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DISCRETE PRODUCT SYSTEMS WITH TWISTED UNITS

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The spectral C*-algebra of the discrete product systems of H.T. Dinh is shown to be a twisted semigroup crossed product whenever the product system has a twisted unit. The covariant representations of the corresponding dynamical system are always faithful, implying the simplicity of these crossed products; an application of a recent theorem of G.J. Murphy gives their nuclearity. Furthermore, a semigroup of endomorphisms of $\mathcal{B}(H)$ having an intertwining projective semigroup of isometries can be extended to a group of automorphisms of a larger Type I factor.

INTRODUCTION

Discrete product systems and their C*-algebras were introduced by Dinh in [4] in connection with discrete semigroups of endomorphisms of type I factors; the theory was further developed in [5, 6] and a few basic facts are listed in Section 1.

Dinh showed that in general the C*-algebra of a product system is simple [4] and, assuming the existence of a unit, that it is a full corner in a classical crossed product, from which nuclearity follows, [6]. The key fact behind his proof of simplicity is the existence of a dual action. This, together with some quite technical results concerning the fixed point algebra, makes it possible to follow an argument similar to Cuntz's [3].

Since the known examples of product systems have twisted units, it is natural to ask whether their C*-algebras are twisted semigroup crossed products in some sense. With this question in mind we give in Section 2 a brief introduction to twisted semigroup crossed products developed around a universal property for covariant representations along the lines of [10, 11, 1, 8], and then state a twisted version of a theorem about faithful representations of semigroup crossed products [2, 1].

In Section 3 we show how, assuming the existence of a twisted unit, the spectral C*-algebra of a semigroup of endomorphisms of $\mathcal{B}(H)$ can be seen as a semigroup crossed product. The hard bits of the argument involving the fixed point algebra of the dual action are actually independent of the existence of units and are borrowed from Dinh's work. The main point here is to make the crossed product structure explicit in the hope that it will lead to a better understanding of the role of units in a discrete

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product system. As applications of the twisted crossed product structure of the spectral C*-algebras, we derive their simplicity from the material in Section 5 and nuclearity from a recent theorem of Murphy.

Another consequence of the existence of twisted units is obtained in Section 4 where a dilation theorem of Phillips and Raeburn [9, Theorem 2.1] (see also [7, Corollary 2.4]) is used to extend a semigroup of endomorphisms of $\mathcal{B}(H)$ having an intertwining projective semigroup of isometries to a group of automorphisms of a Type I factor, generalising [5, Theorem 3.1].

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1. Discrete Product Systems

Let Γ^+ denote the positive cone of a countable discrete subgroup Γ of $\mathbb{R}$. An abstract *discrete product system* over Γ^+ is the disjoint union of a family of separable Hilbert spaces $\{E_t : t \in \Gamma^+\}$ on which there is an associative, bilinear multiplication $(x,y) \in E_s \times E_t \mapsto xy \in E_{s+t}$ which acts like tensoring in the sense that

(i) $E_s E_t$ spans a dense subset of E_{s+t}, and
(ii) $\langle xx', yy'\rangle = \langle x,y\rangle\langle x',y'\rangle$ whenever $x,y \in E_t$ and $x',y' \in E_{t'}$.

This establishes a natural Hilbert space isomorphism between $E_s \otimes E_t$ and E_{s+t}.

A discrete product system is *concrete* if it consists of operators in $\mathcal{B}(H)$ with operator multiplication and inner product given by $T^*S = \langle S,T\rangle I$ for $S,T \in E_t \subset \mathcal{B}(H)$. In this case,

$$E_s^* E_t = \begin{cases} E_{t-s} & \text{if } s \leqslant t, \\ E_{s-t}^* & \text{otherwise.} \end{cases}$$

A *representation* of a discrete product system E on a Hilbert space H is a map $\phi : E \to \mathcal{B}(H)$ such that

(i) $\phi(xy) = \phi(x)\phi(y)$ for $x \in E_s$ and $y \in E_t$.
(ii) $\langle x,y\rangle I = \phi(y)^*\phi(x)$ when $x,y \in E_t$.

This is enough to imply that ϕ is linear and, in fact, isometric on each fiber E_t. Hence $\phi(E)$ is a concrete product system isomorphic to E.

If $\gamma : \Gamma^+ \to \operatorname{end}(\mathcal{B}(H))$ is a representation of Γ^+ by $*$-endomorphisms of $\mathcal{B}(H)$ then

$$E_t = \{T \in \mathcal{B}(H) : TA = \gamma_t(A)T \quad \text{for all } A \in \mathcal{B}(H)\}$$

is a concrete product system over Γ^+, and every concrete product system over Γ^+ arises this way. The endomorphisms can be retrieved from E by taking an orthonormal basis $\{V_n^t\}_{n\in\mathbb{N}}$ for each E_t and letting

$$\gamma_t(A) = \sum_n V_n^t A V_n^{t*}.$$

Thus γ_t is unital if and only if $\sum_n V_n^t V_n^{t*} = I$, which happens if and only if $E_t H$ spans a dense subspace of H, in which case we say that E is essential. Since Γ^+ is Archimedean, if γ_t is unital for a single $t \neq 0$ then it is unital for every $t \in \Gamma^+$.

DEFINITION: A function $\sigma : \Gamma^+ \times \Gamma^+ \to \mathbb{T}$ is a *multiplier* on Γ^+ if $\sigma(r,s)\sigma(r+s,t) = \sigma(r,s+t)\sigma(s,t)$ for $r,s,t \in \Gamma^+$.

A *σ-unit* (or *twisted unit*) for E is a nonzero cross-section $w : \Gamma^+ \to E$, $w_t \in E_t$, such that $w_s w_t = \sigma(s,t) w_{s+t}$ for $s,t \in \Gamma^+$.

Since $\|w_{s+t}\| = \|w_s\|\,\|w_t\|$, $\|w\|$ is never zero and $v_t = w_t/\|w_t\|$ is a normalised twisted unit. In a concrete product system such units are the projective representations of Γ^+ by isometries W_t which intertwine the associated semigroup of endomorphisms, that is, $W_t A = \gamma_t(A) W_t$. Dinh has constructed a product system in which the only units are twisted by σ [4].

A product system E over Γ^+ generates a universal C^*-algebra $C^*(E)$ such that for any representation ϕ of E, $x \mapsto \phi(x)$ extends to a C*-algebra homomorphism of $C^*(E)$ onto $C^*(\phi(E))$. $C^*(\phi(E))$ is canonically independent of ϕ because $C^*(E)$ is simple [4].

2. Twisted semigroup crossed products

Let Γ^+ be a countable dense subsemigroup of $\mathbb{R}^+$, and suppose $\sigma : \Gamma^+ \times \Gamma^+ \to \mathbb{T}$ is a multiplier on Γ^+. A twisted covariant representation of the semigroup dynamical system (A, Γ^+, α) with multiplier σ is a pair (π, V) in which π is a unital representation of the C*-algebra A, V is an isometric σ-representation of Γ^+, that is, $V_s V_t = \sigma(s,t) V_{s+t}$, and the covariance condition $\pi(\alpha_t(a)) = V_t \pi(a) V_t^*$ for $a \in A$ and $t \in \Gamma^+$ is satisfied. When concerned only with twisted covariant pairs with a specific multiplier σ, we shall refer to the dynamical system as a twisted dynamical system and denote it by $(A, \Gamma^+, \alpha, \sigma)$.

The crossed product of the system $(A, \Gamma^+, \alpha, \sigma)$ is defined in a manner similar to the way in which the crossed product by a group action is defined in [**10**], by way of a universal property with respect to twisted covariant pairs. If there exists at least one nontrivial covariant pair for the system $(A, \Gamma^+, \alpha, \sigma)$, an essentially unique C*-algebra $A \rtimes_{\alpha,\sigma} \Gamma^+$ can be constructed together with a unital homomorphism $i_A : A \to A \rtimes_{\alpha,\sigma} \Gamma^+$ and a twisted embedding of Γ^+ as isometries $i_{\Gamma^+} : \Gamma^+ \to A \rtimes_{\alpha,\sigma} \Gamma^+$ such that

(1) (i_A, i_{Γ^+}) is a covariant pair for $(A, \Gamma^+, \alpha, \sigma)$,

(2) for any other covariant pair (π, U) there is a representation $\pi \times U$ of $A \rtimes_{\alpha,\sigma} \Gamma^+$ such that $\pi = (\pi \times U) \circ i_A$ and $U = (\pi \times U) \circ i_{\Gamma^+}$, and

(3) $A \rtimes_{\alpha,\sigma} \Gamma^+$ is generated by $i_A(A)$ and $i_{\Gamma^+}(\Gamma^+)$ as a C*-algebra.

The details are similar to those for the untwisted case [**1**]: the difficulty resides in showing that a given system actually has covariant representations. This is the case in Section 3 because covariant pairs for the dynamical system correspond to representations of a product system, which are known to exist by [**4**, Section 3].

Faithful representations of the twisted crossed product can be characterised as in Theorem 1.2 of [**1**]; the same proof works because the multiplier cancels out in all the crucial places.

Theorem 2.1. *Let σ be a multiplier on Γ^+, and suppose (π, V) is a covariant representation for the twisted system $(A, \Gamma^+, \alpha, \sigma)$ such that*

(i) *π is faithful, and*

(ii) *for all finite subsets F of Γ^+ and all choices of $a_{x,y} \in A$,*

$$\left\| \sum_{x \in F} V_x^* \pi(a_{x,x}) V_x \right\| \leqslant \left\| \sum_{x,y \in F} V_x^* \pi(a_{x,y}) V_y \right\|.$$

Then $\pi \times V$ is a faithful representation of $A \rtimes_{\alpha,\sigma} \Gamma^+$.

3. The twisted system $(\mathcal{F}_E, \Gamma^+, \alpha, \sigma)$

Let E be a discrete product system over Γ^+. The maps

$$\begin{aligned} \beta_{ts} : \mathcal{B}(E_s) &\to \mathcal{B}(E_t) \cong \mathcal{B}(E_s) \otimes \mathcal{B}(E_{t-s}) \\ X &\mapsto X \otimes I_{t-s}, \end{aligned}$$

where $s \leqslant t$ and I_{t-s} denotes the identity operator on E_{t-s}, form a system of unital embeddings which is coherent because of the associativity of multiplication on E. Thus the product system gives a directed system of C^*-algebras, each one isomorphic to $\mathcal{B}(E_t)$. The direct limit of this system will be denoted by B_∞. Since each embedding is injective, each $\mathcal{B}(E_t)$ embeds as a subalgebra B_t of B_∞. The corresponding copy of the compact operators $\mathcal{K}(E_t)$ in B_∞ will be denoted by $\mathcal{K}_t$. Following [**4**, Definition 2.6], we define a C*-algebra

$$\mathcal{F}_E = \overline{\operatorname{span}} \bigcup_t \mathcal{K}_t \subset B_\infty.$$

Since Γ^+ is countable, $\mathcal{F}_E$ is an AF algebra, generated by the 'rank-one' elements $R_{x,y} = \langle \cdot, y \rangle x$ for $x, y \in E_t$ at each level $t \in \Gamma^+$.

From now on we assume that E has a normalised twisted unit v, with multiplier $\sigma \in Z^2(\Gamma^+,\mathbb{T})$, that is, for each $t \in \Gamma^+$ there exists $v_t \in E_t$, with $\|v_t\| = 1$ and $v_s v_t = \sigma(s,t)v_{s+t}$ for $s,t \in \Gamma^+$.

Denote by e_s the rank-one projection R_{v_s,v_s} for $s \in \Gamma^+$, and define 'tensoring on the left by e_s' by $(e_s \otimes \cdot) : X \in \mathcal{B}(E_t) \mapsto e_s \otimes X \in \mathcal{B}(E_{s+t})$, using the identification of E_{s+t} with $E_s \otimes E_t$.

Since for every $s,t \in \Gamma^+$ tensoring on the left by e_s commutes with tensoring on the right by I_t, we obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{B}(E_r) & \xrightarrow{\cdot\otimes I_{t-r}} & \mathcal{B}(E_t) & \xrightarrow{\cdot\otimes I} & \cdots B_\infty \\ \big\downarrow{\scriptstyle e_s\otimes\cdot} & & \big\downarrow{\scriptstyle e_s\otimes\cdot} & & \big\downarrow{\scriptstyle \alpha_s} \\ \mathcal{B}(E_{s+r}) & \xrightarrow{\cdot\otimes I_{t-r}} & \mathcal{B}(E_{s+t}) & \xrightarrow{\cdot\otimes I} & \cdots B_\infty \end{array} \tag{3.1}$$

which gives an endomorphism α_s of B_∞ for each $s \in \Gamma^+$. Since $K \in \mathcal{K}_t$ implies $\alpha_s(K) \in \mathcal{K}_s \otimes \mathcal{K}_t \cong \mathcal{K}_{s+t}$, α_s restricts to an endomorphism of $\mathcal{F}_E$. In particular, $\alpha_t(R_{x,y}) = R_{v_t x, v_t y}$ and α_0 is the identity.

The semigroup property for $\{\alpha_s\}_{s\in\Gamma^+}$ reduces to $e_s \otimes e_t = e_{s+t}$, which can be proved by applying the left hand side to an elementary tensor $f \otimes g$ with $f \in E_s$ and $g \in E_t$,

$$(e_s \otimes e_t)(f \otimes g) = \langle f, v_s\rangle\langle g, v_t\rangle(v_s \otimes v_t) = \langle f \otimes g, v_s \otimes v_t\rangle v_s \otimes v_t,$$

and then using multiplication on E instead of tensor products to obtain

$$\langle fg, v_s v_t\rangle v_s v_t = \langle fg, \sigma(s,t)v_{s+t}\rangle\sigma(s,t)v_{s+t} = e_{s+t}(fg).$$

The following two propositions establish the relation between covariant representations of the twisted semigroup dynamical system $(\mathcal{F}_E, \Gamma^+, \alpha, \sigma)$ and representations of the product system E.

Proposition 3.1. *Suppose E is a product system over Γ^+ with a normalised σ-unit v. If $\phi : E \to \mathcal{B}(H)$ is a representation of E on a Hilbert space H, then there exists a covariant representation (π_ϕ, V_ϕ) of $(\mathcal{F}_E, \Gamma^+, \alpha, \sigma)$ on H such that for each $t \in \Gamma^+$, $V_{\phi,t} = \phi(v_t)$ and $\pi_\phi(R_{x,y}) = \phi(x)\phi(y)^*$ for $x,y \in E_t$. Moreover, the pair (π_ϕ, V_ϕ) satisfies conditions (i) and (ii) of Theorem 2.1.*

Proof: The elements $R_{x,y} = \langle\cdot, y\rangle x$ for $x,y \in E_t$ span a dense subspace of $\mathcal{K}(E_t)$. The map π_ϕ defined by $\pi_\phi(R_{x,y}) = \phi(x)\phi(y)^*$ is multiplicative and contractive on linear combinations [4, Proposition 2.10], so it extends to a representation of $\mathcal{F}_E$.

It is clear that $V_{\phi,t} = \phi(v_t)$ defines a σ-representation of Γ^+ by isometries on H, and it suffices to show covariance on the rank-one generators at each level;

$$\begin{aligned}\pi_\phi(\alpha_s(R_{x,y})) &= \pi_\phi(e_s \otimes R_{x,y}) = \pi_\phi(R_{v_s x, v_s y}) = \phi(v_s x)\phi(v_s y)^* \\ &= \phi(v_s)\phi(x)\phi(y)^*\phi(v_s)^* = V_s \pi_\phi(R_{x,y}) V_s^*.\end{aligned}$$

Since $\mathcal{F}_E = \overline{\operatorname{span}} \bigcup_t \mathcal{K}_t$, in order to show that π_ϕ is faithful it suffices to show that it is faithful on each of the subalgebras $\overline{\operatorname{span}} \bigcup_{j=1}^n K_{t_j}$ where $t_1 < t_2 < \ldots < t_n$ (by convention assume $\mathcal{K}_0 = \mathbb{C}I$). The ideals of $\overline{\operatorname{span}} \bigcup_{j=1}^n K_{t_j}$ are nested and the smallest one is $\mathcal{K}_{t_n}$, [4, Lemma 2.14]. Taking $R_{x,x}$ with $0 \neq x \in E_{t_n}$ shows that π_ϕ does not vanish on $\mathcal{K}_{t_n}$.

Let $x, y \in E_t$; if $s \geqslant t$ the operator

$$V_s^* \pi_\phi(R_{x,y}) V_s = \phi(v_s)^*\phi(x)\phi(y)^*\phi(v_s)$$

is in $\phi(E_{s-t})^*\phi(E_{s-t})$, hence it is a scalar, while if $s \leqslant t$, it is in $\phi(E_{t-s})\phi(E_{t-s})^*$. In any case $V_s^* \pi_\phi(\mathcal{F}_E) V_s \subseteq \pi_\phi(\mathcal{F}_E)$ and condition (ii) becomes

$$\|\pi_\phi(a_0)\| \leqslant \left\| \sum_{i=-1}^{-n} V_{t_i}^* \pi_\phi(a_i) + \pi_\phi(a_0) + \sum_{i=1}^{n} \pi_\phi(a_i) V_{t_i} \right\|$$

for $a_i \in \mathcal{F}_E$ and $t_i \in \Gamma^+$ for $i = 0, \pm 1, \pm 2, \ldots, \pm n$. This key fact is proved in [4, Proposition 2.16] by constructing a projection Q (denoted $\gamma_{t_n}(P)$ there) with the familiar two properties:

$$\|Q\pi(a_0)Q\| = \|\pi(a_o)\| \text{ and } Q\pi(a_i)V_{t_i}Q = 0 \text{ if } t_i \neq 0,$$

which make Cuntz's argument work. □

Proposition 3.2. *Suppose E is a product system over Γ^+ with a normalised σ-unit v. If (π, V) is a covariant pair for $(\mathcal{F}_E, \Gamma^+, \alpha, \sigma)$ then*

$$\phi : f \mapsto \pi(R_{f,v_t})V_t \qquad \text{for } f \in E_t,$$

extends to a representation of E such that $\pi = \pi_\phi$ and $V = V_\phi$.

Proof: If $f \in E_s$ and $g \in E_t$, then

$$\begin{aligned}\phi(g)^*\phi(f) &= (\pi(R_{g,v_t})V_t)^*\pi(R_{f,v_t})V_t = V_t^*\pi(R_{v_t,g}R_{f,v_t})V_t \\ &= V_t^*\langle f,g\rangle \pi(e_t)V_t = \langle f,g\rangle I,\end{aligned}$$

and

$$\begin{aligned}\phi(f)\phi(g) &= \pi(R_{f,v_s})V_s\pi(R_{g,v_t})V_t = \pi(R_{f,v_s})V_s\pi(R_{g,v_t})V_s^*V_sV_t \\ &= \pi(R_{f,v_s})\pi(R_{v_sg,v_sv_t})V_sV_t = \pi(R_{f,v_s})\pi\Big(\overline{\sigma(s,t)}R_{v_sg,v_{s+t}}\Big)\sigma(s,t)V_{s+t} \\ &= \pi\big(R_{f,v_s}R_{v_sg,v_{s+t}}\big)V_{s+t} = \pi\big(R_{fg,v_{s+t}}\big)V_{s+t} \\ &= \phi(fg).\end{aligned}$$

Thus ϕ is a representation of E.

By definition, $\phi(v_t) = \pi(R_{v_t,v_t})V_t = \pi(\alpha_t(I))V_t$, which by covariance equals $V_tIV_t^*V_t = V_t$. For $f,g \in E_t$,

$$\pi_\phi(R_{f,g}) = \phi(f)\phi(g)^* = \pi(R_{f,v_t})V_tV_t^*\pi(R_{g,v_t})^* = \pi(R_{f,v_t}R_{v_t,v_t}R_{v_t,g}) = \pi(R_{f,g}),$$

hence $(\pi_\phi, V_\phi) = (\pi, V)$. ∎

Theorem 3.3. *If E is a product system over Γ^+ having a σ-unit then $C^*(E)$ is isomorphic to $\mathcal{F}_E \rtimes_{\alpha,\sigma} \Gamma^+$, hence nuclear and simple.*

Proof: By the corresponding universal properties, the bijection between representations of E and covariant representations of $(\mathcal{F}_E, \Gamma^+, \alpha, \sigma)$ gives an isomorphism between $C^*(E)$ and $\mathcal{F}_E \rtimes_{\alpha,\sigma} \Gamma^+$. Since every covariant representation arises as in Proposition 3.1, every representation of $\mathcal{F}_E \rtimes_{\alpha,\sigma} \Gamma^+$ is faithful, hence $C^*(E)$ is simple. For nuclearity it suffices to observe that since the endomorphisms α_s are injective and the algebra $\mathcal{F}_E$ is AF, the system $(\mathcal{F}_E, \Gamma^+, \alpha, \sigma)$ satisfies the hypothesis of [**8**, Theorem 3.1]. ∎

Although $\alpha_s(\mathcal{F}_E)$ is hereditary for every $s \in S$, we are unable to use [**8**, Theorem 4.2] to conclude the faithfulness of every covariant pair because it is not clear how to construct a dual action at the level of the represented crossed product. Thus the key reference to [**4**, Proposition 2.16] in the proof of Proposition 3.1 seems unavoidable. Furthermore, [**8**, Theorem 5.2] does not apply to the present situation because $\mathcal{F}_E$ contains copies of the compact operators, and cannot have a faithful tracial state.

4. Minimal automorphic extensions

Dinh proved in [**5**] that a semigroup of endomorphisms of a type I factor with an intertwining semigroup of isometries can be extended in a minimal way to a group of automorphisms of a larger type I factor $\mathcal{M}$. His proof uses the intertwining isometries to construct a directed system of Hilbert spaces whose inductive limit is the Hilbert space on which $\mathcal{M}$ acts naturally.

The goal of this section is to derive a relatively short proof of a more general result from a dilation theorem for projective isometric representations of semigroups [**9**, 7]. The extra generality is obtained by requiring the existence of only an intertwining *projective* representation of the semigroup by isometries, and by observing that the argument works for the normal cancellative semigroups discussed in [7].

A cancellative semigroup S is *normal* if $xS = Sx$ for every $x \in S$, in which case it can be embedded as a generating subsemigroup of a group G in an essentially unique way. Defining $x \succ y$ to mean $x \in Sy$ gives a partial directed preorder on G which is invariant under multiplication by elements of S, [7, Remark 1.2].

THEOREM 4.1. *Let S be a countable normal subsemigroup generating the group G and suppose α is a representation of S by unital endomorphisms of $\mathcal{B}(H)$, with H separable, such that*

$$\alpha_s(T)V_s = V_sT, \qquad \text{for } s \in S, \quad T \in \mathcal{B}(H),$$

for some projective isometric representation V of S with multiplier $\sigma \in Z^2(S, \mathbb{T})$.

Then there are a separable Hilbert space $\mathcal{H}$, a representation $\widetilde{\alpha}$ of G by automorphisms of $\mathcal{B}(\mathcal{H})$ and a unital embedding φ of $\mathcal{B}(H)$ as a subfactor of $\mathcal{B}(\mathcal{H})$, such that

- (i) *$\widetilde{\alpha}$ extends α, in the sense that $\widetilde{\alpha}_s$ leaves $\varphi(\mathcal{B}(H))$ invariant and $\widetilde{\alpha}_s \circ \varphi = \varphi \circ \alpha_s$ for $s \in S$, and*
- (ii) *the extension is minimal, in the sense that $\left(\bigcup_{x \in G} \widetilde{\alpha}_x \varphi(\mathcal{B}(H))\right)'' = \mathcal{B}(\mathcal{H})$.*

PROOF: The intertwining family of isometries $\{V_s : s \in S\}$ satisfies the hypothesis of Theorem 2.1 and Corollary 2.4 of [7], thus, retaining the notation from there, there exists a Hilbert space $\mathcal{H}$ and a dilation of the isometries from H to a projective unitary representation U of G on $\mathcal{H}$, whose multiplier extends σ. Define automorphisms of $\mathcal{B}(\mathcal{H})$ by $\widetilde{\alpha}_x(A) = U_x A U_x^*$ for $A \in \mathcal{B}(\mathcal{H})$. It all reduces to defining the right embedding of $\mathcal{B}(H)$ into $\mathcal{B}(\mathcal{H})$.

From the proof of [7, Theorem 2.1], recall that $\mathcal{H}$ is the completion of the pre-Hilbert space H_0 of functions $f : S \to H$ for which there is some element $s \in S$, called admissible for f, such that

$$f(y) = \overline{\sigma(ys^{-1}, s)} V_{ys^{-1}}(f(s)) \qquad \text{for } y \in sS,$$

under the pre-inner product defined by $\langle f, g\rangle = \langle f(s), g(s)\rangle$ for s admissible for both f and g. For $A \in \mathcal{B}(H)$ and f in H_0 let

$$(\varphi(A)f)(s) = \alpha_s(A)f(s), \qquad s \in S.$$

The following computation shows that $\varphi(A)f$ is in H_0 as well, by showing that any value of s admissible for f turns out to be admissible for $\varphi(A)f$,

$$\begin{aligned}(\varphi(A)f)(y) = \alpha_y(A)f(y) &= \alpha_y(A)\overline{\sigma(ys^{-1},s)}V_{ys^{-1}}f(s)\\ &= \overline{\sigma(ys^{-1},s)}V_{ys^{-1}}\alpha_s(A)f(s)\\ &= \overline{\sigma(ys^{-1},s)}V_{ys^{-1}}(\varphi(A)f)(s).\end{aligned}$$

The third equality holds because of the intertwining property of V. By the definition of the inner product on H_0, $\langle\varphi(A)f,\varphi(A)f\rangle_{\mathcal{H}} = \langle\alpha_s(A)f(s),\alpha_s(A)f(s)\rangle_H \leqslant \|A\|^2\,\|f\|^2$, where s is any admissible value for f. Thus $\|\varphi(A)\| \leqslant \|A\|$ and $\varphi(A)$ extends uniquely to all of $\mathcal{H}$. Routine computations, which depend on α_s being a unital $*$-endomorphism for each $s \in S$, show that φ is in fact a unital representation of $\mathcal{B}(H)$ on $\mathcal{H}$. Since $\mathcal{H}$ is separable, φ is an embedding of $\mathcal{B}(H)$ as a subfactor of $\mathcal{B}(\mathcal{H})$.

To check that $\widetilde{\alpha}_s$ is an extension of α_s for every $s \in S$, let $f \in H_0$ and $s,t \in S$ and compute

$$\begin{aligned}(U_s\varphi(A)f)(t) = \sigma(t,s)(\varphi(A)f)(ts) &= \sigma(t,s)\alpha_{ts}(A)f(ts)\\ &= \alpha_t(\alpha_s(A))(U_sf)(t) = (\varphi(\alpha_s(A))(U_sf))(t).\end{aligned}$$

This proves that $\widetilde{\alpha}_s(\varphi(A)) = \varphi(\alpha_s(A))$ for every $s \in S$ and $A \in \mathcal{B}(H)$.

The embedding $\xi \mapsto \widehat{\xi}$ of H in $\mathcal{H}$ defined by $\widehat{\xi}(s) = V_s\xi$ is compatible with φ in that $\varphi(T)\widehat{\xi}(s) = \alpha_s(T)V_s\xi = V_sT\xi = \widehat{T\xi}$ for every $T \in \mathcal{B}(H)$.

To prove (ii) we shall show that every unit vector $g \in \mathcal{H}$ is cyclic for the action of $\bigcup_x \widetilde{\alpha}_x(\varphi(\mathcal{B}(H)))$ on $\mathcal{H}$.

Suppose ξ is an arbitrary nonzero element of H and $\varepsilon > 0$. Choose a unit vector $f \in H_0$ with $\|f-g\| < \varepsilon/\|\xi\|$. Let s be admissible for f, thus $\|f(s)\| = \|f\| = 1$ and denote by T the rank-one operator $\langle\cdot,f(s)\rangle\xi$ on H, so $Tf(s) = \xi$ and $\|T\| = \|\xi\|$. By the last paragraph in the proof of [7, Theorem 2.1], $f = U_s^*\widehat{f(s)}$ and

$$\widetilde{\alpha}_s^{-1}(\varphi(T))f = U_s^*\varphi(T)U_sU_s^*\widehat{f(s)} = U_s^*\varphi(T)\widehat{f(s)} = U_s^*\widehat{Tf(s)} = U_s^*\widehat{\xi},$$

hence

$$\left\|\widetilde{\alpha}_s^{-1}(\varphi(T))g - U_s^*\widehat{\xi}\right\| \leqslant \|\varphi(T)\|\,\|g-f\| < \varepsilon.$$

Thus the closure of $\bigcup_{s\in S}\widetilde{\alpha}_s^{-1}(\varphi(\mathcal{B}(H)))g$ contains $\bigcup_{s\in S}U_s^*\widehat{H}$, which by [7, Theorem 2.1 (ii)] is dense in $\mathcal{H}$. Therefore $\bigcup_x \widetilde{\alpha}_x(\varphi(\mathcal{B}(H)))'$ consists of scalars only. ∎

Since the dilation Hilbert space $\mathcal{H}$ constructed in Theorem 4.1 generalises the direct limit considered in [5] to the case in which the embeddings are twisted by a 2-cocycle, one may view $\mathcal{H}$ as the 'direct limit of a twisted system' of Hilbert spaces.

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