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ON A PROBLEM OF PONGSRIIAM ON THE SUM OF DIVISORS

RUI-JING WANG

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Abstract

For any positive integer n, let $\sigma(n)$ be the sum of all positive divisors of n. We prove that for every integer k with $1 \le k \le 29$ and $(k, 30) = 1$,

$$\sum_{n\le K} \sigma(30n) > \sum_{n\le K} \sigma(30n + k)$$

for all $K \in \mathbb{N}$, which gives a positive answer to a problem posed by Pongsriiam ['Sums of divisors on arithmetic progressions', *Period. Math. Hungar.* **88** (2024), 443–460].

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1. Introduction

For any positive integer n, let $\sigma(n)$ be the sum of all positive divisors of n. We always assume that x is a real number, m and n are positive integers, p is a prime, p_n is the nth prime and $\phi(n)$ denotes the Euler totient function. Jarden [4, page 65] observed that $\phi(30n + 1) > \phi(30n)$ for all $n \le 10^5$ and later the inequality was calculated to be true up to 10^9. However, Newman [8] proved that there are infinitely many n such that $\phi(30n + 1) < \phi(30n)$ and the smallest one is

$$\frac{p_{385}p_{388}\prod_{j=4}^{383} p_j - 1}{30},$$

which was given by Martin [7]. For related work, see [3, 5, 6, 9, 11]. It is certainly natural to consider the analogous problem for the sum of divisors function. Recently, Pongsriiam [10, Theorem 2.4] proved that $\sigma(30n) - \sigma(30n + 1)$ also has infinitely many sign changes. He found that $\sigma(30n) > \sigma(30n + 1)$ for all $n \le 10^7$ and posed the following problem.

PROBLEM 1.1 [10, Problem 3.8(ii)]. Is it true that

$$\sum_{n\le K}\sigma(30n) > \sum_{n\le K}\sigma(30n+1)$$

for all $K\in\mathbb{N}$?

Recently, Ding *et al.* [2] solved several problems of Pongsriiam. Inspired by their ideas, we answer affirmatively the above Problem 1.1. In fact, we prove a slightly stronger result.

THEOREM 1.2. *For every integer k with $1\le k\le 29$ and $(k,30)=1$,*

$$\sum_{n\le K}\sigma(30n) > \sum_{n\le K}\sigma(30n+k)$$

for all $K\in\mathbb{N}$.

2. Estimations

Let

$$\beta_0 = \sum_{d=1}^{\infty}\frac{B_0(d)}{d^2},$$

where $B_0(d)$ denotes the number of solutions, not counting multiplicities, of the congruence $30m\equiv 0 \pmod d$. For every integer k with $1\le k\le 29$ and $(k,30)=1$, let

$$\beta_k = \sum_{d=1}^{\infty}\frac{B_k(d)}{d^2},$$

where $B_k(d)$ denotes the number of solutions, not counting multiplicities, of the congruence $30m+k\equiv 0 \pmod d$. By the Chinese remainder theorem, both $B_0(d)$ and $B_k(d)$ are multiplicative. Note that for $p\nmid 30$, we have $B_0(p^\alpha)=1$ and $B_k(p^\alpha)=1$ for any positive integer α. It is obvious that $B_0(p^\alpha)=p$ and $B_k(p^\alpha)=0$ for $p=2,3,5$ and any positive integer α. It follows that $B_0(d)\le 30$ and $B_k(d)\le 1$ for any positive integer d. By [1, Theorem 11.7] and

$$\frac{\pi^2}{6}=\zeta(2)=\prod_p\frac{1}{1-p^{-2}},$$

we have

$$\beta_0=\prod_p\Big(1+\frac{B_0(p)}{p^2}+\frac{B_0(p^2)}{p^4}+\cdots\Big)=\frac{5}{3}\,\frac{11}{8}\,\frac{29}{24}\prod_{p\nmid 30}\frac{1}{1-p^{-2}}=\frac{319\pi^2}{1080}$$

and

$$\beta_k=\prod_p\Big(1+\frac{B_k(p)}{p^2}+\frac{B_k(p^2)}{p^4}+\cdots\Big)=\prod_{p\nmid 30}\frac{1}{1-p^{-2}}=\frac{8\pi^2}{75}.$$

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It can be checked that there are large oscillations of the error terms which influence the main terms if one tries to calculate the sums in Problem 1.1 directly. Therefore, we first manipulate the weighted sums and then transform them to the original sums via summations by parts.

We always assume that $x \geq 1000,\ 1 \leq k \leq 29,\ (k, 30) = 1$ in the following lemmas.

LEMMA 2.1. *We have*

$$\sum_{m\leq x} \frac{\sigma(30m)}{30m} = \beta_0 x + g(x),$$

where

$$-30\log 30x - 32 < g(x) < 30\log 30x + 32.$$

PROOF. By the definition of $B_0(d)$,

$$\begin{aligned}
\sum_{m\leq x} \frac{\sigma(30m)}{30m} = \sum_{m\leq x}\sum_{d\mid 30m} \frac{1}{d} &= \sum_{d\leq 30x} \frac{1}{d} \sum_{\substack{m\leq x\\ 30m\equiv 0 \pmod d}} 1\\
&= \sum_{d\leq 30x} \frac{B_0(d)}{d}\Big(\frac{x}{d} + \alpha_0(x,d)\Big)\\
&= x\sum_{d\leq 30x} \frac{B_0(d)}{d^2} + \sum_{d\leq 30x} \frac{B_0(d)}{d}\alpha_0(x,d)\\
&= \beta_0 x - x\sum_{d>30x} \frac{B_0(d)}{d^2} + \sum_{d\leq 30x} \frac{B_0(d)}{d}\alpha_0(x,d)\\
&= \beta_0 x + g(x),
\end{aligned}$$

where $-1 \leq \alpha_0(x,d) \leq 1$ and

$$g(x) = -x\sum_{d>30x} \frac{B_0(d)}{d^2} + \sum_{d\leq 30x} \frac{B_0(d)}{d}\alpha_0(x,d).$$

By [1, Theorem 3.2],

$$0 \leq \sum_{d>30x} \frac{B_0(d)}{d^2} \leq 30\sum_{d>30x} \frac{1}{d^2} \leq 30\Big(\frac{1}{30x} + \frac{30x - [30x]}{(30x)^2}\Big)$$

and

$$0 \leq \sum_{d\leq 30x} \frac{B_0(d)}{d} \leq 30\sum_{d\leq 30x} \frac{1}{d} < 30(\log 30x + 1).$$

It follows that

$$-30\log 30x - 32 < g(x) < 30\log 30x + 32.$$

This completes the proof of Lemma 2.1. □

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LEMMA 2.2. *We have*

$$\sum_{m\le x}\frac{\sigma(30m+k)}{30m+k}=\beta_k x+h_k(x),$$

where

$$-\log(30x+k)-2<h_k(x)<\log(30x+k)+2.$$

PROOF. By the definition of $B_k(d)$,

$$\begin{aligned}\sum_{m\le x}\frac{\sigma(30m+k)}{30m+k}&=\sum_{m\le x}\sum_{d\mid 30m+k}\frac{1}{d}=\sum_{d\le 30x+k}\frac{1}{d}\sum_{\substack{m\le x\\ 30m+k\equiv 0\ (\mathrm{mod}\ d)}}1\\&=\sum_{d\le 30x+k}\frac{B_k(d)}{d}\Big(\frac{x}{d}+\alpha_k(x,d)\Big)\\&=x\sum_{d\le 30x+k}\frac{B_k(d)}{d^2}+\sum_{d\le 30x+k}\frac{B_k(d)}{d}\alpha_k(x,d)\\&=\beta_k x-x\sum_{d>30x+k}\frac{B_k(d)}{d^2}+\sum_{d\le 30x+k}\frac{B_k(d)}{d}\alpha_k(x,d)\\&=\beta_k x+h_k(x),\end{aligned}$$

where $-1\le\alpha_k(x,d)\le 1$ and

$$h_k(x)=-x\sum_{d>30x+k}\frac{B_k(d)}{d^2}+\sum_{d\le 30x+k}\frac{B_k(d)}{d}\alpha_k(x,d).$$

By [1, Theorem 3.2],

$$0\le\sum_{d>30x+k}\frac{B_k(d)}{d^2}\le\sum_{d>30x+k}\frac{1}{d^2}\le\frac{1}{30x+k}+\frac{30x+k-[30x+k]}{(30x+k)^2}$$

and

$$0\le\sum_{d\le 30x+k}\frac{B_k(d)}{d}\le\sum_{d\le 30x+k}\frac{1}{d}<\log(30x+k)+1.$$

It follows that

$$-\log(30x+k)-2<h_k(x)<\log(30x+k)+2.$$

This completes the proof of Lemma 2.2. □

LEMMA 2.3. *We have*

$$\sum_{m\le x}\sigma(30m)>15\beta_0x^2-2000x\log 30x.$$

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PROOF. Let

$$S(x) = \sum_{m \le x} \frac{\sigma(30m)}{30m}.$$

By [1, Theorem 3.1],

$$\begin{aligned}
\sum_{m \le x} \sigma(30m) &= \sum_{m \le x} 30m(S(m) - S(m-1)) \\
&= 30[x]S([x]) - \sum_{m \le x-1} (30(m+1) - 30m)S(m) - 30S(0) \\
&> 30(x-1)(\beta_0 x - 30\log 30x - 32) - 30 \sum_{m \le x-1} (\beta_0 m + 30\log 30m + 32) \\
&> 30\beta_0 x^2 - 900x\log 30x - 960x - 15\beta_0 x^2 - 900x\log 30x - 960x \\
&\quad - 30\beta_0 x + 900\log 30x + 960 \\
&> 15\beta_0 x^2 - 2000x\log 30x.
\end{aligned}$$

This completes the proof of Lemma 2.3. □

LEMMA 2.4. *We have*

$$\sum_{m \le x} \sigma(30m + k) < 15\beta_k x^2 + 100x\log(30x + k).$$

PROOF. Let

$$T_k(x) = \sum_{m \le x} \frac{\sigma(30m + k)}{30m + k}.$$

By [1, Theorem 3.1],

$$\begin{aligned}
\sum_{m \le x} \sigma(30m + k) &= \sum_{m \le x} (30m + k)(T_k(m) - T_k(m-1)) \\
&= (30[x] + k)T_k([x]) - \sum_{m \le x-1} (30(m+1) + k - 30m - k)T_k(m) \\
&< (30x + k)(\beta_k x + \log(30x + k) + 2) - 30 \sum_{m \le x-1} (\beta_k m - \log(30m + k) - 2) \\
&< 30\beta_k x^2 + 30x\log(30x + k) + 60x - 15\beta_k (x-2)^2 + 30x\log(30x + k) \\
&\quad + 60x + k\beta_k x + k\log(30x + k) + 2k \\
&< 15\beta_k x^2 + 100x\log(30x + k).
\end{aligned}$$

This completes the proof of Lemma 2.4. □

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3. Proof of Theorem 1.2

PROOF OF THEOREM 1.2. For every integer k with $1 \le k \le 29$, $(k, 30) = 1$, by Lemmas 2.3 and 2.4,

$$\sum_{m \le x} \sigma(30m) > 15\beta_0 x^2 - 2000x \log 30x > 15\beta_k x^2 + 100x \log(30x + k) > \sum_{m \le x} \sigma(30m + k)$$

provided that $x \ge 1000$. It is easy to verify that

$$\sum_{m \le K} \sigma(30m) > \sum_{m \le K} \sigma(30m + k)$$

for every positive integer $K < 1000$ and for every integer k with $1 \le k \le 29$ and $(k, 30) = 1$, by programming. This completes the proof of Theorem 1.2. □

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References

[1] T. M. Apostol, *Introduction to Analytic Number Theory* (Springer-Verlag, New York, 1976).
[2] Y. Ding, H. Pan and Y.-C. Sun, 'Solutions to some sign change problems on the functions involving sums of divisors', Preprint, 2024, arXiv:2401.09842.
[3] P. Erdős, 'On a problem of Chowla and some related problems', *Math. Proc. Cambridge Philos. Soc.* **32** (1936), 530–540.
[4] D. Jarden, *Recurring Sequences* (Riveon Lematematika, Jerusalem, 1973).
[5] M. Kobayashi and T. Trudgian, 'On integers n for which $\sigma(2n + 1) \ge \sigma(2n)$', *J. Number Theory* **215** (2020), 138–148.
[6] F. Luca and C. Pomerance, 'The range of the sum-of-proper-divisors function', *Acta Arith.* **168** (2015), 187–199.
[7] G. Martin, 'The smallest solution of $\phi(30n + 1) < \phi(30n)$ is . . . ', *Amer. Math. Monthly* **106** (1999), 449–451.
[8] D. J. Newman, 'Euler's ϕ function on arithmetic progressions', *Amer. Math. Monthly* **104** (1997), 256–257.
[9] P. Pollack and C. Pomerance, 'Some problems of Erdős on the sum-of-divisors function', *Trans. Amer. Math. Soc. Ser. B* **3** (2016), 1–26.
[10] P. Pongsriiam, 'Sums of divisors on arithmetic progressions', *Period. Math. Hungar.* **88** (2024), 443–460.
[11] R.-J. Wang and Y.-G Chen, 'On positive integers n with $\sigma_l(2n + 1) < \sigma_l(2n)$', *Period. Math. Hungar.* **85** (2022), 210–224.

RUI-JING WANG, School of Mathematical Sciences,
Jiangsu Second Normal University, Nanjing 210013, PR China
e-mail: wangruijing271@163.com

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