or}\quad &\partial_t^2\zeta(t) + 2d\partial_t\zeta(t) + (\epsilon^2 + d^2)\zeta(t) = 0, \end{aligned} \tag{18.32}$$

where $d = -d^*$ is anti-self-adjoint on $\mathcal{X}$. Note that this equation is no longer invariant under time-reversal. Examples of (18.32) are wave or Klein–Gordon equations on *stationary space-times* (see Example 19.43). Setting

$$\varsigma(t) := \zeta(t), \qquad \vartheta(t) := \partial_t\zeta(t) + d\zeta(t), \qquad a := \begin{bmatrix} -d & -\epsilon^2 \\ \mathbb{1} & -d \end{bmatrix},$$

we can rewrite (18.32) as a first-order equation,

$$\partial_t \begin{bmatrix} \vartheta(t) \\ \varsigma(t) \end{bmatrix} = a \begin{bmatrix} \vartheta(t) \\ \varsigma(t) \end{bmatrix},$$

with a Hamiltonian

$$\frac{1}{2}(\vartheta - d\varsigma)\cdot(\vartheta - d\varsigma) - \frac{1}{2}(d\varsigma)\cdot d\varsigma + \frac{1}{2}(\epsilon\varsigma)\cdot\epsilon\varsigma.$$

If $\epsilon^2 + d^2 > 0$, then the dynamics is weakly stable.

Note that the associated complex structure j does not have a simple expression anymore.

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18.3.4 Splitting into complex configuration and momentum spaces

This subsection is the charged version of Subsect. 18.3.1. Suppose that $(\mathcal{Y}, \omega)$ is a charged symplectic space equipped with a Racah time reversal satisfying $\kappa^2 = \mathbb{1}$. Again, we can split the dual phase space into the direct sum of Lagrangian subspaces $\mathcal{Y} = \mathcal{Y}^{\kappa} \oplus \mathcal{Y}^{-\kappa}$, where $\mathcal{Y}^{\pm\kappa} := \{y \in \mathcal{Y} \ : \ y = \pm\kappa y\}$. (Note that spaces $\mathcal{Y}^{\kappa}$ and $\mathcal{Y}^{-\kappa}$ are both complex.)

$\mathcal{Y}^{\kappa}$ has the interpretation of the *dual configuration space* and $\mathcal{Y}^{-\kappa}$ of the *dual momentum space*.

$(\mathcal{Y}^{\kappa}, \mathcal{Y}^{-\kappa})$ can be interpreted as an anti-dual pair, so that the charged symplectic form can be written

$$\overline{(\vartheta_1, \varsigma_1)} \cdot \omega(\vartheta_2, \varsigma_2) = \overline{\vartheta}_1 \cdot \varsigma_2 - \overline{\varsigma}_1 \cdot \vartheta_2, \qquad (\vartheta_i, \varsigma_i) \in \mathcal{Y}^{\kappa} \oplus \mathcal{Y}^{-\kappa}, \quad i = 1, 2.$$

The Racah time reversal acts as

$$\kappa(\vartheta, \varsigma) = (\vartheta, -\varsigma), \quad (\vartheta, \varsigma) \in \mathcal{Y}^{\kappa} \oplus \mathcal{Y}^{-\kappa}. \tag{18.33}$$

Let $\{r_t\}_{t \in \mathbb{R}}$ be a dynamics invariant w.r.t the Racah time reversal. For $(\vartheta, \varsigma) \in \mathcal{Y}^{\kappa} \oplus \mathcal{Y}^{-\kappa}$ write $r_t(\vartheta, \varsigma) = (\vartheta(t), \varsigma(t))$. Then there exist $f \in L(\mathcal{Y}^{\kappa}, \mathcal{Y}^{-\kappa})$, $g \in L(\mathcal{Y}^{-\kappa}, \mathcal{Y}^{\kappa})$ such that $g = g^*$, $f = f^*$ and

$$\partial_t \varsigma(t) = f\vartheta(t), \quad \partial_t \vartheta(t) = -g\varsigma(t).$$

The Hamiltonian of the dynamics is

$$\overline{\vartheta} \cdot g\vartheta + \overline{\varsigma} \cdot f\varsigma.$$

18.3.5 Charged Klein–Gordon equation

Now we describe the charged version of Subsect. 18.3.2. Let $\mathcal{X}$ be a complex Hilbert space. For $\zeta_1, \zeta_2 \in \mathcal{X}$, the scalar product will be denoted by $\overline{\zeta}_1 \cdot \zeta_2$. Consider again a strictly positive self-adjoint operator ϵ on $\mathcal{X}$ and the equation (18.24).

Definition 18.42 *If the space $\mathcal{X}$ is complex, the equation (18.24) will be called an* abstract charged Klein–Gordon equation.

Thus the only difference between the charged and neutral Klein–Gordon equations is the presence of the $U(1)$ symmetry given by the multiplication by $\mathrm{e}^{\mathrm{i}\theta}$, $\theta \in [0, 2\pi]$.

The Racah time reversal consists in replacing $t \mapsto \zeta(t)$ with $t \mapsto \zeta(-t)$. The charged Klein–Gordon equation is always invariant w.r.t. the Racah time reversal.

Let us fix a complex conjugation on $\mathcal{X}$, denoted by $\zeta \mapsto \overline{\zeta}$, which defines the charge reversal. The Wigner time reversal involves replacing a function $t \mapsto \zeta(t)$ with $t \mapsto \overline{\zeta(-t)}$. If $\overline{\epsilon} = \epsilon$, then (18.24) is also invariant w.r.t. the charge and Wigner time reversal.

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Consider the Cauchy problem (18.25). We introduce the space $\mathcal{X} \oplus \mathcal{X}$, equipped with the charged symplectic form

$$\overline{(\vartheta_1, \varsigma_1)} \cdot \omega(\vartheta_2, \varsigma_2) = \overline{\vartheta}_1 \cdot \varsigma_2 - \overline{\varsigma}_1 \cdot \vartheta_2, \qquad (\vartheta_i, \varsigma_i) \in \mathcal{X} \oplus \mathcal{X}, \quad i = 1, 2.$$

The Hamiltonian is

$$\overline{\vartheta} \cdot \vartheta + \overline{\epsilon\varsigma} \cdot \epsilon\varsigma.$$

$\mathcal{Y}_{\mathrm{en}}$, $\mathcal{Y}_{\mathrm{dyn}}$, j, a are given by the same expressions as in Subsect. 18.3.2. In particular, it is natural to replace the original dual phase space $\mathcal{X} \oplus \mathcal{X}$ by $\mathcal{Y}_{\mathrm{dyn}} = \epsilon^{\frac{1}{2}} \mathcal{X} \oplus \epsilon^{-\frac{1}{2}} \mathcal{X}$.

In terms of the Cauchy data, the Racah time reversal is given by (18.33). The charge reversal and the Wigner time reversal are given by

$$\chi(\vartheta, \varsigma) \ = \ (\overline{\vartheta}, \overline{\varsigma}), \quad \tau(\vartheta, \varsigma) \ = \ (\overline{\vartheta}, -\overline{\varsigma}).$$

We can "diagonalize" the dynamics by introducing the map

$$W : \mathcal{Y}_{\mathrm{dyn}} = \epsilon^{\frac{1}{2}} \mathcal{X} \oplus \epsilon^{-\frac{1}{2}} \mathcal{X} \ni (\vartheta, \varsigma) \mapsto (\vartheta + \mathrm{i}\epsilon\varsigma, \overline{\vartheta} + \mathrm{i}\epsilon\overline{\varsigma}) \in (2\epsilon)^{\frac{1}{2}} \mathcal{X} \oplus (2\epsilon)^{\frac{1}{2}} \mathcal{X}.$$

W is a unitary operator satisfying

$$\begin{aligned} W \mathrm{e}^{ta} W^{-1} &= \mathrm{e}^{\mathrm{i}t(\epsilon \oplus \epsilon)}, \ W \mathrm{i} W^{-1} = \mathrm{i}\mathbb{1} \oplus (-\mathrm{i}\mathbb{1}), \\ W \mathrm{j} W^{-1} &= \mathrm{i}\mathbb{1} \oplus \mathrm{i}\mathbb{1}, \ W q W^{-1} = \mathbb{1} \oplus (-\mathbb{1}). \end{aligned}$$

Thus if we interpret W as an operator on $\mathcal{Z}$ (which differs from $\mathcal{Y}_{\mathrm{dyn}}$ only by treating j as the basic complex structure), then $W : \mathcal{Z} \to (2\epsilon)^{\frac{1}{2}} \mathcal{X} \oplus (2\epsilon)^{\frac{1}{2}} \mathcal{X}$ is unitary.

After conjugation by W, the charge and Wigner time reversal become

$$\chi(h_1, \overline{h_2}) \ = \ (\overline{h_2}, h_1), \quad \tau(h_1, \overline{h_2}) \ = \ (-\overline{h_1}, h_2).$$

Remark 18.43 *This remark is analogous to Remark 18.41 from the neutral case. A charged abstract Klein–Gordon equation describes the most general stable dynamics invariant w.r.t. the Racah time reversal.*

18.3.6 Charged Klein–Gordon equation in an external potential

Again we can consider the complex analog of (18.32). It is more natural to write it as

$$\begin{aligned} &(\partial_t + \mathrm{i}V)^2 \zeta(t) + \epsilon^2 \zeta(t) = 0, \qquad (18.34) \\ \text{or} \quad &\partial_t^2 \zeta(t) + 2\mathrm{i}V \partial_t \zeta(t) + (\epsilon^2 - V^2)\zeta(t) = 0, \end{aligned}$$

where $V = V^*$. An example is obtained by minimally coupling (18.24) to an external electric field.

Setting

$$\varsigma(t) := \zeta(t), \quad \vartheta(t) := \partial_t \zeta(t) + \mathrm{i}V\zeta(t),$$

we can rewrite (18.34) as

$$\partial_t \begin{bmatrix} \vartheta(t) \\ \varsigma(t) \end{bmatrix} = \begin{bmatrix} -\mathrm{i}V & -\epsilon^2 \\ \mathbb{1} & -\mathrm{i}V \end{bmatrix} \begin{bmatrix} \varsigma(t) \\ \vartheta(t) \end{bmatrix}$$

with the Hamiltonian

$$\begin{aligned} &\overline{\vartheta} \cdot \vartheta + \overline{\epsilon\varsigma} \cdot \epsilon\varsigma - \mathrm{i}\overline{\vartheta} \cdot V\varsigma + \mathrm{i}\overline{\varsigma} \cdot V\vartheta \\ = &\overline{(\vartheta - \mathrm{i}V\varsigma)} \cdot (\vartheta - \mathrm{i}V\varsigma) - \overline{V\varsigma} \cdot V\varsigma + \overline{\epsilon\varsigma} \cdot \epsilon\varsigma. \end{aligned}$$

If $\epsilon^2 - V^2 > 0$, then the dynamics is weakly stable.

Note that if $\mathcal{X}$ is equipped with a conjugation such that V and ϵ are real, then (18.34) is invariant under the Wigner time reversal.

18.3.7 Quantization of the Klein–Gordon equation

Until the end of this section, we would like to treat the neutral and charged cases together. We do this by embedding the neutral case in the charged case.

More precisely, until the end of the section $\mathcal{X}$ is always a complex Hilbert space with a positive self-adjoint operator ϵ. We consider the abstract charged Klein–Gordon equation for $t \mapsto \zeta(t) \in \mathcal{X}$:

$$\partial_t^2 \zeta(t) + \epsilon^2 \zeta(t) = 0. \tag{18.35}$$

We consider the charged symplectic space of solutions of (18.35), denoted $\mathcal{Y}$. Recall that every such solution can be parametrized by its Cauchy data (ϑ, ς). The space $\mathcal{Y}$ is equipped with a charged symplectic dynamics r_t.

If we want to consider the neutral case, we assume that there exists a conjugation χ on $\mathcal{X}$ that commutes with ϵ. Thus we can restrict the abstract Klein–Gordon equation to $\mathcal{X}^\chi = \{\zeta \in \mathcal{X} \ : \ \chi\zeta = \zeta\}$, obtaining the symplectic space of solutions $\mathcal{Y}^\chi$. The space $\mathcal{Y}^\chi$ is equipped with a symplectic dynamics $r_t\big|_{\mathcal{Y}^\chi}$, which satisfies the abstract neutral Klein–Gordon equation.

We apply the positive energy quantization described in Subsects. 18.1.1, resp. 18.2.1, obtaining operator-valued functions

$$\begin{aligned} \mathcal{Y}^\chi \ni (\vartheta, \varsigma) &\mapsto \phi(\vartheta, \varsigma), \quad \text{in the neutral case,} \\ \mathcal{Y} \ni (\vartheta, \varsigma) &\mapsto \psi(\vartheta, \varsigma), \quad \text{in the charged case,} \end{aligned}$$

and the Hamiltonian H such that

$$\begin{aligned} \mathrm{e}^{\mathrm{i}tH}\phi(\vartheta, \varsigma)\mathrm{e}^{-\mathrm{i}tH} &= \phi\big(r_t(\vartheta, \varsigma)\big), \quad (\vartheta, \varsigma) \in \mathcal{Y}^\chi, \\ \mathrm{e}^{\mathrm{i}tH}\psi(\vartheta, \varsigma)\mathrm{e}^{-\mathrm{i}tH} &= \psi\big(r_t(\vartheta, \varsigma)\big), \quad (\vartheta, \varsigma) \in \mathcal{Y}. \end{aligned}$$

Definition 18.44 *The* time t configuration space field *is defined as*

$$\begin{aligned} \phi_t(\vartheta) &:= \phi\big(r_{-t}(\vartheta, 0)\big), \quad \vartheta \in \epsilon^{\frac{1}{2}}\mathcal{X}^\chi, \\ \psi_t(\vartheta) &:= \psi\big(r_{-t}(\vartheta, 0)\big), \quad \vartheta \in \epsilon^{\frac{1}{2}}\mathcal{X}. \end{aligned}$$

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18.3.8 Two-point functions for the Klein–Gordon equation

The remaining part of this section is devoted to various functions related to the Klein–Gordon equation, which are often used in its quantization.

Consider the Klein–Gordon equation (18.35), where $\zeta(t) \in \mathcal{X}$ is replaced with an operator $G(t) \in B(\mathcal{X})$:

$$\partial_t^2 G(t) + \epsilon^2 G(t) = 0. \tag{18.36}$$

The functions $\frac{\sin \epsilon t}{\epsilon}$ and $\frac{\mathrm{e}^{\mathrm{i}t\epsilon}}{2\epsilon}$ solve (18.36) and appear naturally in the quantized theory:

$$\begin{aligned}
[\varphi_{t_1}(\vartheta_1), \varphi_{t_2}(\vartheta_2)] &= \mathrm{i}\vartheta_1 \cdot \frac{\sin \epsilon(t_1 - t_2)}{\epsilon} \vartheta_2 \, 1\!\!1, \\
\big(\Omega|\varphi_{t_1}(\vartheta_1)\varphi_{t_2}(\vartheta_2)\Omega\big) &= \vartheta_1 \cdot \frac{1}{2\epsilon} \mathrm{e}^{\mathrm{i}\epsilon(t_1 - t_2)} \vartheta_2, \quad \vartheta_1, \vartheta_2 \in \mathcal{X}^{\chi}, \\
[\psi_{t_1}(\vartheta_1), \psi_{t_2}^*(\vartheta_2)] &= \mathrm{i}\overline{\vartheta}_1 \cdot \frac{\sin \epsilon(t_1 - t_2)}{\epsilon} \vartheta_2 \, 1\!\!1, \\
\big(\Omega|\psi_{t_1}(\vartheta_1)\psi_{t_2}^*(\vartheta_2)\Omega\big) &= \overline{\vartheta}_1 \cdot \frac{1}{2\epsilon} \mathrm{e}^{\mathrm{i}\epsilon(t_1 - t_2)} \vartheta_2, \quad \vartheta_1, \vartheta_2 \in \mathcal{X}.
\end{aligned}$$

Definition 18.45 $\frac{\sin \epsilon t}{\epsilon}$ *is called the* Pauli–Jordan *or* commutator function.

18.3.9 Green's functions of the abstract Klein–Gordon equation

Let us now consider an inhomogeneous version of Eq. (18.36).

In what follows we will often use the Heaviside function $\theta(t) := 1\!\!1_{[0,+\infty[}(t)$.

Definition 18.46 $\mathbb{R} \ni t \mapsto G(t) \in B(\mathcal{X})$ *is a* Green's function *or a* fundamental solution of Eq. (18.35) *if it solves*

$$\partial_t^2 G(t) + \epsilon^2 G(t) = \delta(t) 1\!\!1. \tag{18.37}$$

In particular, we introduce the following Green's functions:

retarded	$G^+(t) := \theta(t) \frac{\sin \epsilon t}{\epsilon}$,
advanced	$G^-(t) := -\theta(-t) \frac{\sin \epsilon t}{\epsilon}$,
Feynman *or* causal	$G_{\mathrm{F}}(t) := \frac{1}{2\mathrm{i}\epsilon}\big(\mathrm{e}^{\mathrm{i}t\epsilon}\theta(t) + \mathrm{e}^{-\mathrm{i}t\epsilon}\theta(-t)\big)$,
anti-Feynman *or* anti-causal	$G_{\overline{\mathrm{F}}}(t) := -\frac{1}{2\mathrm{i}\epsilon}\big(\mathrm{e}^{-\mathrm{i}t\epsilon}\theta(t) + \mathrm{e}^{\mathrm{i}t\epsilon}\theta(-t)\big)$,
principal value *or* Dirac	$G_{\mathrm{Pv}}(t) := \frac{\mathrm{sgn}(t)}{2} \frac{\sin \epsilon t}{\epsilon}$.

Note the identities

$$\begin{aligned}
\frac{\sin \epsilon t}{\epsilon} &= G^+(t) - G^-(t), \\
G_{\mathrm{Pv}}(t) &= \frac{1}{2}\big(G^+(t) + G^-(t)\big), \\
G^+(t) + G^-(t) &= G_{\mathrm{F}}(t) + G_{\overline{F}}(t).
\end{aligned}$$

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The importance of the Feynman Green's functions in the quantum theory comes from the identities

$$\begin{aligned}\big(\Omega|\mathrm{T}\left(\varphi_{t_2}(\vartheta_2)\varphi_{t_1}(\vartheta_1)\right)\Omega\big) &= \mathrm{i}\vartheta_2\cdot G_{\mathrm{F}}(t_2-t_1)\vartheta_1, \quad \vartheta_1,\vartheta_2\in\mathcal{X}^\chi,\\ \big(\Omega|\mathrm{T}\left(\psi_{t_2}(\vartheta_2)\psi_{t_1}^*(\vartheta_1)\right)\Omega\big) &= \mathrm{i}\overline{\vartheta}_2\cdot G_{\mathrm{F}}(t_2-t_1)\vartheta_1, \quad \vartheta_1,\vartheta_2\in\mathcal{X},\end{aligned}$$

where we have used the time-ordering operation

$$\mathrm{T}\big(A_{t_2}A_{t_1}\big) := \theta(t_2-t_1)A_{t_2}A_{t_1} + \theta(t_1-t_2)A_{t_1}A_{t_2}.$$

18.3.10 Green's functions of the Klein–Gordon equation as operators

Let $\mathcal{X}$ be as above. For simplicity, we assume that $\mathcal{X}$ is separable. We will need the space

$$L^2(\mathbb{R})\otimes\mathcal{X}\simeq L^2(\mathbb{R},\mathcal{X}). \tag{18.38}$$

Note that the unitary identification $\simeq$ in (18.38) is possible thanks to the fact that $\mathcal{X}$ is separable. It means that we can represent elements of $L^2(\mathbb{R})\otimes\mathcal{X}$ with measurable a.e. defined functions, which e.g. in the temporal representation are written as $\mathbb{R}\ni t\mapsto\zeta(t)\in\mathcal{X}$ and satisfy

$$\int \|\zeta(t)\|^2\mathrm{d}t<\infty.$$

Clearly, the subspace $(L^1\cap L^2)(\mathbb{R},\mathcal{X})$ is dense in $L^2(\mathbb{R},\mathcal{X})$.

We will distinguish two physical meanings of the variable in $\mathbb{R}$ that appears in (18.38). The first meaning is the time, and the corresponding generic variable in $\mathbb{R}$ will be denoted t. We will then say that we use the *temporal representation* of the extended space. The second meaning will be the energy. Its generic name will be τ and we will speak about the *energy representation*. To pass from one representation to the other we apply the Fourier transformation $\mathcal{F}$, so that $\mathcal{F}^{-1}\tau\mathcal{F}=\mathrm{i}^{-1}\partial_t$.

Green's functions of the abstract Klein–Gordon equation can be interpreted as quadratic forms on $(L^1\cap L^2)(\mathbb{R},\mathcal{X})$ given (in the temporal representation) by

$$\overline{\zeta}_1\cdot G\zeta_2 := \int \overline{\zeta_1(t)}\cdot G(t-s)\zeta_2(s)\mathrm{d}t\mathrm{d}s, \tag{18.39}$$

for $\zeta_1,\zeta_2\in(L^1\cap L^2)(\mathbb{R},\mathcal{X})$. In the energy representation they are multiplication operators. Here we list the most important Green's functions in the momentum representation:

$$\begin{aligned}G^+(\tau) &= (\epsilon^2-(\tau-\mathrm{i}0)^2)^{-1},\\ G^-(\tau) &= (\epsilon^2-(\tau+\mathrm{i}0)^2)^{-1},\\ G_{\mathrm{F}}(\tau) &= (\epsilon^2-\tau^2+\mathrm{i}0)^{-1} = \big(\epsilon^2-(\tau^2-\mathrm{i}0\,\mathrm{sgn}(\tau))^2\big)^{-1},\\ G_{\overline{\mathrm{F}}}(\tau) &= (\epsilon^2-\tau^2-\mathrm{i}0)^{-1} = \big(\epsilon^2-(\tau^2+\mathrm{i}0\,\mathrm{sgn}(\tau))^2\big)^{-1},\\ G_{\mathrm{Pv}}(\tau) &= \mathrm{Pv}(\epsilon^2-\tau^2)^{-1}.\end{aligned}$$

18.3.11 Euclidean Green's function of the Klein–Gordon equation

Let us introduce the *imaginary time*, that is, let us replace $t \in \mathbb{R}$ with $\mathrm{i}s \in \mathrm{i}\mathbb{R}$. The abstract Klein–Gordon equation is then transformed into

$$-\partial_s^2 \zeta + \epsilon^2 \zeta = 0. \tag{18.40}$$

Definition 18.47 *Equation (18.40) will be called the* abstract Euclidean Klein–Gordon equation.

The use of (18.40) instead of the Klein–Gordon equation is the main feature of the so-called Euclidean approach to quantum field theory.

Definition 18.48 *The* Euclidean Green's function of the abstract Klein–Gordon equation *is defined as*

$$G_{\mathrm{E}}(s) = \frac{1}{2\epsilon}\big(\mathrm{e}^{-s\epsilon}\theta(s) + \mathrm{e}^{s\epsilon}\theta(-s)\big).$$

Clearly, G_{E} solves

$$-\partial_s^2 G_{\mathrm{E}}(s) + \epsilon^2 G_{\mathrm{E}}(s) = \delta(s)\mathbb{1}. \tag{18.41}$$

The function $G_{\mathrm{E}}(s)$ extends to a continuous function for complex s with $\mathrm{Re}\, s \geq 0$, holomorphic for $\mathrm{Re}\, s > 0$. We have

$$\frac{1}{\mathrm{i}}G_{\mathrm{E}}(\mathrm{i}t) = G_{\mathrm{F}}(t), \quad -\frac{1}{\mathrm{i}}G_{\mathrm{E}}(-\mathrm{i}t) = G_{\overline{\mathrm{F}}}(t).$$

Consider the self-adjoint operator $-\partial_s^2 + \epsilon^2$ on $L^2(\mathbb{R}, \mathcal{X})$. Set

$$G_{\mathrm{E}} := (-\partial_s^2 + \epsilon^2)^{-1}.$$

We then have

$$G_{\mathrm{E}}\zeta(s) = \int_{\mathbb{R}} G_{\mathrm{E}}(s - s_1)\zeta(s_1)\mathrm{d}s_1$$

for $\zeta \in (L^1 \cap L^2)(\mathbb{R}, \mathcal{X})$. In the energy representation it is the operator of multiplication by

$$G_{\mathrm{E}}(\tau) = (\tau^2 + \epsilon^2)^{-1}.$$

Note that if $\epsilon \geq m > 0$, then G_{E} is bounded.

We will use the standard notation for operators on $L^2(\mathbb{R})$. In particular, the operator of multiplication by t in the temporal representation is denoted by t and $D_t := -\mathrm{i}\partial_t$. A similar notation will be used for the energy representation, with τ replacing t.

Introduce the following operator on $L^2(\mathbb{R})$ (where we give its form in both the temporal and the energy representation):

$$A := -\frac{1}{2}(tD_t + D_t t) = \frac{1}{2}(\tau D_\tau + D_\tau \tau). \tag{18.42}$$

Clearly,

$$\mathrm{e}^{\mathrm{i}\theta A} t \mathrm{e}^{-\mathrm{i}\theta A} = \mathrm{e}^{-\theta} t, \qquad \mathrm{e}^{\mathrm{i}\theta A} \tau \mathrm{e}^{-\mathrm{i}\theta A} = \mathrm{e}^{\theta}\tau.$$

Note that

$$\mathbb{R} \ni \theta \mapsto \mathrm{e}^{\mathrm{i}\theta A} G_{\mathrm{E}} \mathrm{e}^{-\mathrm{i}\theta A} =: G_{\mathrm{E}}^{\theta}$$

extends to an analytic function in the strip $-\frac{\pi}{2} < \mathrm{Im}\,\theta < \frac{\pi}{2}$ given in the momentum representation by

$$G_{\mathrm{E}}^{\theta}(\tau) = (\mathrm{e}^{2\theta}\tau^2 + \epsilon^2)^{-1}.$$

Its boundary values coincide with the Feynman and anti-Feynman Green's functions:

$$G_{\mathrm{E}}^{\mathrm{i}\frac{\pi}{2}} = G_{\mathrm{F}}, \quad G_{\mathrm{E}}^{-\mathrm{i}\frac{\pi}{2}} = G_{\overline{\mathrm{F}}}.$$

This is the famous *Wick rotation.*

18.3.12 Thermal Green's function of the Klein–Gordon equation

Recall that one of the steps of the quantization of symplectic, resp. charged symplectic dynamics is the construction of the one-particle space $\mathcal{Z}$ and the one-particle Hamiltonian h, as described in Subsect. 18.1.1, resp. 18.2.1. For the zero temperature, the main requirement is the positivity of the Hamiltonian, and as the result of the quantization we obtain the Hilbert space $\Gamma_{\mathrm{s}}(\mathcal{Z})$ and the Hamiltonian $\mathrm{d}\Gamma(h)$.

If we are interested in positive temperatures, we can apply the formalism described in Subsect. 17.1.7, obtaining a β-KMS state ω_β and the corresponding Araki–Woods CCR representation. In particular, we can apply this formalism to the abstract Klein–Gordon equation.

In this subsection we describe the 2-point correlation functions for the abstract Klein–Gordon equation at positive temperatures.

Definition 18.49 *The* thermal Euclidean Green's function at inverse temperature β of the abstract Klein–Gordon equation *is defined for* $s \in [0, \beta]$ *as*

$$G_{\mathrm{E},\beta}(s) := \frac{\mathrm{e}^{-s\epsilon} + \mathrm{e}^{(s-\beta)\epsilon}}{2\epsilon(\mathbb{1} - \mathrm{e}^{-\beta\epsilon})}.$$

Note that $G_{\mathrm{E},\beta}$ is the unique solution of the problem

$$\begin{aligned}
&-\partial_s^2 G_{\mathrm{E},\beta}(s) + \epsilon^2 G_{\mathrm{E},\beta}(s) = 0, \qquad s \in]0, \beta[,\\
&G_{\mathrm{E},\beta}(0) = G_{\mathrm{E},\beta}(\beta), \qquad \partial_s^- G_{\mathrm{E},\beta}(\beta) - \partial_s^+ G_{\mathrm{E},\beta}(0) = \mathbb{1}_{\mathcal{X}},
\end{aligned}$$

where $\partial_s^{\pm}$ denotes the derivative from the right, resp. from the left. In fact, we have

$$\begin{aligned}
G_{\mathrm{E},\beta}(\beta) \;=\; G_{\mathrm{E},\beta}(0) &= \frac{\mathbb{1} + \mathrm{e}^{-\beta\epsilon}}{2\epsilon(\mathbb{1} - \mathrm{e}^{-\beta\epsilon})},\\
\partial_s^- G_{\mathrm{E},\beta}(\beta) = -\partial_s^+ G_{\mathrm{E},\beta}(0) &= \frac{1}{2}\mathbb{1}_{\mathcal{X}}.
\end{aligned}$$

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Let $S_\beta := [0,\beta]$ (with the endpoints identified) be the circle of length β. Clearly, $G_{\mathrm{E},\beta}$ can be interpreted as a function on S_β that solves the equation

$$-\partial_s^2 G_{\mathrm{E},\beta}(s) + \epsilon^2 G_{\mathrm{E},\beta}(s) = \delta(s)\mathbb{1}_{\mathcal{X}}, \text{ on } S_\beta.$$

We denote by $\mathcal{F}_\beta : L^2(S_\beta) \to l^2\big(\frac{2\pi}{\beta}\mathbb{Z}\big)$ the discrete Fourier transform

$$\mathcal{F}_\beta\zeta(\sigma) = \int_0^\beta \mathrm{e}^{-\mathrm{i}s\sigma}\zeta(s)\mathrm{d}s, \quad \zeta \in L^2(S_\beta).$$

Its inverse is given by

$$\mathcal{F}_\beta^{-1}v(s) = \beta^{-1}\sum_{\sigma\in\frac{2\pi}{\beta}\mathbb{Z}} \mathrm{e}^{\mathrm{i}s\sigma}v(\sigma), \quad v \in l^2\big(\tfrac{2\pi}{\beta}\mathbb{Z}\big).$$

If we denote by $\partial_s^{\mathrm{per}}$ the operator ∂_s with periodic boundary conditions, defined by

$$\mathrm{Dom}\,\partial_s^{\mathrm{per}} := \big\{\zeta \in L^2([0,\beta]),\ \partial_s\zeta \in L^2([0,\beta]),\ \zeta(0) = \zeta(\beta)\big\},$$

then

$$\mathcal{F}_\beta\partial_s^{\mathrm{per}} = \mathrm{i}\sigma\mathcal{F}_\beta. \tag{18.43}$$

Introduce the space

$$L^2(S_\beta)\otimes\mathcal{X} \simeq L^2(S_\beta,\mathcal{X}). \tag{18.44}$$

Consider the self-adjoint operator $-(\partial_s^{\mathrm{per}})^2 + \epsilon^2$ on $L^2(S_\beta,\mathcal{X})$. Set

$$G_{\mathrm{E},\beta} := \big(-(\partial_s^{\mathrm{per}})^2 + \epsilon^2\big)^{-1}.$$

We then have

$$G_{\mathrm{E},\beta}\zeta(s) = \int_{S_\beta} G_{\mathrm{E},\beta}(s-s_1)\zeta(s_1)\mathrm{d}s_1, \quad \text{for } \zeta \in L^2(S_\beta,\mathcal{X}).$$

In the energy representation obtained by applying the discrete Fourier transform $\mathcal{F}_\beta$, $G_{\mathrm{E},\beta}$ becomes the multiplication operator on $l^2(\frac{2\pi}{\beta}\mathbb{Z},\mathcal{X})$ by the Fourier transform of $s\mapsto G_{\mathrm{E},\beta}(s)$, the so-called *Matsubara coefficients*:

$$G_{\mathrm{E},\beta}(\sigma) := (\sigma^2+\epsilon^2)^{-1}, \quad \sigma \in \frac{2\pi}{\beta}\mathbb{Z}. \tag{18.45}$$

Let us now describe the role of thermal Green's functions in the quantum theory. The function $s\mapsto G_{\mathrm{E},\beta}(s)$ extends to a function continuous in the strip $\mathrm{Re}\, s\in[0,\beta]$ and holomorphic inside this strip. Its boundary values express the 2-point correlation functions for the state ω_β. More precisely, we have the following identities (first in the neutral and then in the charged case):

$$\begin{aligned}
\omega_\beta\big(\varphi_t(\vartheta_1)\varphi_0(\vartheta_2)\big) &= \vartheta_1\cdot G_{\mathrm{E},\beta}(\mathrm{i}t)\vartheta_2,\\
\omega_\beta\big(\varphi_0(\vartheta_2)\varphi_t(\vartheta_1)\big) &= \vartheta_1\cdot G_{\mathrm{E},\beta}(\beta+\mathrm{i}t)\vartheta_2, \quad \vartheta_1,\vartheta_2\in\mathcal{X}^\chi,
\end{aligned}$$

$$\begin{aligned}
\omega_\beta\big(\psi_t(\vartheta_1)\psi_0^*(\vartheta_2)\big) &= \overline{\vartheta}_1\cdot G_{\mathrm{E},\beta}(\mathrm{i}t)\vartheta_2,\\
\omega_\beta\big(\psi_0^*(\vartheta_2)\psi_t(\vartheta_1)\big) &= \overline{\vartheta}_1\cdot G_{\mathrm{E},\beta}(\beta+\mathrm{i}t)\vartheta_2, \quad \vartheta_1,\vartheta_2\in\mathcal{X}.
\end{aligned}$$

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18.4 Abstract Dirac equation and its quantization

In Subsects. 18.1.2, resp. 18.2.2 we described how to quantize orthogonal, resp. unitary dynamics. The most important orthogonal or unitary dynamics used in quantum field theory is given by the *Dirac equation*. We will study this equation and its quantization in the next chapter. In this section we describe various constructions related to the quantization of the Dirac equation in an abstract setting.

An orthogonal dynamics can be defined by an equation of the form

$$(\partial_t - a)\zeta(t) = 0, \tag{18.46}$$

with an anti-self-adjoint a. In the charged case (18.46) can be replaced with

$$(\partial_t - \mathrm{i}b)\zeta(t) = 0, \tag{18.47}$$

where b is self-adjoint. Many of the constructions of this section involve only a or b. We choose, however, to use more structure in our presentation. In particular, we multiply (18.46) and (18.47) from the left by an anti-self-adjoint operator γ_0 satisfying $\gamma_0^2 = -\mathbb{1}$. This is used in the relativistic formulation of the Dirac equation to make it covariant.

This section is parallel to Sect. 18.3 about the abstract Klein–Gordon equation. We will see, in particular, that with every abstract Dirac equation we can associate an abstract Klein–Gordon equation. The knowledge of Green's functions for the abstract Klein–Gordon equation can be used to compute Green's functions of the abstract Dirac equation.

18.4.1 Abstract Dirac equation

Let $\mathcal{Y}$ be a real or complex Hilbert space, which will have the meaning of a fermionic dual phase space. Let Γ and γ_0 be anti-self-adjoint operators on $\mathcal{Y}$ such that

$$\gamma_0^2 = -\mathbb{1}, \qquad \gamma_0\Gamma + \Gamma\gamma_0 = 0. \tag{18.48}$$

Let $m \geq 0$ be a number called the *mass*.

Definition 18.50 *An equation of the form*

$$(\gamma_0\partial_t + \Gamma - m\mathbb{1})\zeta(t) = 0, \tag{18.49}$$

where $\zeta(t)$ is a function from $\mathbb{R}$ to $\mathcal{Y}$, will be called an abstract Dirac equation.

Multiplying (18.49) with $-\gamma_0$, we obtain the equation (18.46) with an anti-self-adjoint operator

$$a := \gamma_0\Gamma - m\gamma_0.$$

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Introducing the Cauchy problem

$$\begin{cases} (\gamma_0 \partial_t + \Gamma - m\mathbb{1})\zeta(t) = 0, \\ \zeta(0) = \vartheta, \end{cases} \tag{18.50}$$

we see that (18.49) is solved by $\zeta(t) = \mathrm{e}^{ta}\vartheta$.

Using (18.48), we obtain that $a^{\#} a = \Gamma^{\#}\Gamma + m^2\mathbb{1}$, so the dynamics $r_t = \mathrm{e}^{ta}$ is non-degenerate if

$$\operatorname{Ker}\Gamma = \{0\} \text{ or } m \neq 0. \tag{18.51}$$

18.4.2 Neutral Dirac equation

If $\mathcal{Y}$ is a real Hilbert space, (18.49) will be called a *abstract neutral Dirac equation.* A time reversal for the neutral Dirac equation is $\tau \in O(\mathcal{Y})$ satisfying $\tau^2 = \pm\mathbb{1}$,

$$\tau\gamma_0 = -\gamma_0\tau, \qquad \tau\Gamma = \Gamma\tau, \tag{18.52}$$

or, if $m = 0$,

$$\tau\gamma_0 = \gamma_0\tau, \qquad \tau\Gamma = -\Gamma\tau. \tag{18.53}$$

In both cases, $\tau a = -a\tau$, hence if $\zeta(t)$ is a solution of (18.49), then so is $\tau\zeta(-t)$.

18.4.3 Charged Dirac equation

Assume now that $\mathcal{Y}$ is a (complex) Hilbert space. Thanks to the complex structure, we can introduce the self-adjoint operator

$$b := -\mathrm{i}\gamma_0\Gamma + \mathrm{i}m\gamma_0.$$

The Cauchy problem (18.50) is solved by $\zeta(t) = \mathrm{e}^{\mathrm{i}tb}\vartheta$.

A *(Wigner) time reversal* is an anti-unitary τ on $\mathcal{Y}$ satisfying (18.52), or, if $m = 0$, (18.53). In both cases, $\tau b = b\tau$, hence if $\zeta(t)$ is a solution of (18.49), then so is $\tau\zeta(-t)$.

A *charge reversal* is an anti-unitary χ on $\mathcal{Y}$ such that $\chi^2 = \pm\mathbb{1}$ and

$$\chi\gamma_0 = \gamma_0\chi, \quad \chi\Gamma = \Gamma\chi, \tag{18.54}$$

or, if $m = 0$,

$$\chi\gamma_0 = -\gamma_0\chi, \quad \chi\Gamma = -\Gamma\chi. \tag{18.55}$$

In both cases $b\chi = -\chi b$ and the dynamics $\mathrm{e}^{\mathrm{i}tb}$ is invariant under χ.

If $\chi^2 = \mathbb{1}$, then χ is a conjugation on $\mathcal{Y}$ and by restriction to the real subspace $\mathcal{Y}^\chi$ we obtain a neutral Dirac equation, as in Subsect. 18.4.2.

18.4.4 Quantization of the Dirac equation

Until the end of this section we would like to treat the neutral and charged cases together. We will treat the charged case as the basic one and $\mathcal{Y}$ will always be a complex Hilbert space. $\mathcal{Y}$ is identified with the space of solutions of the abstract Dirac equation by considering the initial conditions: $\vartheta = \zeta(0)$. The space $\mathcal{Y}$ is equipped with a unitary dynamics $r_t = \mathrm{e}^{\mathrm{i}tb}$.

If we want to consider the neutral case, we assume that there exists in $\mathcal{Y}$ a conjugation χ that anti-commutes with b. Thus we can restrict the Dirac equation to $\mathcal{Y}^\chi$ equipped with the Euclidean structure and an orthogonal dynamics $\mathrm{e}^{\mathrm{i}bt}\big|_{\mathcal{Y}^\chi} = \mathrm{e}^{at}$.

We apply the positive energy quantization described in Subsects. 18.1.2 and 18.2.2, obtaining operator-valued functions

$$\begin{aligned} \mathcal{Y}^\chi \ni \vartheta \mapsto \phi(\vartheta), &\quad \text{in the neutral case,} \\ \mathcal{Y} \ni \vartheta \mapsto \psi(\vartheta), &\quad \text{in the complex case,} \end{aligned}$$

and the Hamiltonian H such that

$$\begin{aligned} \mathrm{e}^{\mathrm{i}tH}\phi(\vartheta)\mathrm{e}^{-\mathrm{i}tH} &= \phi(r_t(\vartheta)), \quad \vartheta \in \mathcal{Y}^\chi, \\ \mathrm{e}^{\mathrm{i}tH}\psi(\vartheta)\mathrm{e}^{-\mathrm{i}tH} &= \psi(r_t(\vartheta)), \quad \vartheta \in \mathcal{Y}. \end{aligned}$$

The fermionic fields satisfy the anti-commutation relations

$$\begin{aligned} [\phi(\vartheta_1), \phi(\vartheta_2)]_+ &= 2\vartheta_1 \cdot \vartheta_2, \quad \vartheta_1, \vartheta_2 \in \mathcal{Y}^\chi, \\ [\psi(\vartheta_1), \psi^*(\vartheta_2)]_+ &= \overline{\vartheta}_1 \cdot \vartheta_2, \quad \vartheta_1, \vartheta_2 \in \mathcal{Y}. \end{aligned}$$

In the following part of the section, for simplicity we restrict ourselves to the charged case.

18.4.5 Two-point functions for the Dirac equation

Consider the abstract Dirac equation (18.35), where $\zeta(t) \in \mathcal{Y}$ is replaced with an operator $S(t) \in B(\mathcal{Y})$:

$$(\gamma_0\partial_t + \Gamma - m\mathbb{1})S(t) = 0. \tag{18.56}$$

Recall that θ denotes the Heaviside function. The functions $\mathrm{e}^{\mathrm{i}bt}$ and $\mathrm{e}^{\mathrm{i}bt}\theta(b)$ solve (18.56) and appear naturally in the quantized theory:

$$\begin{aligned} [\psi_{t_2}(\vartheta_2), \psi^*_{t_1}(\vartheta_1)]_+ &= \overline{\vartheta}_2 \cdot \mathrm{e}^{\mathrm{i}b(t_2-t_1)}\vartheta_1\, \mathbb{1}, \\ \big(\Omega|\psi_{t_2}(\vartheta_2)\psi^*_{t_1}(\vartheta_1)\Omega\big) &= \overline{\vartheta}_2 \cdot \mathrm{e}^{\mathrm{i}b(t_2-t_1)}\theta(b)\vartheta_1, \quad \vartheta_1, \vartheta_2 \in \mathcal{Y}. \end{aligned}$$

18.4.6 Green's functions of the Dirac equation

Consider the abstract charged Dirac equation as in Subsect. 18.4.3.

Definition 18.51 *We say that* $\mathbb{R} \ni t \mapsto S(t) \in B(\mathcal{Y})$ *is a* Green's function *or* fundamental solution of Eq. (18.49) *if it solves*

$$(\gamma_0\partial_t + \Gamma - m\mathbb{1})S(t) = \delta(t) \otimes \mathbb{1}_{\mathcal{Y}}. \tag{18.57}$$

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We introduce in particular the following Green's functions:

$$
\begin{aligned}
&\text{retarded} && S^{+}(t) := -\theta(t)\mathrm{e}^{\mathrm{i}tb}\gamma_0,\\
&\text{advanced} && S^{-}(t) := -\theta(-t)\mathrm{e}^{\mathrm{i}tb}\gamma_0,\\
&\text{Feynman} && S_{\mathrm{F}}(t) := -\mathrm{e}^{\mathrm{i}tb}\left(\theta(t)\theta(b)-\theta(-t)\theta(-b)\right)\gamma_0,\\
&\text{anti-Feynman} && S_{\overline{\mathrm{F}}}(t) := -\mathrm{e}^{\mathrm{i}tb}\left(\theta(t)\theta(-b)-\theta(-t)\theta(b)\right)\gamma_0,\\
&\text{principal value} && S_{\mathrm{Pv}}(t) := -\frac{\operatorname{sgn}(t)}{2}\mathrm{e}^{\mathrm{i}tb}\gamma_0.
\end{aligned}
$$

Set

$$\epsilon := |b| = \sqrt{\Gamma^*\Gamma + m^2\mathbb{1}}.$$

We then have

$$(\gamma_0\partial_t + \Gamma - m\mathbb{1})(\gamma_0\partial_t + \Gamma + m\mathbb{1}) = -(\partial_t^2 + \epsilon^2).$$

Thus if $G(t)$ is a Green's function for $\partial_t^2 + \epsilon^2$, then $-(\gamma_0\partial_t + \Gamma + m\mathbb{1})G(t)$ is a Green's function for the Dirac equation. In fact we have the identities

$$
\begin{aligned}
S^{+}(t) &= -(\gamma_0\partial_t + \Gamma + m\mathbb{1})G^{+}(t),\\
S^{-}(t) &= -(\gamma_0\partial_t + \Gamma + m\mathbb{1})G^{-}(t),\\
S_{\mathrm{F}}(t) &= -(\gamma_0\partial_t + \Gamma + m\mathbb{1})G_{\mathrm{F}}(t),\\
S_{\overline{\mathrm{F}}}(t) &= -(\gamma_0\partial_t + \Gamma + m\mathbb{1})G_{\overline{\mathrm{F}}}(t),\\
S_{\mathrm{Pv}}(t) &= -(\gamma_0\partial_t + \Gamma + m\mathbb{1})G_{\mathrm{Pv}}(t),
\end{aligned}
$$

which easily follow from

$$\mathrm{e}^{\mathrm{i}tb} = \cos(\epsilon t) + \mathrm{i}\operatorname{sgn}(b)\sin(\epsilon t), \quad \Gamma + m\mathbb{1} = \mathrm{i}b\gamma_0.$$

The Feynman Green's functions arise in the quantum theory in the following way:

$$\big(\Omega|\mathrm{T}\big(\psi_{t_2}(\vartheta_2)\psi_{t_1}^*(\vartheta_1)\big)\Omega\big) = \overline{\vartheta}_2\cdot S_{\mathrm{F}}(t_2-t_1)\vartheta_1, \quad \vartheta_1,\vartheta_2\in\mathcal{Y},$$

where we have used the fermionic time-ordering operation: if A_1, A_2 are odd fermionic operators, then

$$\mathrm{T}\left(A_{t_2}A_{t_1}\right) := \theta(t_2-t_1)A_{t_2}A_{t_1} - \theta(t_1-t_2)A_{t_1}A_{t_2}. \tag{18.58}$$

18.4.7 Green's functions of the Dirac equation as operators

Let $\mathcal{Y}$ be as above. For simplicity, we assume that $\mathcal{Y}$ is separable. Similarly to Subsect. 18.3.10, we will use the space

$$L^2(\mathbb{R})\otimes\mathcal{Y} \simeq L^2(\mathbb{R},\mathcal{Y}). \tag{18.59}$$

We will use both the *temporal representation* and the *energy representation* of $L^2(\mathbb{R},\mathcal{Y})$. Green's functions of the abstract Dirac equation can be interpreted

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as quadratic forms on $(L^1 \cap L^2)(\mathbb{R}, \mathcal{Y})$, denoted S and given (in the temporal representation) by

$$\overline{\zeta}_1 \cdot S\zeta_2 := \int \overline{\zeta_1(t)} \cdot S(t-s)\zeta_2(s)\mathrm{d}t\mathrm{d}s. \tag{18.60}$$

In the energy representation they are multiplication operators. Here we list the most important Green's functions of the Dirac operator in the energy representation:

$$\begin{aligned}
S^+(\tau) &= \big(\mathrm{i}\gamma_0(\tau - \mathrm{i}0) + \Gamma - m\mathbb{1}\big)^{-1},\\
S^-(\tau) &= \big(\mathrm{i}\gamma_0(\tau + \mathrm{i}0) + \Gamma - m\mathbb{1}\big)^{-1},\\
S_{\mathrm{F}}(\tau) &= \big(\mathrm{i}\gamma_0(\tau - \mathrm{i}0\,\mathrm{sgn}(\tau)) + \Gamma - m\mathbb{1}\big)^{-1},\\
S_{\overline{\mathrm{F}}}(\tau) &= \big(\mathrm{i}\gamma_0(\tau + \mathrm{i}0\,\mathrm{sgn}(\tau)) + \Gamma - m\mathbb{1}\big)^{-1},\\
S_{\mathrm{Pv}}(\tau) &= \mathrm{Pv}\big(\mathrm{i}\gamma_0\tau + \Gamma - m\mathbb{1}\big)^{-1}.
\end{aligned}$$

18.4.8 Euclidean Green's function of the Dirac equation

Definition 18.52 *The* Euclidean Green's function for the abstract Dirac equation *is defined as*

$$S_{\mathrm{E}}(s) = \big(-\theta(s)\theta(b) + \theta(-s)\theta(-b)\big)\mathrm{e}^{-bs}\mathrm{i}\gamma_0.$$

Note that S_{E} solves

$$(-\mathrm{i}\gamma_0\partial_s + \Gamma - m\mathbb{1})S_{\mathrm{E}}(s) = \delta(s)\mathbb{1}_{\mathcal{Y}}.$$

It is related to the Green's function of the abstract Klein–Gordon equation by

$$S_{\mathrm{E}}(s) = (-\mathrm{i}\gamma_0\partial_s + \Gamma + m\mathbb{1})G_{\mathrm{E}}(s).$$

The function $S_{\mathrm{E}}(s)$ extends to an analytic function for complex s. We have

$$\frac{1}{\mathrm{i}}S_{\mathrm{E}}(\mathrm{i}t) = S_{\mathrm{F}}(t), \quad -\frac{1}{\mathrm{i}}S_{\mathrm{E}}(-\mathrm{i}t) = S_{\overline{\mathrm{F}}}(t).$$

Consider the operator on $L^2(\mathbb{R}, \mathcal{Y})$

$$-\mathrm{i}\gamma_0\partial_s + \Gamma - m\mathbb{1}.$$

In the energy representation, this becomes the operator of multiplication by

$$\gamma_0\tau + \Gamma - m\mathbb{1}.$$

It is closed on its natural domain. Moreover, we have

$$\begin{aligned}
(\gamma_0\tau + \Gamma - m\mathbb{1})^*(\gamma_0\tau + \Gamma - m\mathbb{1}) &= (\gamma_0\tau + \Gamma - m\mathbb{1})(\gamma_0\tau + \Gamma - m\mathbb{1})^*\\
&= \tau^2 + \Gamma^*\Gamma + m^2\mathbb{1} \ > \ 0,
\end{aligned}$$

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which implies that $-\mathrm{i}\gamma_0\partial_s + \Gamma - m\mathbb{1}$ is invertible if (18.51) holds. If this is the case, set $S_\mathrm{E} := (-\mathrm{i}\gamma_0\partial_s + \Gamma - m\mathbb{1})^{-1}$. Then,

$$S_\mathrm{E}\zeta(s) = \int_\mathbb{R} S_\mathrm{E}(s-s_1)\zeta(s_1)\mathrm{d}s_1, \quad \zeta \in (L^1\cap L^2)(\mathbb{R},\mathcal{Y}).$$

In the energy representation, this is the operator of multiplication by

$$S_\mathrm{E}(\tau) = (\gamma_0\tau + \Gamma - m\mathbb{1})^{-1}.$$

Using the notation in Subsect. 18.3.11, and in particular the generator of dilations A, we see that

$$\mathbb{R}\ni\theta\mapsto \mathrm{e}^{\mathrm{i}\theta A}S_\mathrm{E}\mathrm{e}^{-\mathrm{i}\theta A} =: S_\mathrm{E}^\theta$$

extends to an analytic function in the strip $-\frac{\pi}{2} < \operatorname{Im}\theta < \frac{\pi}{2}$, given in the momentum representation by

$$S_\mathrm{E}^\theta(\tau) = (\gamma_0\mathrm{e}^\theta\tau + \Gamma - m\mathbb{1})^{-1}.$$

Its boundary values coincide with the Feynman and anti-Feynman propagators:

$$S_\mathrm{E}^{\mathrm{i}\frac{\pi}{2}} = S_\mathrm{F}, \quad S_\mathrm{E}^{-\mathrm{i}\frac{\pi}{2}} = S_{\overline{\mathrm{F}}}.$$

This is the *Wick rotation* in the fermionic case.

18.4.9 Thermal Green's function for the abstract Dirac equation

Clearly, we can apply the positive temperature formalism of Subsect. 17.2.7 to a system described by an abstract Dirac equation, obtaining a β-KMS state ω_β and the corresponding Araki–Wyss CAR representation. In this subsection, parallel to Subsect. 18.3.12, we describe the 2-point functions given by this state.

Definition 18.53 *The* thermal Euclidean Green's function at inverse temperature β of the abstract Dirac equation *is defined for $s\in[0,\beta]$ as*

$$S_{\mathrm{E},\beta}(s) := -\mathrm{i}\frac{\mathrm{e}^{-sb}}{\mathbb{1}+\mathrm{e}^{-\beta b}}\gamma_0.$$

Note that $S_{\mathrm{E},\beta}$ is the unique solution of the problem

$$\begin{aligned}&(-\mathrm{i}\gamma_0\partial_s + \Gamma - m\mathbb{1})S_{\mathrm{E},\beta}(s) = 0, \quad s\in]0,\beta[,\\ &-\mathrm{i}\gamma_0 S_{E,\beta}(0) = \mathrm{i}\gamma_0 S_{E,\beta}(\beta) + \mathbb{1}_\mathcal{Y}.\end{aligned} \tag{18.61}$$

(18.61) can be interpreted as

$$(-\mathrm{i}\gamma_0\partial_s + \Gamma - m\mathbb{1})S_{\mathrm{E},\beta}(s) = \delta(s)\mathbb{1}_\mathcal{Y}, \quad \text{on } S_\beta,$$

where we look for functions with anti-periodic boundary conditions at $\beta = 0$. More precisely, we consider functions ζ on S_β such that $\zeta(\beta) = -\zeta(0)$, and the Dirac delta function is defined as $\int_{S_\beta}\delta(s)\zeta(s)\mathrm{d}s = \zeta(0) = -\zeta(\beta)$ (the "right hand side delta function").

$S_\beta \ni s \mapsto S_{\mathrm{E},\beta}(s)$ extends to a function continuous in the strip $\mathrm{Re}\, s \in [0, \beta]$ and holomorphic inside this strip. If ω_β is the β-KMS state, then

$$\omega_\beta\big(\psi_t(\vartheta_1)\psi_0^*(\vartheta_2)\big) = \overline{\vartheta}_1 \cdot S_{\mathrm{E},\beta}(\mathrm{i}t)(-\mathrm{i}\gamma_0)\vartheta_2,$$
$$\omega_\beta\big(\psi_0^*(\vartheta_2)\psi_t(\vartheta_1)\big) = \overline{\vartheta}_1 \cdot S_{\mathrm{E},\beta}(\beta + \mathrm{i}t)(-\mathrm{i}\gamma_0)\vartheta_2, \quad \vartheta_1, \vartheta_2 \in \mathcal{Y}.$$

Let $\partial_s^{\mathrm{ant}}$ denote the operator ∂_s on the Hilbert space $L^2(S_\beta)$ with the anti-periodic boundary conditions. Its domain is given by

$$\mathrm{Dom}\, \partial_s^{\mathrm{ant}} := \big\{\zeta \in L^2(S_\beta),\ \partial_s\zeta \in L^2(S_\beta),\ \zeta(0) = -\zeta(\beta)\big\}.$$

Note that $\partial_s^{\mathrm{ant}}$ is anti-self-adjoint.

Define the *anti-periodic discrete Fourier transform*

$$\mathcal{F}_\beta^{\mathrm{ant}} : L^2(S_\beta) \to l^2\left(\frac{2\pi}{\beta}\left(\mathbb{Z} + \frac{1}{2}\right)\right)$$

by

$$\mathcal{F}_\beta\zeta(\sigma) = \int_0^\beta \mathrm{e}^{-\mathrm{i}s\sigma}\zeta(s)\mathrm{d}s, \quad \zeta \in L^2(S_\beta).$$

Its inverse is

$$(\mathcal{F}_\beta^{\mathrm{ant}})^{-1}v(s) = \beta^{-1} \sum_{\sigma \in \frac{2\pi}{\beta}(\mathbb{Z}+\frac{1}{2})} \mathrm{e}^{\mathrm{i}s\sigma}v(\sigma), \quad v \in l^2\left(\frac{2\pi}{\beta}\left(\mathbb{Z} + \frac{1}{2}\right)\right).$$

Clearly, $\partial_s^{\mathrm{ant}} = (\mathcal{F}_\beta^{\mathrm{ant}})^{-1}(\mathrm{i}\sigma)\mathcal{F}_\beta^{\mathrm{ant}}$.

On the Hilbert space $L^2(S_\beta) \otimes \mathcal{Y} \simeq L^2(S_\beta, \mathcal{Y})$, we define the closed operator

$$(-\mathrm{i}\gamma_0\partial_s^{\mathrm{ant}} + \Gamma - m\mathbb{1}) = (-\mathrm{i}\gamma_0)(\partial_s^{\mathrm{ant}} + b).$$

Note that $\partial_s^{\mathrm{ant}} + b$ is a normal operator on its natural domain. We set $S_{\mathrm{E},\beta} := (-\mathrm{i}\gamma_0\partial_s^{\mathrm{ant}} + \Gamma - m\mathbb{1})^{-1}$. We then have

$$S_{\mathrm{E},\beta}\zeta(s) = \int_{S_\beta} S_{\mathrm{E},\beta}(s_1)\zeta(s - s_1)\mathrm{d}s_1, \quad \zeta \in L^2(S_\beta, \mathcal{Y}).$$

In the energy representation, obtained by applying $\mathcal{F}_\beta^{\mathrm{ant}}$, this becomes the operator of multiplication by the *fermionic Matsubara coefficients*:

$$S_{\mathrm{E},\beta}(\sigma) = (\gamma_0\sigma + \Gamma - m)^{-1}, \quad \sigma \in \frac{2\pi}{\beta}\left(\mathbb{Z} + \frac{1}{2}\right).$$

Set

$$G_{\mathrm{E},\beta}^{\mathrm{ant}}(s) = \frac{-\mathrm{e}^{-s\epsilon} + \mathrm{e}^{(s-\beta)\epsilon}}{2\epsilon(\mathbb{1} + \mathrm{e}^{-\beta\epsilon})}.$$

Note that $G_{\mathrm{E},\beta}^{\mathrm{ant}}$ is the unique solution of

$$-\partial_s^2 G_{\mathrm{E},\beta}^{\mathrm{ant}}(s) + \epsilon^2 G_{\mathrm{E},\beta}^{\mathrm{ant}}(s) = 0, \quad s \in]0, \beta[,$$
$$G_{\mathrm{E},\beta}^{\mathrm{ant}}(0) = -G_{\mathrm{E},\beta}^{\mathrm{ant}}(\beta), \quad \partial_s^+ G_{\mathrm{E},\beta}^{\mathrm{ant}}(0) + \partial_s^- G_{\mathrm{E},\beta}^{\mathrm{ant}}(\beta) = \mathbb{1}. \tag{18.62}$$

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In fact, we have

$$G^{\mathrm{ant}}_{\mathrm{E},\beta}(0) = -G^{\mathrm{ant}}_{\mathrm{E},\beta}(\beta) = \frac{-\mathbb{1} + \mathrm{e}^{-\beta\epsilon}}{2\epsilon(\mathbb{1} + \mathrm{e}^{-\beta\epsilon})},$$

$$\partial_s^+ G^{\mathrm{ant}}_{\mathrm{E},\beta}(0) = \partial_s^- G^{\mathrm{ant}}_{\mathrm{E},\beta}(\beta) = \frac{1}{2}\mathbb{1}.$$

Thus $G^{\mathrm{ant}}_{\mathrm{E},\beta}$ can be interpreted as the solution of the equation on S_β

$$-\partial_s^2 G^{\mathrm{ant}}_{\mathrm{E},\beta}(s) + \epsilon^2 G^{\mathrm{ant}}_{\mathrm{E},\beta}(s) = \delta(s)\mathbb{1}_{\mathcal{X}},$$

with anti-periodic boundary conditions at $\beta = 0$ and the right hand side Dirac delta function. Then we can express $S_{\mathrm{E},\beta}$ in terms of $G^{\mathrm{ant}}_{\mathrm{E},\beta}$ as

$$S_{\mathrm{E},\beta}(s) = (-\mathrm{i}\gamma_0\partial_s + \Gamma + m\mathbb{1})G^{\mathrm{ant}}_{\mathrm{E},\beta}(s).$$

18.5 Notes

The topics discussed in this chapter in the context of concrete systems, usually based on relativistic equations, can be found in every textbook on quantum field theory, such as Jauch–Röhrlich (1976), Schweber (1962), Weinberg (1995) and Srednicki (2007). The Racah and Wigner time reversals were introduced by Racah (1927) and Wigner (1932a), respectively. Our presentation, in spite of its abstract mathematical language, follows very closely the usual exposition; see in particular Srednicki (2007), Sect. 22, for complex bosons and Srednicki (2007), Sect. 49, for neutral fermions.

The idea of positive quantization of classical linear dynamics goes back to the early days of the quantum field theory. In the fermionic case it was first formulated in terms of the "Dirac sea"; see Dirac (1930). This approach hides the particle–anti-particle symmetry. Its modern formulation is attributed to Fock (1933) and Furry–Oppenheimer (1934).

In the case of bosons, the "Dirac sea" approach is not available. The bosonic positive energy quantization was described by Pauli–Weisskopf (1934).

An interesting outline of the history of quantum field theory, which in particular discusses the topic of the positive energy quantization, is contained in the introduction to the monograph of Weinberg (1995).

The role of complex structures in positive energy quantization was emphasized by Segal (1964) and Weinless (1969).

An abstract formulation of the positive energy quantization was given by Segal, and can be found e.g. in Baez–Segal–Zhou (1991).

Positive temperature Green's functions can be found e.g. in Fetter–Walecka (1971), and from a more mathematical point of view in Birke–Fröhlich (2002).