ON CENTRAL SERIES

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Let

$$1 = Z_0 \leqq Z_1 \leqq Z_2 \leqq \dots$$

and

$$G = \Gamma_1 \geqq \Gamma_2 \geqq \Gamma_3 \geqq \dots$$

be, respectively, the upper and lower central series of a group G. Our purpose in this note is to extend known results and find some information as to which of the factors Z_k/Z_{k-1} and Γ_k/Γ_{k+1} may be infinite. Though our conclusions about the lower central series will be quite general we assume in the other case that the group is f.n., i.e. an extension of a finite group by a nilpotent group. The essential facts about f.n. groups are to be found in P. Hall's paper **(4)**. We also refer to **(4)** for general notation; we reserve the letter k for positive integers.

We note two well-known theorems, the first of which is proved in both **(1)** and **(2)**:

Theorem (R. Baer). *Let G be a group such that, for some k, G/Z_k is finite. Then Γ_{k+1} is finite.*

The other result is taken from **(4)**:

Theorem (P. Hall). *Let G be a group such that, for some k, Γ_{k+1} is finite. Then G/Z_{2k} is finite.*

Our main tool will be:

Theorem 1. *Let G be a group such that, for some k, $\Gamma_k/(Z_1 \cap \Gamma_k)$ is finite. Then Γ_{k+1} is finite.*

Proof. We shall take $k \geqq 2$, for, when $k = 1$, the result is a special case of Baer's Theorem.

Let us show first that $\Gamma_{k+1}/\Gamma_{k+2}$ is finite, for which purpose we may assume that $\Gamma_{k+2} = 1$. Now modulo Z_1 there is only a finite number of left-normed commutators of weight k, say $c_1Z_1, c_2Z_1, \dots, c_nZ_1$, so we choose and keep fixed elements a_{ij} for which

$$c_jZ_1 = [a_{1j}, a_{2j}, \dots, a_{kj}]Z_1,$$

with $1 \leqq j \leqq n$.

Let $c = [a_1, a_2, \dots, a_k, b]$ be a commutator of weight $k+1$ in which the $a_1, a_2, \dots, a_k$ are certain of the a_{ij}. By the lemma of **(5)**, p. 272, it is possible to express c as the product of commutators of the form $[b, a_1^*, a_2^*, \dots, a_k^*]^{\pm 1}$ where each a_i^* is a conjugate of some $a_j^{\pm 1}$. Since $\Gamma_{k+2} = 1$ each a_i^* is here one of the $a_j^{\pm 1}$, and in particular we see that the possible values of a_k^* are finite in number. Because $[b, a_1^*, a_2^*, \ldots, a_{k-1}^*]Z_1$ has only a finite number of possible values as $\Gamma_k/(Z_1 \cap \Gamma_k)$ is finite, c has only a finite number of possible values, and so Γ_{k+1} is generated by a finite number of commutators of weight $k+1$.

Next we show that $[a_1, a_2, \ldots, a_k, b]$ has finite order. Since $\Gamma_k/(Z_1 \cap \Gamma_k)$ is finite we have $[a_1, a_2, \ldots, a_k]^m \in Z_1$ for some positive integer m, and therefore

$$[[a_1, a_2, \ldots, a_k]^m, b] = 1.$$

To expand the left-hand side of this equation we use the well-known identity

$$[xy, b] = [x, b]^y[y, b]$$

with $y = [a_1, a_2, \ldots, a_k]$ and $x = y^{m-1}$, and because $\Gamma_{k+2} = 1$ we have $[x, b]$ central in G:

$$[xy, b] = [x, b][y, b].$$

An easy induction therefore gives

$$1 = [[a_1, a_2, \ldots, a_k]^m, b] = [a_1, a_2, \ldots, a_k, b]^m,$$

and so $[a_1, a_2, \ldots, a_k, b]$ has finite order. As the abelian subgroup Γ_{k+1} is generated by a finite number of elements each of finite order, it is itself finite.

We have established that if $\Gamma_k/(Z_1 \cap \Gamma_k)$ is finite then $\Gamma_{k+1}/\Gamma_{k+2}$ is finite, and it is also true that if Γ_k/Γ_{k+1} is finite then $\Gamma_{k+1}/\Gamma_{k+2}$ is finite. The proof is easy for $k = 1$ and the above arguments may be used when $k > 1$. But now induction shows that if $\Gamma_{k+1}/\Gamma_{k+2}$ is finite then so are $\Gamma_{k+2}/\Gamma_{k+3}$, $\Gamma_{k+3}/\Gamma_{k+4}, \ldots, \Gamma_{2k-1}/\Gamma_{2k}$; therefore Γ_{k+1}/Γ_{2k} is finite.

Next we apply Hall's Theorem to G/Z_1 and use the fact that $\Gamma_k/(Z_1 \cap \Gamma_k)$ is finite. Thus G/Z_{2k-1} is finite. Then Baer's Theorem shows that Γ_{2k} is finite, which, with an earlier result, establishes the finiteness of Γ_{k+1} and completes the proof of the theorem.

Corollary 1. *Let G be a group such that, for some k and j,* $\Gamma_k/(Z_j \cap \Gamma_k)$ *is finite. Then* $\Gamma_{k+i}/(Z_{j-i} \cap \Gamma_{k+i})$ *is finite for* $i = 1, 2, \ldots, j$.

Proof. This result follows from Theorem 1, induction on j being used.

Corollary 2. *Let G be a group such that, for some k,* $\Gamma_{k+1}/(Z_1 \cap \Gamma_{k+1})$ *is finite. Then* G/Z_{2k} *is finite.*

Proof. This corollary depends heavily on the proof of Hall's Theorem in (4). First we note that G/H is finite, H being the centraliser of Γ_{k+1} in G. For H is the intersection of the centralisers of a finite number of elements, namely a transversal of Γ_{k+1} modulo $Z_1 \cap \Gamma_{k+1}$; each centraliser has finite index in G because Γ_{k+2} is finite by Theorem 1. This, and finiteness of $\Gamma_{k+1}/(Z_1 \cap \Gamma_{k+1})$, are sufficient for Hall's method of proof to establish the finiteness of G/Z_{2k}; the details are omitted.

Corollary 3. *Let G be a group such that* Γ_k/Γ_{k+1}, *for some k, is finite. Then* $\Gamma_{k+i}/\Gamma_{k+i+1}$ *is finite for every positive integer i.*

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Proof. This corollary was established in the course of the proof of Theorem 1.

We note that in the infinite dihedral group we have Γ_k/Γ_{k+1} finite but Γ_{k+1} infinite, for each k.

Corollary 4. *Let G be a group such that Γ_k/Γ_{k+1}, for some k, is the first finite factor in the lower central series. Then the infinite factors Γ_i/Γ_{i+1}, with i an integer, are just those with $i = 1, 2, \ldots, k-1$.*

Proof. This follows from Corollary 3.

The special case $k = 1$ was given by Baer in (**1**), p. 358.

Just as Corollary 4 concerns the position of infinite factors in the lower central series, so the next result gives some information on such factors in the upper central series of f.n. groups.

Corollary 5. *Let G be an f.n. group with just k infinite factors in the upper central series. Then G/Z_{2k} is finite.*

Proof. Since G is f.n. we have G/Z_u finite for some positive integer u, by Hall's Theorem, and u may be taken so that Z_u/Z_{u-1} is infinite. Application of Theorem 1 to G/Z_{u-1} shows that $\Gamma_2/(Z_{u-1} \cap \Gamma_2)$ is finite. If for some v we have Z_{u-1}/Z_v finite but Z_v/Z_{v-1} infinite, then $\Gamma_2/(Z_v \cap \Gamma_2)$ is clearly finite. Considering G/Z_{v-1}, we see by Theorem 1 that $\Gamma_3/(Z_{v-1} \cap \Gamma_3)$ is finite. Proceeding in this way we find eventually that Γ_{k+1} is finite, and then Hall's Theorem shows that G/Z_{2k} is finite. This proves the corollary, and in addition:

Corollary 6. *In an f.n. group the number of infinite factors in the lower central series cannot exceed the number of infinite factors in the upper central series.*

Next we comment upon some interesting examples relevant to Corollary 5, which were constructed by Hall in (**4**). These groups have class $2k$, for any given k, and Z_k is finite while $Z_{k+1}/Z_k, Z_{k+2}/Z_{k+1}, \ldots, Z_{2k}/Z_{2k-1}$ are all infinite. So the hypotheses of Corollary 5 alone cannot imply that G/Z_{2k-1} is finite.

Corollary 7. *None of Hall's examples can be presented as the central factor group of any group.*

Proof. If the corollary is false then there exists a group G of class $2k+1$ for which Z_{k+1}/Z_1 is finite and G/Z_{2k} is infinite. Since $\Gamma_k \leqq Z_{k+1}$, we have $\Gamma_k/(Z_1 \cap \Gamma_k)$ finite. By Corollary 2 this implies that G/Z_{2k-2} is finite and we reach a contradiction, which proves the corollary.

Next we examine this question: what additional conditions on the group G satisfying the hypotheses of Corollary 5 will make G/Z_k finite? It is clear from the proof of that corollary that this will be the effect of any condition which causes the finiteness of Γ_{k+1} to imply the finiteness of G/Z_k. Here are some such conditions:

(i) $\Gamma_{k+1} \cap Z_1 = 1$;
(ii) G contains an abelian subgroup of finite index;
(iii) G is finitely generated;
(iv) G satisfies the minimum condition for subgroups;
(v) G is residually finite.

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The first three of these were given by Hall in (4). Suppose condition (iv) applies. Since Γ_{k+1} is finite we know that G/Z_{2k} is finite by Hall's Theorem, and Z_{2k} is nilpotent with minimum condition for subgroups. So a well-known result implies that the centre of Z_{2k} has finite index in Z_{2k}. Since G has an abelian subgroup of finite index, (ii) shows that G/Z_k is finite. Finally suppose that (v) holds. Then there is a normal subgroup N of G for which $N \cap \Gamma_{k+1}$ is trivial and G/N is finite, and we need show only that $N \leqq Z_k$. This follows from the fact that

$$N^{(k)} \leqq N \cap \Gamma_{k+1} = 1.$$

Condition (iv) for the case $k = 1$ is due to Baer; see (3), p. 521.

We note a further result in the same direction: *if G is metabelian and Γ_{k+1} is finite then G/Z_{k+1} is finite.* This is established by again adapting the proof of Hall's Theorem in (4), a process of which the details are omitted. The main step is to replace $[H^{(k-s-1)}, H] \leqq Z_{k+s}$ deduced from Lemma 2 of (4) by $[H^{(k-s-1)}, H] \leqq Z_{1+s}$; this depends on the fact that in a metabelian group the elements g_i with $i \geqq 3$ in the commutator $[g_1, g_2, \ldots, g_n]$ can be permuted at will.

Hence in a metabelian f.n. group with k infinite factors in the upper central series we have G/Z_{k+1} finite; this result and its proof are analogous to Corollary 5 and its proof. As an example of such a group we refer to the central product of infinitely many copies of the dihedral group of order 2^{k+2}.

Hall mentions in (4) that his examples cannot satisfy conditions (i), (ii) and (iii) above. To this list we have now added another three conditions, in general.

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