Canad. J. Math. Vol. **61** (6), 2009 pp. 1201–1213

# Invariant Einstein Metrics on Some Homogeneous Spaces of Classical Lie Groups

Andreas Arvanitoyeorgos, V. V. Dzhepko, and Yu. G. Nikonorov

*Abstract.* A Riemannian manifold $(M, \rho)$ is called Einstein if the metric $\rho$ satisfies the condition $\mathrm{Ric}(\rho) = c \cdot \rho$ for some constant $c$. This paper is devoted to the investigation of $G$-invariant Einstein metrics, with additional symmetries, on some homogeneous spaces $G/H$ of classical groups. As a consequence, we obtain new invariant Einstein metrics on some Stiefel manifolds $SO(n)/SO(l)$. Furthermore, we show that for any positive integer $p$ there exists a Stiefel manifold $SO(n)/SO(l)$ that admits at least $p$ $SO(n)$-invariant Einstein metrics.

## 1 Introduction

A Riemannian manifold $(M, \rho)$ is called Einstein if the metric $\rho$ satisfies the condition $\mathrm{Ric}(\rho) = c \cdot \rho$ for some real constant $c$. A detailed exposition on Einstein manifolds can be found in A. Besse [4], and more recent results on homogeneous Einstein manifolds can be found in the survey by M. Wang [20]. General existence results are hard to obtain. Among the first important attempts are the works of G. Jensen [10] and M. Wang and W. Ziller [21]. Recently, a new existence approach was introduced by C. Böhm, M. Wang, and W. Ziller [5, 7]. The above existence results were used by C. Böhm and M. Kerr [6] to show that every compact simply connected homogeneous space up to dimension 11 admits at least one invariant Einstein metric. It is known [4, 6, 21] that in dimension 12 there are examples of compact simply connected homogeneous spaces that do not admit any invariant Einstein metrics.

The structure of the set of invariant Einstein metrics on a given homogeneous space is still not very well understood in general. The situation is only clear for a few classes of homogeneous spaces, such as isotropy irreducible homogeneous spaces, low dimensional examples, certain flag manifolds, and some other special types of homogeneous spaces [2, 4, 14, 15, 18]. For an arbitrary compact homogeneous space $G/H$ it is not clear if the set of invariant Einstein metrics (up to isometry and up to scaling) is finite or not (see [22]). A finiteness conjecture states that this set is, in fact, finite if the isotropy representation of $G/H$ consists of pairwise inequivalent irreducible components [7, p. 683].

Received by the editors February 28, 2007.

The first author was partially supported by the C. Carathéodory grant # C.161 (2007-10), University of Patras. The second and the third authors are supported in part by the Russian Foundation for Basic Research(grant N 05-01-00611-a) and by the State Maintenance Program for the Leading Scientific Schools of the Russian Federation (grants NSH-8526.2006.1) The third author is supported by the State Maintenance Program for Young Russian Scientists of the Russian Federation (grant MD-5179.2006.1)

AMS subject classification: Primary: 53C25; secondary: 53C30.

Keywords: Riemannian manifolds, homogeneous spaces, Einstein metrics, Stiefel manifolds.

Let $G$ be a compact Lie group and $H$ a closed subgroup so that $G$ acts almost effectively on $G/H$. In this paper we investigate $G$-invariant metrics on $G/H$ with additional symmetries. More precisely, let $K$ be a closed subgroup of $G$ with $H \subset K \subset G$, and suppose that $K = L' \times H'$, where $\{e_{L'}\} \times H' = H$. It is clear that $K \subset N_G(H)$, the normalizer of $H$ in $G$. If we denote $L = L' \times \{e_{H'}\}$, then the group $\widetilde{G} = G \times L$ acts on $G/H$ by $(a, b) \cdot gH = agb^{-1}H$, and the isotropy subgroup at $eH$ is $\widetilde{H} = \{(a, b) : ab^{-1} \in H\}$.

Later on it will be shown that the set $\mathcal{M}^{\widetilde{G}}$ of $\widetilde{G}$-invariant metrics on $\widetilde{G}/\widetilde{H}$ is a subset of $\mathcal{M}^G$, the set of $G$-invariant metrics on $G/H$. Therefore, it would be simpler to search for invariant Einstein metrics on $\mathcal{M}^{\widetilde{G}}$. In this way we obtain existence results for Einstein metrics for certain quotients.

We apply this method for the case of Stiefel manifolds $SO(n)/SO(n-k)$. Note that the simplest case $S^{n-1} = SO(n)/SO(n-1)$ is an irreducible symmetric space, therefore it admits up to scale a unique invariant Einstein metric. Concerning history, it was S. Kobayashi [13] who first proved the existence of an invariant Einstein metric on $T_1S^n = SO(n)/SO(n-2)$. Later, A. Sagle [16] proved that the Stiefel manifolds $SO(n)/SO(n-k)$ admit at least one homogeneous invariant Einstein metric. For $k \geq 3$, G. Jensen [11] found a second metric. Einstein metrics on $SO(n)/SO(n-2)$ are completely classified. If $n = 3$, the group $SO(3)$ has a unique Einstein metric. If $n \geq 5$, A. Back and W. Y. Hsiang [3] showed that $SO(n)/SO(n-2)$ admits exactly one homogeneous invariant Einstein metric. The same result was obtained by M. Kerr [12]. The Stiefel manifold $SO(4)/SO(2)$ admits exactly two invariant Einstein metrics, which follows from the classification of 5-dimensional homogeneous Einstein manifolds due to D. V. Alekseevsky, I. Dotti, and C. Ferraris [1]. We also refer to [7, p. 727–728] for further discussion. For $k \geq 3$ there is no obstruction for existence of more than two homogeneous invariant Einstein metrics on Stiefel manifolds $SO(n)/SO(n-k)$.

In particular we prove the following.

***Theorem 1.1*** *If $s > 1$ and $l > k \geq 3$, then the Stiefel manifold $SO(sk+l)/SO(l)$ admits at least four $SO(sk+l) \times (SO(k))^s$-invariant Einstein metrics, two of which are Jensen's metrics.*

We also prove the following.

***Theorem 1.2*** *For any positive integer $p$ there exists a Stiefel manifold $SO(n)/SO(l)$ that admits at least $p$ $SO(n)$-invariant Einstein metrics.*

We remark that, in fact, there are other homogeneous spaces for which the number of invariant Einstein metrics can be at least a prescribed number. Indeed, if a compact homogeneous space $G/H$ admits two distinct invariant Einstein metrics, then the product of $m$ copies of this space admits at least $m+1$ distinct Einstein metrics invariant under the natural action of $G^m$. There are analogous examples in the class of non-product homogeneous spaces. For instance, in [9, p. 62] it was shown in particular that the groups $SU(2n)$, $SU(2n+3)$, $Sp(2n)$, and $Sp(2n+1)$ admit at least $n+1$ distinct left-invariant Einstein metrics, whereas the groups $SO(2n)$ and $SO(2n+1)$ admit at least $3n-2$ left-invariant Einstein metrics.

We also note that the methods of this paper (after some minor revisions) can be used for obtaining new invariant Einstein metrics on homogeneous spaces of the groups $Sp(n)$ and $SU(n)$. It is interesting to note that the case of unitary groups $SU(n)$ is more tractable (in some sense) than the cases of the orthogonal groups $SO(n)$ and the symplectic groups $Sp(n)$. For instance, one can compare the Einstein equations and its solutions for the spaces $SU(k_1+k_2+k_3)/S(U(k_1)\times U(k_2)\times U(k_3))$ (see [2, 14]) with the Einstein equations and its solutions for the spaces

$$SO(k_1+k_2+k_3)\big/ SO(k_1)\times SO(k_2)\times SO(k_3)$$

and

$$Sp(k_1+k_2+k_3)\big/ Sp(k_1)\times Sp(k_2)\times Sp(k_3)$$

(see [15]).

This paper is organized as follows: the basic construction for searching for invariant Einstein metrics with additional symmetries on $G/H$ is presented in Section 2, where we also clarify the meaning of such symmetries. Metrics with this property are described for some homogeneous spaces of the group $SO(n)$ (Section 3, Lemma 3.2). In Section 4 we compute the scalar curvature for these metrics (Proposition 4.3), and the variational approach to the Einstein metrics is given in Proposition 4.5. In Section 5, as an application of our construction, we obtain Jensen's invariant Einstein metrics on the Stiefel manifold $SO(k_1+k_2)/SO(k_2)$. In Section 6 we investigate invariant Einstein metrics on $SO(sk+l)/SO(l)$. Finally, in Section 7 the proofs of the main results are given.

## 2 The Main Construction

Let $G$ be a compact Lie group and $H$ a closed subgroup so that $G$ acts almost effectively on $G/H$. Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of $G$ and $H$, and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be a reductive decomposition of $\mathfrak{g}$ with respect to some $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. The orthogonal complement $\mathfrak{p}$ can be identified with the tangent space $T_{eH}G/H$. Any $G$-invariant metric $\rho$ of $G/H$ corresponds to an $\mathrm{Ad}(H)$-invariant inner product $(\,\cdot\,,\,\cdot\,)$ on $\mathfrak{p}$ and vice versa. For $G$ semisimple, the negative of the Killing form $B$ of $\mathfrak{g}$ is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, therefore we can choose the above decomposition with respect to this form. We will use such a decomposition later on. Moreover, the restriction $\langle\,\cdot\,,\,\cdot\,\rangle = -B|_{\mathfrak{p}}$ is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{p}$, which generates a $G$-invariant metric on $G/H$ called *standard*.

The normalizer $N_G(H)$ of $H$ in $G$ acts on $G/H$ by $(a, gH) \mapsto ga^{-1}H$. For a fixed $a$ this action induces a $G$-equivariant diffeomorphism $\varphi_a\colon G/H \to G/H$. Note that if $a \in H$ this diffeomorphism is trivial, so the action of the gauge group $N_G(H)/H$ is well defined. However, it is simpler from a technical point of view to use the action of $N_G(H)$. Let $\rho$ be a $G$-invariant metric of $G/H$ with corresponding inner product $(\,\cdot\,,\,\cdot\,)$. Then the diffeomorphism $\varphi_a$ is an isometry of $(G/H, \rho)$ if and only if the operator $\mathrm{Ad}(a)|_{\mathfrak{p}}$ is orthogonal with respect to $(\,\cdot\,,\,\cdot\,)$.

Let $K$ be a closed subgroup of $G$ with $H \subset K \subset G$ such that $K = L' \times H'$, where $\{e_{L'}\} \times H' = H$, and consider $L = L' \times \{e_{H'}\}$. It is clear that $K \subset N_G(H)$. The

https://doi.org/10.4153/CJM-2009-056-2 Published online by Cambridge University Press

group $\widetilde{G} = G \times L$ acts on $G/H$ by $(a, b) \cdot gH = agb^{-1}H$, and the isotropy at $eH$ is given as follows.

***Lemma 2.1*** *The isotropy subgroup $\widetilde{H}$ is isomorphic to $K$.*

**Proof** It is clear that $\widetilde{H} = \{(a, b) \in G \times L : ab^{-1} \in H\}$. Let $i: K \hookrightarrow G$ be the inclusion of $K$ in $G$. Then $i(\{e_{L'}\} \times H') = H$ and $i(L' \times \{e_{H'}\}) = L$. Let $(a, b) \in G \times L$ be such that $ab^{-1} = h \in H$. Then $a = hb$, so

$$\begin{aligned}(a, b) = (hb, b) &= (i(b', e_{H'})i(e_{L'}, h'), i(b', e_{H'}))((b', h'), b') \\ &\in K \times L' = L' \times H' \times L'.\end{aligned}$$

Thus $\widetilde{H}$ is identified with a subgroup of $L' \times H' \times L'$, and it is then obvious that $\widetilde{H}$ is isomorphic to $L' \times H' = K$. ■

The set $\mathcal{M}^G$ of $G$-invariant metrics on $G/H$ is finite dimensional. We consider the subset $\mathcal{M}^{G,K}$ of $\mathcal{M}^G$ corresponding to $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{p}$ (and not only $\mathrm{Ad}(H)$-invariant).

Let $\rho \in \mathcal{M}^{G,K}$ and $a \in K$. The above diffeomorphism $\varphi_a$ is an isometry of $(G/H, \rho)$. The action of $\widetilde{G}$ on $(G/H, \rho)$ is isometric, so any metric form $\mathcal{M}^{G,K}$ can be identified as a metric in $\mathcal{M}^{\widetilde{G}}$ and vice versa. Therefore, we may think of $\mathcal{M}^{\widetilde{G}}$ as $\mathcal{M}^{G,K}$, which is a subset of $\mathcal{M}^G$.

Since metrics in $\mathcal{M}^{G,K}$ correspond to $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{p}$, we call these metrics $\mathrm{Ad}(K)$*-invariant metrics on* $G/H$.

The aim of this work is to apply the above construction for $G = SO(n)$ and prove the existence of Einstein metrics in the set $\mathcal{M}^{G,K}$ for choices of the subgroup $K = L' \times H'$.

Let $n \in \mathbb{N}$ and $k_1, k_2, \dots, k_s, k_{s+1}, \dots, k_{s+t}$ be natural numbers such that $k_i \geq 2$, $k_1 + \cdots + k_s = l$, $k_{s+1} + \cdots + k_{s+t} = m$, $l + m = n$. Let $G = SO(n)$ and $K = L' \times H'$, where $L' = SO(k_1) \times \cdots \times SO(k_s)$ and $H' = SO(k_{s+1}) \times \cdots \times SO(k_{t+s})$. The embedding of $K$ in $G$ is the standard one.

We note that for $s = 0$ we obtain a flag manifold of the group. Invariant Einstein metrics on $SO(k_1 + k_2 + k_3)/SO(k_1) \times SO(k_2) \times SO(k_3)$, were studied in [15, 17].

## 3 $\mathrm{Ad}(K)$-Invariant Metrics on the Space $G/H$

Let $\mathfrak{p}_i$ be the subalgebra $so(k_i)$ in $\mathfrak{g}$, $1 \leq i \leq s + t$. We note that for $1 \leq 1 \leq s$ the submodule $\mathfrak{p}_i$ of $\mathfrak{p}$ is an $\mathrm{Ad}(K)$-invariant and an $\mathrm{Ad}(K)$-irreducible submodule. For $1 \leq i < j \leq s + t$ we denote by $\mathfrak{p}_{(i,j)}$ the $\mathrm{Ad}(K)$-invariant and $\mathrm{Ad}(K)$-irreducible submodule of $\mathfrak{p}$ that is determined by the equality

$$so(k_i + k_j) = so(k_i) \oplus so(k_j) \oplus \mathfrak{p}_{(i,j)},$$

where $\mathfrak{p}_{(i,j)}$ is orthogonal to $so(k_i) \oplus so(k_j)$ with respect to the Killing form $B$.

Denote by $d_i$ and $d_{(i,j)}$ the dimensions of the modules $\mathfrak{p}_i$ and $\mathfrak{p}_{(i,j)}$, respectively. It is easy to obtain that $d_i = \frac{k_i(k_i-1)}{2}$ and $d_{(i,j)} = k_i k_j$.

We have a decomposition of $\mathfrak{p}$ into a sum of $\mathrm{Ad}(K)$-invariant and $\mathrm{Ad}(K)$-irreducible submodules:

$$\mathfrak{p} = \bigoplus_{i=1}^{s} \mathfrak{p}_i \oplus \bigoplus_{1 \le i < j \le s+t} \mathfrak{p}_{(i,j)} \,. \tag{3.1}$$

***Lemma 3.1*** *If for all $1 \le i \le s+t$ we have $k_i \ge 2$, and there is at most one $1 \le i \le s$ such that $k_i = 2$, then there are no pairwise $\mathrm{Ad}(K)$-isomorphic submodules among $\mathfrak{p}_i$ $(i = 1, \dots, s)$ and $\mathfrak{p}_{(i,j)}$ $(1 \le i < j \le s+t)$.*

**Proof** It is clear that $\mathfrak{p}_i$ and $\mathfrak{p}_j$ act on $\mathfrak{p}_{(i,j)}$, and that each $\mathfrak{p}_i$ acts on itself. Moreover, the last action is trivial if and only if $k_i = 2$. Therefore, there are no pairwise $\mathrm{Ad}(K)$-isomorphic submodules. ∎

If the assumptions of Lemma 3.1 are satisfied, then we have a complete description of the $\mathrm{Ad}(K)$-invariant metrics on $G/H$.

Let $\rho$ be any $\mathrm{Ad}(K)$-invariant metric on $G/H$ with corresponding $\mathrm{Ad}(K)$-invariant inner product $(\,\cdot\,,\,\cdot\,)$ on $\mathfrak{p}$.

***Lemma 3.2*** *If there are no pairwise $\mathrm{Ad}(K)$-isomorphic submodules among $p_i$ and $p_{(i,j)}$, then*

$$(\,\cdot\,,\,\cdot\,) = \sum_{i=1}^{s} x_i \cdot \langle\,\cdot\,,\,\cdot\,\rangle|_{\mathfrak{p}_i} + \sum_{1 \le i < j \le s+t} x_{(i,j)} \cdot \langle\,\cdot\,,\,\cdot\,\rangle|_{\mathfrak{p}_{(i,j)}} \tag{3.2}$$

*for positive constants $x_i > 0$ and $x_{(i,j)} > 0$, where $\langle\,\cdot\,,\,\cdot\,\rangle = -B|_{\mathfrak{p}}$. Therefore, the set of $\mathrm{Ad}(K)$-invariant metrics on $G/H$ depends on $(s+t)(s+t-1)/2 + s$ parameters.*

In the case of pairwise $\mathrm{Ad}(K)$-isomorphic modules $\mathfrak{p}_\alpha$ and $\mathfrak{p}_\beta$ the set of $\mathrm{Ad}(K)$-invariant metrics has a more complicated structure [21].

## 4 The Scalar Curvature and the Einstein Condition

Let $\{e_\alpha^j\}$ be an orthonormal basis of $\mathfrak{p}_\alpha$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle$, where $1 \le j \le d_\alpha$ (here $\alpha$ means any of the symbols of type $i$ or $(k,i)$). We define the numbers (see [21]) $[\alpha\beta\gamma]$ by the equation

$$[\alpha\beta\gamma] = \sum_{i,j,k} \left\langle \left[e_\alpha^i, e_\beta^j\right], e_\gamma^k \right\rangle^2,$$

where $i, j, k$ vary from 1 to $d_\alpha, d_\beta, d_\gamma$, respectively. The symbols $[\alpha\beta\gamma]$ are symmetric with respect to all three indices, as follows from the $\mathrm{Ad}(G)$-invariance of $\langle\,\cdot\,,\,\cdot\,\rangle$.

For any Lie algebra $\mathfrak{q}$ we shall use the symbol $B_{\mathfrak{q}}$ for the Killing form of $\mathfrak{q}$. If a simple algebra $\mathfrak{q}$ is a subalgebra of a Lie algebra $\mathfrak{r}$, then we denote by $\alpha_{\mathfrak{r}}^{\mathfrak{q}}$ a real number that satisfies the equality $B_{\mathfrak{q}} = \alpha_{\mathfrak{r}}^{\mathfrak{q}} \cdot B_{\mathfrak{r}}|_{\mathfrak{q}}$.

***Lemma 4.1*** *Let $\mathfrak{q} \subset \mathfrak{r}$ be arbitrary subalgebras in $\mathfrak{g}$ with $\mathfrak{q}$ simple. Consider in $\mathfrak{q}$ an orthonormal (with respect to $-B_{\mathfrak{r}}$) basis $\{f_j\}$ ($1 \leq j \leq \dim(\mathfrak{q})$). Then*

$$\sum_{j,k=1}^{\dim(\mathfrak{q})} (-B_{\mathfrak{r}}([f_i, f_j], f_k))^2 = \alpha_{\mathfrak{r}}^{\mathfrak{q}}, \quad i = 1, \dots, \dim(\mathfrak{q}),$$

$$\sum_{i,j,k=1}^{\dim(\mathfrak{q})} (-B_{\mathfrak{r}}([f_i, f_j], f_k))^2 = \alpha_{\mathfrak{r}}^{\mathfrak{q}} \cdot \dim(\mathfrak{q}),$$

*where $\alpha_{\mathfrak{r}}^{\mathfrak{q}}$ is determined by the equation $B_{\mathfrak{q}} = \alpha_{\mathfrak{r}}^{\mathfrak{q}} \cdot B_{\mathfrak{r}}|_{\mathfrak{q}}$.*

**Proof** By direct computations it follows that

$$\sum_{j,k} (-B_{\mathfrak{r}}([f_i, f_j], f_k))^2 = \sum_j -B_{\mathfrak{r}}([f_i, f_j], [f_i, f_j]) = \frac{1}{\alpha_{\mathfrak{r}}^{\mathfrak{q}}} \sum_j -B_{\mathfrak{q}}([f_i, f_j], [f_i, f_j]).$$

The vectors $\widetilde{f_j} = \left(1/\sqrt{\alpha_{\mathfrak{r}}^{\mathfrak{q}}}\right) f_j$ form an orthonormal basis in $\mathfrak{q}$ with respect to $-B_{\mathfrak{q}}$. Then for $i = 1, \dots, \dim(\mathfrak{q})$, and by use of the properties of the Killing form, we have that

$$1 = -B_{\mathfrak{q}}(\widetilde{f_i}, \widetilde{f_i}) = \sum_j B_{\mathfrak{q}}([\widetilde{f_i}, [\widetilde{f_i}, \widetilde{f_j}]], \widetilde{f_j}) = \frac{1}{(\alpha_{\mathfrak{r}}^{\mathfrak{q}})^2} \sum_j -B_{\mathfrak{q}}([f_i, f_j], [f_i, f_j]),$$

which proves the first statement of the lemma. The second statement is a direct consequence of the first one. ∎

Using this lemma we obtain an explicit expression for $[\alpha\beta\gamma]$. It is clear that the only non-zero symbols (up to permutation of indices) are

$$[aaa], \quad [a(a,b)(a,b)], \quad [b(a,b)(a,b)],$$

with $1 \leq a < b \leq s + t$, and $[(a,b)(b,c)(a,b)]$ with $1 \leq a < b < c \leq s + t$.

***Lemma 4.2*** *The following relations hold:*

$$[aaa] = \frac{k_a(k_a - 1)(k_a - 2)}{2(n-2)}, \quad [a(a,b)(a,b)] = \frac{k_a k_b (k_a - 1)}{2(n-2)},$$

$$[b(a,b)(a,b)] = \frac{k_a k_b (k_b - 1)}{2(n-2)}, \quad [(a,b)(b,c)(a,c)] = \frac{k_a k_b k_c}{2(n-2)}.$$

**Proof** For the standard embedding $so(k) \subset so(n)$ we have $\alpha_{so(n)}^{so(k)} = \frac{k-2}{n-2}$ (see[9]).

The first equality $[aaa] = \frac{k_a(k_a-1)(k_a-2)}{2(n-2)}$ follows from Lemma 4.1. In fact, $d_a = \dim(so(k_a)) = k_a(k_a - 1)/2$ and $\alpha_{so(n)}^{so(k_a)} = \frac{k_a-2}{n-2}$.

To prove the second equality we consider the subalgebra $so(k_a + k_b) \subset so(n)$. It is clear that $[\mathfrak{p}_a, \mathfrak{p}_b] = 0$, $[\mathfrak{p}_a, \mathfrak{p}_{(a,b)}] \subset \mathfrak{p}_{(a,b)}$. According to Lemma 4.1 we have that

$$[aaa] + [a(a,b)(a,b)] = \dim(\mathfrak{p}_a) \cdot \alpha_{so(n)}^{so(k_a+k_b)} = \frac{k_a(k_a-1)(k_a+k_b-2)}{2(n-2)},$$

which proves the second equality. The third equality can be obtained analogously.

To prove the fourth equality we consider the subalgebra $so(k_a + k_b + k_c) \subset so(n)$. It is clear that

$$\dim(\mathfrak{p}_{(a,b)}) \cdot \alpha_{so(n)}^{so(k_a+k_b+k_c)} = 2\left([(a,b)a(a,b)] + [(a,b)b(a,b)] + [(a,b)(b,c)(a,c)]\right),$$

from which we obtain the last equality. ∎

According to [21], the scalar curvature $S$ of $(\cdot,\cdot)$ is given by

$$S((\cdot,\cdot)) = \frac{1}{2}\sum_{\alpha} \frac{d_\alpha}{x_\alpha} - \frac{1}{4}\sum_{\alpha,\beta,\gamma}[\alpha\beta\gamma]\frac{x_\gamma}{x_\alpha x_\beta},$$

where $\alpha$, $\beta$, and $\gamma$ are arbitrary symbols of the type $i$ $(1 \le i \le s)$ or of the type $(i, j)$ $(1 \le i < j \le s+t)$.

For the metric (3.2) this formula takes the following form.

***Proposition 4.3*** *The scalar curvature $S$ of an* $\mathrm{Ad}(K)$*-invariant metric* (3.2) *has the form*

(4.1)

$$\begin{aligned} S = {} & \sum_{a=1}^{s} \frac{k_a(k_a-1)(k_a-2)}{8(n-2)} \cdot \frac{1}{x_a} + \frac{1}{2}\sum_{1\le a<b\le s+t} \frac{k_ak_b}{x_{(a,b)}} \\ & - \frac{1}{8(n-2)} \sum_{\substack{1\le a\le s \\ a+1\le b\le s+t}} k_ak_b(k_a-1)\frac{x_a}{x_{(a,b)}^2} \\ & - \frac{1}{8(n-2)} \sum_{1\le a<b\le s} k_ak_b(k_b-1)\frac{x_b}{x_{(a,b)}^2} \\ & - \frac{1}{4(n-2)} \sum_{1\le a<b<c\le s+t} k_ak_bk_c\left(\frac{x_{(a,b)}}{x_{(a,c)}x_{(b,c)}} + \frac{x_{(a,c)}}{x_{(a,b)}x_{(b,c)}} + \frac{x_{(b,c)}}{x_{(a,b)}x_{(a,c)}}\right). \end{aligned}$$

Denote by $\mathcal{M}_1^G$ the set of all $G$-invariant metrics with a fixed volume element on the space $G/H$. The following variational principle for invariant Einstein metrics is well known.

***Proposition 4.4*** [4] *Let $G/H$ be a homogeneous space where $G$ and $H$ are compact. Then the $G$-invariant Einstein metrics on the homogeneous space $G/H$ are precisely the critical points of the scalar curvature functional $S$ restricted to $\mathcal{M}_1^G$.*

For the general construction as described in Section 2, the above variational principle implies the following.

***Proposition 4.5*** *Let $\mathcal{M}_1^{G,K}$ be the subset of $\mathcal{M}^{G,K}$ with fixed volume element. Then a metric in $\mathcal{M}_1^{G,K}$ is Einstein if and only if it is a critical point of the scalar curvature functional $S$ restricted to $\mathcal{M}_1^{G,K}$.*

**Proof** The set $\mathcal{M}_1^{G,K}$ is precisely the set of $\widetilde{G}$-invariant metrics with fixed volume element on $\widetilde{G}/\widetilde{H}$. ■

The volume condition for the metric (3.2) takes the form

$$\prod_{i=1}^{s} x_i^{d_i} \cdot \prod_{1 \le i < j \le s+t} x_{(i,j)}^{d_{(i,j)}} = \text{constant}\,. \tag{4.2}$$

By using Proposition 4.5 the problem of searching for $\mathrm{Ad}(K)$-invariant Einstein metrics on $G/H$ reduces to a Lagrange-type problem for the scalar curvature functional $S$ under the constraint (4.2).

## 5 Jensen's Metrics

As a first simple illustration of Proposition 4.5, we will show that Jensen's metrics [11] can be obtained on the Stiefel manifold $SO(k_1+k_2)/SO(k_2)$ ($k_1 \ge 2$). We apply Proposition 4.3, formula (4.1) for $s = 1$ and $t = 1$. Then the scalar curvature reduces to

$$S = \frac{k_1(k_1-1)(k_1-2)}{8(n-2)}\frac{1}{x_1} + \frac{1}{2}\frac{k_1k_2}{x_{12}} - \frac{1}{8(n-2)}k_1k_2(k_1-1)\frac{x_1}{{x_{12}}^2}.$$

The volume condition (4.2) is $V = x_1^{d_1}x_{12}^{d_{12}} = \text{constant}$. By use of the Lagrange method we obtain the equation $(k_1-2)x_{12}^2 - 2(k_1+k_2-2)x_1x_{12} + (k_2+k_1-1)x_1^2 = 0$. If $k_1 = 2$, the above equation has a unique solution $x_{12} = \frac{k_2+1}{2k_2}x_1$. If $k_1 > 2$, the equation has two solutions,

$$x_{12} = \frac{k_1+k_2-2 \pm \sqrt{(k_1+k_2-2)^2 - (k_1-2)(k_1+k_2-1)}}{k_1-2}x_1.$$

These solutions are $SO(k_1+k_2) \times SO(k_1)$-invariant Einstein metrics on $SO(k_1+k_2)/SO(k_2)$, and were found by G. Jensen [11].

## 6 New Examples of Einstein Metrics

Let $t = 1$ and $k_1 = k_2 = \cdots = k_s = k$ ($s \ge 2$), $k_{s+1} = l$. Then $n = sk + l$. We will investigate $SO(sk+l) \times (SO(k))^s$-invariant Einstein metrics on the space $SO(sk+l)/SO(l)$ ($s \ge 2$). If we choose $L' = (SO(k))^s$, then by Lemma 3.2 the set of $SO(n) \times (SO(k))^s$-invariant metrics depends on $(s^2+3s)/2$ parameters, which makes the problem difficult for big values of $s$.

However, if we choose $L' = N_{SO(sk)}((SO(k))^s)$, the normalizer of $(SO(k))^s$ in $SO(sk)$, (this is an extension of $(SO(k))^s$ by a discrete subgroup), then the number of parameters of the corresponding $SO(n) \times L$-invariant metrics reduces to three. More precisely, the following lemma holds.

***Lemma 6.1*** *If $L'$ is chosen as above, and $K = L' \times H'$, where $H' = SO(l)$, then we have a decomposition of $\mathfrak{p}$ into a sum of $\mathrm{Ad}(K)$-invariant and $\mathrm{Ad}(K)$-irreducible submodules*

$$\mathfrak{p} = \widetilde{\mathfrak{p}}_1 \oplus \widetilde{\mathfrak{p}}_2 \oplus \widetilde{\mathfrak{p}}_3, \tag{6.1}$$

*where $\widetilde{\mathfrak{p}}_1 = \bigoplus_{i=1}^{s} \mathfrak{p}_i$, $\widetilde{\mathfrak{p}}_2 = \bigoplus_{1\le i<j\le s} \mathfrak{p}_{(i,j)}$, and $\widetilde{\mathfrak{p}}_3 = \bigoplus_{i=1}^{s} \mathfrak{p}_{(i,s+1)}$* (cf. (3.1))*. The submodules $\widetilde{\mathfrak{p}}_1, \widetilde{\mathfrak{p}}_2$, and $\widetilde{\mathfrak{p}}_3$ are pairwise inequivalent; therefore any $\mathrm{Ad}(K)$-invariant inner product of $\mathfrak{p}$ is given by*

$$(\cdot\,,\cdot) = x\cdot\langle\cdot\,,\cdot\rangle|_{\widetilde{\mathfrak{p}}_1} + y\cdot\langle\cdot\,,\cdot\rangle|_{\widetilde{\mathfrak{p}}_2} + z\cdot\langle\cdot\,,\cdot\rangle|_{\widetilde{\mathfrak{p}}_3}. \tag{6.2}$$

**Proof** For any $1 \le i < j \le s$ any two of the submodules $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are interchanged by $\mathrm{Ad}(a)$, for some $a \in L$. Similarly, any two of $\mathfrak{p}_{(i,s+1)}$ and $\mathfrak{p}_{(j,s+1)}$ $(1 \le i, j \le s)$ are interchanged, and any two of $\mathfrak{p}_{(i,j)}$ and $\mathfrak{p}_{(i',j')}$ $(1 \le i < j \le s, 1 \le i' < j' \le s)$. Therefore decomposition (6.1) follows. The other statements are obvious. ■

Next, we compute the scalar curvature for metric (6.2).

***Proposition 6.2*** *The scalar curvature $S$ of an $\mathrm{Ad}(K)$-invariant metric* (6.2) *has the form*

$$\begin{aligned}\frac{8(n-2)}{sk}\cdot S &= (k-1)(k-2)\cdot\tfrac{1}{x} + (s-1)k((s+2)k-4)\cdot\tfrac{1}{y}\\ &\quad + 4(ks+l-2)l\cdot\tfrac{1}{z}\\ &\quad - \Big((s-1)k(k-1)\cdot\frac{x}{y^2} + (k-1)l\cdot\frac{x}{z^2} + (s-1)kl\cdot\frac{y}{z^2}\Big)\end{aligned} \tag{6.3}$$

*with volume condition $x^{sk(k-1)/2}y^{s(s-1)k^2/2}z^{skl}$* = constant.

**Proof** Metric (6.2) is a special case of metric (3.2) for which the scalar curvature was obtained in Proposition 4.3. We apply these expressions for $t = 1$, $k_1 = \cdots = k_s = k$, $k_{s+1} = l$, and $x_a = x$ $(1 \le a \le s)$, $x_{a,b} = y$ $(1 \le a < b \le s)$, $x_{a,s+1} = z$ $(1 \le a \le s)$ to obtain

$$\sum_{a=1}^{s}\frac{k_a(k_a-1)(k_a-2)}{8(n-2)}\cdot\frac{1}{x_a} = \frac{sk(k-1)(k-2)}{8(n-2)}\cdot\frac{1}{x},$$

$$\sum_{1\le a<b\le s+t}\frac{k_ak_b}{x_{(a,b)}} = \sum_{1\le a<b\le s}\frac{k_ak_b}{x_{(a,b)}} + \sum_{1\le a\le s, b=s+1}\frac{k_ak_b}{x_{(a,b)}} = \frac{s(s-1)k^2}{2}\cdot\frac{1}{y} + skl\cdot\frac{1}{z},$$

$$\begin{aligned}\sum_{\substack{1\le a\le s\\ a+1\le b\le s+t}} k_ak_b(k_a-1)\frac{x_a}{x_{(a,b)}^2} &= \sum_{1\le a<b\le s} k_ak_b(k_a-1)\frac{x_a}{x_{(a,b)}^2} + \sum_{\substack{1\le a\le s\\ b=s+1}} k_ak_b(k_a-1)\frac{x_a}{x_{(a,b)}^2}\\ &= \frac{s(s-1)k^2(k-1)}{2}\cdot\frac{x}{y^2} + sk(k-1)l\cdot\frac{x}{z^2},\end{aligned}$$

https://doi.org/10.4153/CJM-2009-056-2 Published online by Cambridge University Press

$$\sum_{1\le a<b\le s} k_a k_b (k_b-1)\frac{x_b}{x_{(a,b)}^2} = \frac{s(s-1)k^2(k-1)}{2}\cdot\frac{x}{y^2};$$

$$\begin{aligned}
&\sum_{1\le a<b<c\le s+t} k_a k_b k_c\Big(\frac{x_{(a,b)}}{x_{(a,c)}x_{(b,c)}}+\frac{x_{(a,c)}}{x_{(a,b)}x_{(b,c)}}+\frac{x_{(b,c)}}{x_{(a,b)}x_{(a,c)}}\Big)\\
&\quad=\sum_{1\le a<b<c\le s} k_a k_b k_c\Big(\frac{x_{(a,b)}}{x_{(a,c)}x_{(b,c)}}+\frac{x_{(a,c)}}{x_{(a,b)}x_{(b,c)}}+\frac{x_{(b,c)}}{x_{(a,b)}x_{(a,c)}}\Big)\\
&\qquad+\sum_{1\le a<b\le s, c=s+1} k_a k_b k_c\Big(\frac{x_{(a,b)}}{x_{(a,c)}x_{(b,c)}}+\frac{x_{(a,c)}}{x_{(a,b)}x_{(b,c)}}+\frac{x_{(b,c)}}{x_{(a,b)}x_{(a,c)}}\Big)\\
&\quad=\frac{s(s-1)(s-2)k^3}{2}\cdot\frac{1}{y}+\frac{s(s-1)k^2 l}{2}\cdot\Big(\frac{y}{z^2}+\frac{2}{y}\Big).
\end{aligned}$$

Therefore, equation (6.3) is obtained. The dimensions of $\widetilde{\mathfrak{p}}_1, \widetilde{\mathfrak{p}}_2$ and $\widetilde{\mathfrak{p}}_3$ are $\frac{sk(k-1)}{2}$, $\frac{s(s-1)k^2}{2}$, and $skl$, respectively, so the volume condition is obtained, and the proof is complete. ■

In order to find the critical points of the scalar curvature $S$ for the above two cases, note that $\frac{8(n-2)}{sk}\cdot S$ and the volume are functions of the form

$$F(x,y,z)=\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-d\frac{x}{y^2}-e\frac{x}{z^2}-f\frac{y}{z^2},\quad G(x,y,z)=x^p y^q z^r,$$

where the constant $a, b, c, d, e, f, p, q$, and $r$ are positive, and

$$d=\frac{pb-qa}{q+2p},\qquad f=\frac{qe}{p}.$$

We need to consider the following problem: find all the critical points (with positive coordinates) of $F(x,y,z)$ under the constraint $G(x,y,z)=$ constant. This is a Lagrange-type problem.

***Lemma 6.3*** *The critical points of the function $F(x,y,z)$ with positive $x,y,z$ under the constraint $G(x,y,z)=$* constant *satisfy the following equations:*

(i) *If $x=y$, then $r(a+d)z^2-pcxz+e(2p+2q+r)x^2=0$;*

(ii) *If $x\neq y$, then*

$$x=\frac{aqyz^2}{pfy^2+d(q+2p)z^2}$$

*and*

$$\begin{aligned}(2d(q+2p)+bq)drz^4-(q+2p)cdqyz^3+(2d(r+q)(q+2p)\\
+(r+2p)aq)fy^2z^2-cfpqy^3z+(r+2q)f^2py^4=0.\end{aligned}$$

*If in addition $d(q+2p)>aq$, then $y>x$.*

https://doi.org/10.4153/CJM-2009-056-2 Published online by Cambridge University Press

**Proof** It is easy to see that the problem reduces to the following system:

$$(6.4)\qquad \begin{cases} q\Big(-\dfrac{a}{x}-d\dfrac{x}{y^2}-e\dfrac{x}{z^2}\Big) = p\Big(-\dfrac{b}{y}+2d\dfrac{x}{y^2}-f\dfrac{y}{z^2}\Big), \\ r\Big(-\dfrac{a}{x}-d\dfrac{x}{y^2}-e\dfrac{x}{z^2}\Big) = p\Big(-\dfrac{c}{z}+2e\dfrac{x}{z^2}+2f\dfrac{y}{z^2}\Big). \end{cases}$$

From the first equation of (6.4) we get:

$$pf\frac{y-x}{z^2} = \frac{aq(y-x)\big(y-\frac{d(q+2p)}{aq}x\big)}{xy^2}.$$

If $x = y$, we easily obtain that $r(a+d)z^2 - pcxz + e(2p+2q+r)x^2 = 0$. Note that solutions to this equation correspond to Jensen's metrics (see Section 5). If $x \neq y$, we obtain

$$x = \frac{aqyz^2}{pfy^2 + d(q+2p)z^2},$$

which implies that $x > 0$ for any $z > 0$, $y > 0$. If $x \neq y$, then

$$\frac{y}{x} > \frac{d(q+2p)}{aq},$$

therefore, if $d(q+2p) > aq$, then $y > x$.

Substituting the above expression for $x$ in the second equation of (6.4), we obtain the Einstein equation

$$(2d(q+2p)+bq)drz^4 - (q+2p)cdqyz^3 + (2d(r+q)(q+2p) + (r+2p)aq)fy^2z^2 - cfpqy^3z + (r+2q)f^2py^4 = 0. \quad \blacksquare$$

Let

$$(6.5)\qquad P(u) = (2d(q+2p)+bq)dru^4 - (q+2p)cdqu^3 + (2d(r+q)(q+2p) + (r+2p)aq)fu^2 - cfpqu + (r+2q)f^2p.$$

It is clear that the equation $P(z/y) = 0$ is equivalent to the last one of Lemma 6.3. The following simple lemma will be used for the proof of the main theorems.

***Lemma 6.4*** *If $P(1) < 0$, then equation $P(u) = 0$ has at least two positive solutions.*

**Proof** It is evident from the facts that $P(0) = (r+2q)f^2p > 0$ and $P(u) \to +\infty$ when $u \to +\infty$. $\blacksquare$

https://doi.org/10.4153/CJM-2009-056-2 Published online by Cambridge University Press

## 7 Proof of the Main Results

**Proof of Theorem 1.1** We apply Lemma 6.3 for values of $a, b, c, d, e, f, p, q$, and $r$ taken from Proposition 6.2. If $x = y$, then obviously one gets Jensen's metrics as discussed in Section 5. If $x \neq y$, then for the polynomial (6.5) it is that

$$P(1) = \frac{1}{2}s^2k^4l(k-1)(s-1)^2(sk^2 - skl + k - 2 - l^2 + 2l).$$

It it easy to check that $sk^2 - skl + k - 2 - l^2 + 2l < 0$ for $l > k$, $k, s \geq 2$, thus $P(1) < 0$. By Lemma 6.4 the equation $P(u) = 0$ has at least two positive solutions, so we obtain at least two new invariant Einstein metrics. Since in this case $d(q + 2p) > aq$, then $y > x$ for these new metrics. ■

**Proof of Theorem 1.2** Fix a positive integer $p$ and choose positive integers $n, l$ such that $n-l$ has at least $p$ different prime factors $a_1, a_2, \dots, a_p$ with $a_i < l\,(i = 1, \dots, p)$. Take $k$ any of the $a_i$'s, and positive integer $s$ so that $n - l = sk$. For this choice of $k, l, s$ we use Theorem 1.1, and obtain that the homogenous space $SO(n)/SO(l)$ admits at least two $SO(n) \times (SO(k))^s$-invariant Einstein metrics that are not invariant under the group $SO(n) \times SO(n-l)$, that is, they are not Jensen's metrics. It is easy to see that for different choices of $k = a_i$ we obtain pairwise different metrics (because they have different full motion groups). Therefore, we obtain at least $2p$ pairwise different $SO(n)$-invariant Einstein metrics on the Stiefel manifold $SO(n)/SO(l)$. ■

It would be an interesting problem to investigate the nature of the invariant Einstein metrics of given volume on the space $SO(n)/SO(l)$ (see Theorem 1.1), as critical points of the scalar curvature functional curvature $S$, for instance, by analysing the Hessian of $S$ at the critical points. Of course, this would require having explicit solutions of the algebraic systems of equations obtained from the Einstein equation. Another interesting problem is to find metrics with maximal and minimal values of the scalar curvature $S$ among all $SO(n)$-invariant Einstein metrics of fixed volume on the spaces $SO(n)/SO(l)$.

**Acknowledgements** The third author is grateful for the hospitality of the Department of Mathematics of the University of Patras-Greece, where the main part of this paper was prepared. The authors would like to thank the referee for useful comments on the manuscript, and for stimulating suggestions for further research.

## References

[1] D. V. Alekseevsky, I. Dotti, and C. Ferraris, *Homogeneous Ricci positive 5-manifolds*, Pacific J. Math. **175**(1996), no. 1, 1–12.

[2] A. Arvanitoyeorgos, *New invariant Einstein metrics on generalized flag manifolds*. Trans. Amer. Math. Soc. **337**(1993), no. 2, 981–995.

[3] A. Back and W. Y. Hsiang, *Equivariant geometry and Kervaire spheres*. Trans. Amer. Math. Soc. **304**(1987) no. 1, 207–227.

[4] A. Besse, *Einstein Manifolds*. Ergebnisse der mathematik und ihrer Grenzgebiete 10, Springer-Verlag, Berlin, 1987.

[5] C. Böhm, *Homogeneous Einstein metrics and simplicial complexes*. J. Differential Geom. **67**(2004), no. 1, 79–165.

[6] C. Böhm and M. Kerr, *Low-dimensional homogenous Einstein manifolds*. Trans. Amer. Math. Soc. **358**(2006), no. 4, 1455–1468.

[7] C. Böhm, M. Wang, and W. Ziller, *A variational approach for compact homogeneous Einstein manifolds*. Geom. Func. Anal. **14**(2004), no. 4, 681–733.
[8] J. E. D'Atri and N. Nickerson, *Geodesic symmetries in space with special curvature tensors*, J. Diff. Geom. 9 (1974) 251–262.
[9] J. E. D'Atri and W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups.* Memoirs Amer. Math. Soc. **18**(1979) no. 215.
[10] G. Jensen, *The scalar curvature of left-invariant Riemannian metrics*. Indiana J. Math. **20**(1971) 1125–1144.
[11] ———, *Einstein metrics on principal fiber bundles* J. Differential Geom. **8**(1973), 599–614.
[12] M. Kerr, *New examples of homogeneous Einstein metrics*. Michigan J. Math. **45**(1998)no. 1, 115–134.
[13] S. Kobayashi, *Topology of positive pinched Kähler manifolds*. Tôhoku Math. J. **15**(1963), 121–139.
[14] M. Kimura, *Homogeneous Einstein metrics on certain Kähler C-spaces*. In: Recent Topics in Differential and Analytic Geometry. Adv. Stud. Pure Math. 18-I. Academic Press, Boston, MA, 1990, pp. 303–320.
[15] A. V. Lomshakov, Yu. G. Nikonorov, and E. V. Firsov, *Invariant Einstein metrics on three-locally-symmetric spaces*. (Russian) Mat. Tr. **6**(2003), no. 2, 80–101; Engl. transl. in Siberian Adv. Math. **14**(2004), no. 3, 43–62.
[16] A. Sagle, *Some homogeneous Einstein manifolds*. Nagoya Math. J. **39**(1970), 81–106.
[17] Yu. G. Nikonorov, *On a class of homogeneous compact Einstein manifolds.* (Russian) Sibirsk. Mat. Zh. **41**(2000), no. 1, 200–205; Engl. transl. in Siberian Math. J. **41**(2000), no. 1, 168–172.
[18] Y. Sakane, *Homogeneous Einstein metrics on flag manifolds.* Lobachevskii J. Math. **4**(1999), 71–87.
[19] N. Wallach: *Compact homogeneous riemannian manifolds with strictly positive curvature*, Ann. math. 96 (1972) 277–295.
[20] M. Wang, *Einstein metrics from symmetry and Bundle constructions.* In: Surveys in Differential Geometry: Essays on Einstein Manifolds. Surv. Differ. Geom. VI, Int. Press, Boston, MA, 1999, pp. 287–325.
[21] M. Wang and W. Ziller, *Existence and non-existence of homogeneous Einstein metrics.* Invent. Math. **84**(1986), no. 1, 177–194.
[22] ———, *Einstein metrics with positive scalar curvature*. In: Curvature and Topology of Riemannian Manifolds, Springer Lecture Notes in Mathematics 1201, Springer, Berlin, 1986, pp. 319–336.

*University of Patras, Department of Mathematics, GR-26500 Rion, Greece*
*e-mail*: arvanito@math.upatras.gr

*Rubtsovsk Industrial Institute, ul. Traktornaya, 2/6, Rubtsovsk, 658207, Russia*
*e-mail*: J_Valera_V@mail.ru
nik@inst.rubtsovsk.ru