Proceedings of the Edinburgh Mathematical Society (2013) **56**, 449–468
DOI:10.1017/S0013091512000247

COHOMOLOGICAL UNIQUENESS OF SOME p-GROUPS

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(Received 31 May 2010)

Abstract We consider classifying spaces of a family of p-groups and prove that mod p cohomology enriched with Bockstein spectral sequences determines their homotopy type among p-completed CW-complexes.

Keywords: classifying spaces; finite p-groups; cohomological uniqueness; Steenrod algebra; Bockstein spectral sequence

2010 *Mathematics subject classification:* Primary 55R35
Secondary 20D20

1. Introduction

Let p be a prime number. A naive way of describing the Bousfield–Kan p-completion functor [**1**] is to say that it transforms mod p cohomology isomorphisms into actual homotopy equivalences. It is then therefore natural to think that the homotopy type of a p-complete space X should be characterized in some sense by its mod p cohomology ring $H^*(X)$. Classifying spaces of finite p-groups provide nice examples of p-complete spaces. Then the following question arises: given a finite p-group P, and a p-complete space X such that $H^*(X) \cong H^*(BP)$, is $X \simeq BP$?

One would like to give a positive answer to the question above, but the very first step towards that positive answer is to understand, or to give the appropriate meaning to, the isomorphism $H^*(X) \cong H^*(BP)$.

It is well known that there are infinitely many examples of non-isomorphic finite p-groups (hence infinitely many examples of non-homotopic p-complete spaces) having isomorphic mod p cohomology rings, even as unstable algebras (see [**4**] for a general proof of this fact in the case when $p = 2$). This is not surprising, since p-completion does not invert abstract mod p cohomology isomorphisms, but inverts just those which are induced by continuous maps, and these compare unstable algebras plus secondary operations.

In this regard, Broto and Levi [**2**] suggested that mod p cohomology rings of finite p-groups should be considered objects in the category $\mathcal{K}_\beta$ of unstable algebras endowed with Bockstein spectral sequences (see § 2 for precise definitions). Here we follow that line and consider the family of groups studied by Leary in [**7**], proving the following theorem.

Theorem 1.1. *Let p be an odd prime and define the finite p-group*

$$P(p,n) = \langle A, B, C \mid A^p = B^p = C^{p^{n-2}} = [A,C] = [B,C] = 1,\ [A,B] = C^{p^{n-3}} \rangle.$$

Given X, a p-complete CW-complex:

(a) *if $n = 3, 4$ and $H^*(X) \cong H^*(BP(p,n))$ as unstable algebras, then $X \simeq BP(p,n)$;*

(b) *if $n \geqslant 5$ and $H^*_\beta(X) \cong H^*_\beta(BP(p,n))$ as objects in $\mathcal{K}_\beta$, then $X \simeq BP(p,n)$.*

Proof. Statement (a) is proved in Corollary 4.6 for $n = 3$, and Corollary 4.7 (a) for $n = 4$. Statement (b) is proved in Corollary 4.7 (b). □

Besides its own topological interest, the result above and the techniques developed in its proof may be appealing from a group theoretical point of view. First, since the classifying space of a finite p-group is a p-complete CW-complex, Theorem 1.1 provides a cohomological characterization of $P(p,n)$.

Theorem 1.2. *Let p be an odd prime and let G be a finite p-group. Then $G \cong P(p,n)$ if and only if $H^*_\beta(BG) \cong H^*_\beta(BP(p,n))$.*

Second, the ideas in the proof of Theorem 1.1 can be used to obtain a cohomological characterization of $P(p,n)$ as a complement for some $N \trianglelefteq G$. This characterization can be seen as a generalization of Tate's cohomological criteria of p-nilpotency [**9**].

Theorem 1.3. *Let p be an odd prime and let G be a finite group such that $P(p,n) \leqslant G$. Then $P(p,n)$ is a complement for some $N \trianglelefteq G$ if and only if one of the following holds:*

(a) *$n = 3, 4$ and there exists $\psi\colon H^*(BP(p,n)) \to H^*(BG)$ as unstable algebras such that $(\mathrm{res} \circ \psi)|_{H^1_\beta(BP(p,n))}$ is the identity;*

(b) *$n \geqslant 5$ and there exists*

$$\psi\colon H^*_\beta(BP(p,n)) \to H^*_\beta(BG) \quad \text{in } \mathcal{K}_\beta$$

such that $(\mathrm{res} \circ \psi)|_{H^1_\beta(BP(p,n))}$ is the identity.

Proof. If $P(p,n)$ is a complement for some $N \trianglelefteq G$, then the induced projection $G \xrightarrow{\pi} G/N \cong P(p,n)$ gives rise to a map between classifying spaces $BG \xrightarrow{B\pi} BP(p,n)$ that provides the desired cohomological morphism $\psi = B\pi^*$.

The converse is proven in Proposition 5.1 for the case $n = 3$, and in Proposition 5.2 for the case $n > 3$. □

https://doi.org/10.1017/S0013091512000247 Published online by Cambridge University Press

1.1. Organization of the paper

In § 2 we introduce the notation used in the paper. In § 3 the group $P(p,n)$ is defined and the mod p cohomology ring of its classifying space is described. In § 4, we explore endomorphisms of the mod p cohomology ring of $BP(p,n)$ and we conclude that mod p cohomology determines the homotopy type of $BP(p,n)$. Finally, in § 5 we apply the ideas developed in the previous section to the group theoretical framework.

2. Definitions and notation

We follow the notation and conventions in [**2**, § 2]. As our study is done for a fixed odd prime p, we just recall the definitions in this case.

All the spaces considered here have the homotopy type of a p-complete CW-complex. Unless otherwise stated, $H^*(X)$ refers to the cohomology of the space X with trivial coefficients in $\mathbb{F}_p$.

Definition 2.1. Let p be an odd prime and let K be an unstable algebra. A *Bockstein spectral sequence (BSS) for* K is a spectral sequence of differential graded algebras $\{E_i(K), \beta_i\}_{i=1}^{\infty}$ where the differentials have degree 1 and such that

(a) $E_1(K) = K$ and $\beta_1 = \beta$ is the primary Bockstein operator,

(b) if $x \in E_i(K)^{\text{even}}$ and $x^p \neq 0$ in $E_{i+1}(K)$, $i \geqslant 1$, then $\beta_{i+1}(x^p) = x^{p-1}\beta_i(x)$.

We work in the category $\mathcal{K}_\beta$, whose objects are pairs $(K; \{E_i(K), \beta_i\}_{i=1}^{\infty})$, where K is an unstable algebra and $\{E_i(K); \beta_i\}_{i=1}^{\infty}$ is a BSS for K. A morphism $f\colon K \to K'$ in $\mathcal{K}_\beta$ is a family of morphisms $\{f_i\}_{i=1}^{\infty}$, where $f_1\colon K \to K'$ is a morphism of $\mathcal{A}_p$-algebras and for each $i \geqslant 2$, $f_i\colon E_i(K) \to E_i(K')$ is a morphism of differential graded algebras, which, as a morphism of graded algebras, is induced by f_{i-1}.

The mod p cohomology of a space X is an object of $\mathcal{K}_\beta$ that is denoted by $H^*_\beta(X)$.

Definition 2.2. We say that two spaces X and Y are *comparable* if $H^*_\beta(X)$ and $H^*_\beta(Y)$ are isomorphic objects in the category $\mathcal{K}_\beta$. We say that X is *determined by cohomology* if, given a space Y comparable to X, there is a homotopy equivalence $X \simeq Y$.

Definition 2.3. Let K_β be an object in $\mathcal{K}_\beta$. Let K be the underlying unstable algebra over $\mathcal{A}_p$. We say that K_β is *weakly generated by* $x_1, \ldots, x_n$ if any endomorphism f of K_β such that the restriction of f to the vector subspace of K generated by $x_1, \ldots, x_n$ is an isomorphism is an isomorphism in $\mathcal{K}_\beta$.

3. The cohomology of some *p*-groups

In this section, the p-group $P(p,n)$, p an odd prime, $n \geqslant 3$, is introduced, and in what follows the notation in [**7**] is used.

The group

$$P(p,n) = \langle A, B, C \mid A^p = B^p = C^{p^{n-2}} = [A,C] = [B,C] = 1,\ [A,B] = C^{p^{n-3}} \rangle \quad (3.1)$$

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has order p^n and fits in a central extension:

$$0 \to \mathbb{Z}/p^{n-2} \to P(p,n) \to \mathbb{Z}/p \times \mathbb{Z}/p \to 0. \tag{3.2}$$

The cohomology of $P(p,n)$ is calculated in [**7**].

Theorem 3.1 (Leary [7, Propositions 3, 8 and Theorem 7]). *$H^*(BP(3,3))$ is generated by elements y, y', x, x', Y, Y', X, X', z with*

$$\begin{aligned}
\deg(y) &= \deg(y') = 1,\\
\deg(x) &= \deg(x') = \deg(Y) = \deg(Y') = 2,\\
\deg(X) &= \deg(X') = 3,\\
\deg(z) &= 6,
\end{aligned}$$

subject to the following relations:

$$\begin{aligned}
yy' &= 0, & YY' &= xx',\\
xy' &= x'y, & Y^2 &= xY',\\
yY &= y'Y' = xy', & Y'^2 &= x'Y,\\
yY' &= y'Y, & yX &= xY - xx',\\
y'X' &= x'Y' - xx', & XY &= x'X,\\
Xy' &= x'Y - xY', & X'Y' &= xX',\\
X'y &= xY' - x'Y, & XY' &= -X'Y,\\
xX' &= -x'X, & XX' &= 0,\\
x(xY' + x'Y) &= -xx'^2, & x^3y' - x'^3y &= 0,\\
x'(xY' + x'Y) &= -x'x^2, & x^3x' - x'^3x &= 0,\\
x^3Y' + x'^3Y &= -x^2x'^2 & x^3X' + x'^3X &= 0.
\end{aligned}$$

Moreover, the action of the mod 3 Steenrod algebra is determined by

$$\begin{aligned}
\beta(y) &= x, & \mathcal{P}^1(X) &= x^2X + zy,\\
\beta(y') &= x', & \mathcal{P}^1(X') &= x'^2X' - zy'\\
\beta(Y) &= X, & \mathcal{P}^1(z) &= zc_2,\\
\beta(Y') &= X',
\end{aligned}$$

where $c_2 = xY' - x'Y - x^2 - x'^2$.

Theorem 3.2 (Leary [7, Propositions 3, 8 and Theorem 6]). *For an odd prime $p \geqslant 5$, the cohomology $H^*(BP(p,3))$ is generated by elements y, y', x, x', Y, Y', X, X',*

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$d_4, \ldots, d_p$, $c_4, \ldots, c_{p-1}$ *and* z *with*

$$\begin{aligned}\deg(y) &= \deg(y') = 1,\\ \deg(x) &= \deg(x') = \deg(Y) = \deg(Y') = 2,\\ \deg(X) &= \deg(X') = 3,\\ \deg(d_i) &= 2i-1,\\ \deg(c_i) &= 2i,\\ \deg(z) &= 2p\end{aligned}$$

subject to the following relations:

$$yy' = 0, \qquad xy' = x'y, \qquad yY = y'Y' = 0, \qquad yY' = y'Y,$$
$$Y^2 = Y'^2 = YY' = 0, \qquad yX = xY, \qquad y'X' = x'Y',$$
$$Xy' = 2xY' + x'Y, \qquad X'y = 2x'Y + xY',$$
$$XY = X'Y' = 0, \qquad XY' = -X'Y, \qquad xX' = -x'X,$$
$$x(xY' + x'Y) = x'(xY' + x'Y) = 0,$$
$$\begin{aligned}x^p y' - x'^p y &= 0,\\ x^p x' - x'^p x &= 0,\\ x^p Y' + x'^p Y &= 0,\\ x^p X' + x'^p X &= 0,\end{aligned}$$

and

$$c_i y = \begin{cases}0\\ -x^{p-1}y\end{cases} \quad c_i y' = \begin{cases}0 & \text{for } i < p-1,\\ -x'^{p-1}y' & \text{for } i = p-1,\end{cases}$$

$$c_i x = \begin{cases}0\\ -x^{p}\end{cases} \quad c_i x' = \begin{cases}0 & \text{for } i < p-1,\\ -x'^{p} & \text{for } i = p-1,\end{cases}$$

$$c_i Y = \begin{cases}0\\ -x^{p-1}Y\end{cases} \quad c_i Y' = \begin{cases}0 & \text{for } i < p-1,\\ -x'^{p-1}Y' & \text{for } i = p-1,\end{cases}$$

$$c_i X = \begin{cases}0\\ -x^{p-1}X\end{cases} \quad c_i X' = \begin{cases}0 & \text{for } i < p-1,\\ -x'^{p-1}X & \text{for } i = p-1,\end{cases}$$

$$c_i c_j = \begin{cases}0 & \text{for } i + j < 2p-2,\\ x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1} & \text{for } i = j = p-1,\end{cases}$$

$$d_i y = \begin{cases}0\\ -x^{p-1}Y\end{cases} \quad d_i y' = \begin{cases}0 & \text{for } i < p,\\ -x'^{p-1}Y' & \text{for } i = p,\end{cases}$$

$$d_i x = \begin{cases}0\\ -x^{p-1}y\\ x^{p-1}X\end{cases} \quad d_i x' = \begin{cases}0 & \text{for } i < p-1,\\ -x'^{p-1}y' & \text{for } i = p-1,\\ -x'^{p-1}X' & \text{for } i = p,\end{cases}$$

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$$d_iY = 0, \qquad d_iY' = 0,$$

$$d_iX = \begin{cases} 0 \\ -x^{p-1}Y \end{cases} \quad d_iX' = \begin{cases} 0 & \text{for } i \neq p-1, \\ -x'^{p-1}Y' & \text{for } i = p-1, \end{cases}$$

$$d_id_j = \begin{cases} 0 & \text{for } i < p \text{ or } j < p-1, \\ x^{2p-3}Y + x'^{2p-3}Y' + x^{p-1}x'^{p-2}Y' & \text{for } i = p,\ j = p-1, \end{cases}$$

$$d_ic_j = \begin{cases} 0 & \text{for } i < p-1 \text{ or } j < p-1, \\ -x^{2p-3}y + x'^{2p-3}y' - x^{p-1}x'^{p-2}y' & \text{for } i = j = p-1, \\ -x^{2p-3}X + x'^{2p-3}X' - x^{p-1}x'^{p-2}X' & \text{for } i = p,\ j = p-1. \end{cases}$$

Moreover, the action of the mod p *Steenrod algebra is determined by*

$$\beta(y) = x, \qquad \beta(y') = x', \qquad \beta(Y) = X, \qquad \beta(Y') = X',$$

$$\beta(d_i) = \begin{cases} c_i & \text{for } i < p, \\ 0 & \text{for } i = p, \end{cases}$$

$$\mathcal{P}^1(z) = zc_{p-1},$$

$$\mathcal{P}^1(X) = x^{p-1}X + zy,$$

$$\mathcal{P}^1(X') = x'^{p-1}X' - zy',$$

$$\mathcal{P}^1(c_i) = \begin{cases} izc_{i-1} & \text{if } 2 \leqslant i < p-1, \\ -zc_{p-2} + x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1} & \text{if } i = p-1, \end{cases}$$

where $c_1 = yy'$*, and* c_2 *and* c_3 *are non-zero multiples of* $xY' + x'Y$ *and* XX' *respectively.*

Remark 3.3. It is straightforward to check from the relations in Theorems 3.1 and 3.2 that the $\mathbb{F}_p$-vector spaces $H^*BP(p,3)$ for $p \geqslant 3$ and $* = 1,2,3,4$ have as basis

$$\{y, y'\},$$
$$\{x, x', Y, Y'\},$$
$$\{xy, xy', x'y', yY', X, X'\}$$

and

$$\{x^2, x'^2, xx', xY, xY', x'Y, x'Y'\},$$

respectively. Also notice that the generator z is free, i.e.

$$H^*BP(p,3) = \langle z\rangle \otimes (H^*BP(p,3)/\langle z\rangle).$$

Finally, consider the quotient map $p\colon H^*BP(p,3) \twoheadrightarrow H^*BP(p,3)/I$, where I is the ideal generated by all generators but x and x', and consider the map $i\colon \mathbb{F}_p[x,x'] \to H^*BP(p,3)$. As the first relation involving only x and x' occurs at degree $2p+2$, it is clear that $p \circ i$ is an isomorphism in degrees $* < 2p+2$.

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Remark 3.4. It is well known [**3**, Proposition 2.3] that, given a group G, one can make the identification $H^1(G) \cong \hom(G, \mathbb{Z}/p) \cong \hom(G_{\mathrm{ab}}, \mathbb{Z}/p)$, where G_{ab} stands for the abelianization of G. Therefore, it is possible to describe the one-dimensional classes in Theorems 3.1 and 3.2 in terms of group morphisms $P(p,3)_{\mathrm{ab}} \to \mathbb{Z}/p$ or $P(p,3) \to \mathbb{Z}/p$.

Note that $P(p,3)_{\mathrm{ab}} = \langle \bar{A}, \bar{B} \rangle \cong \mathbb{Z}/p \times \mathbb{Z}/p$ where $\bar{g}$ denotes the image of the element $g \in P(3,p)$ by the abelianization morphism. Since $\mathrm{aut}(P(p,3))$ acts transitively on the generators of $P(p,3)_{\mathrm{ab}}$ [**5**, Lemma A.5], the classes y and y' can be identified (up to a change of base) with the morphisms $\bar{A}^*\colon P(p,3) \to \langle \bar{A} \rangle \cong \mathbb{Z}/p$ and $\bar{B}^*\colon P(p,3) \to \langle \bar{B} \rangle \cong \mathbb{Z}/p$ respectively [**7**, pp. 68 and 73].

Remark 3.5. As stated in [**7**, p. 71], one can verify that in the cohomology ring $H^*(BP(p,3))$, $p \geqslant 5$, any product of the generators y, y', x, x', Y, Y', X, X' in degree greater than 6 may be expressed in the form

$$\begin{aligned} &f_1 + f_2 Y + f_3 Y' && \text{for even total degree,} \\ &f_1 y + f_2 y' + f_3 X + f_4 X' && \text{for odd total degree,} \end{aligned}$$

where each f_i is a polynomial in x and x'. So, if we define $d_1 = d_2 = d_3 = 0$, then for $1 \leqslant n \leqslant p$ any element $u \in H^{2n-1}(BP(p,3))$ can be expressed as

$$u = a d_n + f_1 y + f_2 y' + f_3 X + f_4 X',$$

where $a \in \mathbb{F}_p$ and each f_i is a polynomial in x and x'.

Remark 3.6. Note that the product of any two generators other than z can be expressed as a sum of products of the generators y, y', x, x', Y, Y', X and X'. Therefore, any decomposable element in $H^*(BP(p,3))$, $p \geqslant 5$, of degree greater than 6 that does not involve the generator z may be expressed as described in the previous remark.

Theorem 3.7 (Leary [7, Theorem 4]). *For $n \geqslant 4$, $H^*(BP(p,n))$ is generated by elements u, y, y', x, x', $c_2, c_3, \dots, c_{p-1}$, z, with*

$$\begin{aligned} \deg(u) = \deg(y) = \deg(y') &= 1, \\ \deg(x) = \deg(x') &= 2, \\ \deg(c_i) &= 2i, \\ \deg(z) &= 2p, \end{aligned}$$

subject to the following relations:

$$xy' = x'y, \qquad x^p y' = x'^p y, \qquad x^p x' = x'^p x,$$

$$c_i y = \begin{cases} 0 \\ -x^{p-1} y \end{cases} \quad c_i y' = \begin{cases} 0 & \text{for } i < p-1, \\ -x'^{p-1} y' & \text{for } i = p-1, \end{cases}$$

$$c_i x = \begin{cases} 0 \\ -x^p \end{cases} \quad c_i x' = \begin{cases} 0 & \text{for } i < p-1, \\ -x'^p & \text{for } i = p-1, \end{cases}$$

$$c_i c_j = \begin{cases} 0 & \text{for } i + j < 2p-2, \\ x^{2p-2} + x'^{2p-2} - x^{p-1} x'^{p-1} & \text{for } i = j = p-1. \end{cases}$$

Moreover, we have the following operations of the mod p *Steenrod algebra:*

$$\beta(y) = x, \qquad \beta(y') = x', \qquad \beta(u) = \begin{cases} 0 & \text{for } n > 4, \\ y'y & \text{for } n = 4, \end{cases}$$

and

$$\mathcal{P}^1(z) = zc_{p-1}, \qquad \mathcal{P}^1(c_i) = \begin{cases} izc_{i-1} & \text{for } i < p-1, \\ -zc_{p-2} + x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1} & \text{for } i = p-1, \end{cases}$$

where $c_1 = y'y$.

Remark 3.8. Consider for $n \geqslant 4$ and p an odd prime the homomorphism of rings $i\colon \mathbb{F}_p[x, x', c_{p-1}] \to H^*BP(p,n)$ and the quotient map $p\colon H^*BP(p,n) \twoheadrightarrow H^*BP(p,n)/I$, where I is the ideal generated by all generators except x, x' and c_{p-1}. From the relations the map $p \circ i$ is an isomorphism in degrees $* < 2p$ and has kernel $\mathbb{F}_p[c_{p-1}x + x^p, c_{p-1}x' + x'^p]$ in degree $* = 2p$.

We also have a map $i\colon \mathbb{F}_p[c_{p-2}, z] \to H^*BP(p,n)$ and a quotient $p\colon H^*BP(p,n) \twoheadrightarrow H^*BP(p,n)/I$, where I is the ideal generated by all generators except c_{p-2} and z. From the relations we deduce that $p \circ i$ is an isomorphism in all degrees.

Remark 3.9. In order to give a complete description of $H^*_\beta(BP(p,n))$ for $n \geqslant 4$ as an object in $\mathcal{K}_\beta$, we have to describe its Bockstein spectral sequence (Definition 2.1): the Bockstein spectral sequence is completely determined by mod p Steenrod algebra and a higher Bockstein operator (differential) $\beta_{n-3}(u) = yy'$ [**7**, p. 66]. In particular, $\beta_i(u) = 0$ for $i = 1, \dots, n-4$, and u survives to the E_{n-3}-page of the Bockstein spectral sequence.

Remark 3.10. Following the notation presented in Remark 3.4 for $n \geqslant 4$ we have

$$P(p,n)_{\mathrm{ab}} = \langle \bar{C}, \bar{A}, \bar{B} \rangle \cong \mathbb{Z}/p^{n-3} \times \mathbb{Z}/p \times \mathbb{Z}/p$$

(note that $\bar{C}$ has order p^{n-3}), and we can identify the classes y, y' and u with the morphisms

$$\bar{A}^*\colon P(p,n) \to \langle \bar{A} \rangle \cong \mathbb{Z}/p,$$
$$\bar{B}^*\colon P(p,n) \to \langle \bar{B} \rangle \cong \mathbb{Z}/p,$$

and

$$\bar{C}^*\colon P(p,n) \to \langle \bar{C} \rangle / \langle \bar{C}^p \rangle \cong \mathbb{Z}/p,$$

respectively [**7**, p. 66].

The existence of the higher Bockstein of the class u described in Remark 3.9 has its group theoretical interpretation in the fact that the morphism $\bar{C}^*$ can be extended to a group morphism $P(p,n) \to \langle \bar{C} \rangle \cong \mathbb{Z}/p^{n-3}$.

The following result gives a characterization of the cohomology class that determines a central extension by $\mathbb{Z}/p$.

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Lemma 3.11. *Let*

$$0 \to \mathbb{Z}/p \to G \xrightarrow{\pi} K \to 1 \tag{3.3}$$

be the central extension classified by $c \in H^2(BK)$. Then $\ker \pi^*|_{H^2(BK)} = \mathbb{F}_p\{c\}$. Moreover, for any non-zero scalar $\lambda \in \mathbb{F}_p$, the central extension classified by λc gives rise to a group isomorphic to G.

Proof. The proof of the first statement is done by inspection of $(E_*^{*,*}, d_*)$, the Leary–Serre spectral sequence [**8**, Chapters 5 and 6] associated to the exact sequence (3.3). Define $H^*(B\mathbb{Z}/p) = E(u) \otimes \mathbb{F}_p[v]$; then $E_2^{*,*} = H^*(B\mathbb{Z}/p) \otimes H^*(BK)$ and $c \in H^2(BK)$ classifies the central extension (3.3) if and only if $d_2(u) = c$. By dimensional reasons $E_\infty^{2,0} = H^2(BK)/\mathbb{F}_p\{c\}$, and by means of the edge morphism we obtain $\ker \pi^*|_{H^2(BK)} = \mathbb{F}_p\{c\}$ (cf. [**8**, Theorem 6.8]).

Now, let $\lambda \in \mathbb{F}_p$ be a non-zero scalar, and let

$$0 \to \mathbb{Z}/p \to \tilde{G} \xrightarrow{\tilde{\pi}} K \to 1$$

be the central extension classified by λc.

Multiplication by λ in $\mathbb{Z}/p$ induces a group morphism $\_ \cdot \lambda \colon \mathbb{Z}/p \to \mathbb{Z}/p$, and therefore a continuous map $B^2(\_ \cdot \lambda) \colon B^2\mathbb{Z}/p \to B^2\mathbb{Z}/p$ that maps the fundamental class $\iota \in H^2(B^2\mathbb{Z}/p)$ to $\lambda\iota \in H^2(B^2\mathbb{Z}/p)$. At the level of central group extensions, $\_ \cdot \lambda$ gives rise to a group morphism $G \xrightarrow{f} \tilde{G}$ that makes the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & G & \xrightarrow{\pi} & K & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle _\cdot\lambda} & & \downarrow{\scriptstyle f} & & \| & & \\
0 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & \tilde{G} & \xrightarrow{\tilde{\pi}} & K & \longrightarrow & 1
\end{array}$$

This shows that G and $\tilde{G}$ are isomorphic groups. □

The description of the cohomology classes classifying the central extensions involved in the p-central series of $P(p,n)$ follows from the previous lemma.

Proposition 3.12. *Consider the groups $\mathbb{Z}/p^i \times \mathbb{Z}/p \times \mathbb{Z}/p$ and fix the following notation for the cohomology:*

$$H^*(B\mathbb{Z}/p^i \times B\mathbb{Z}/p \times B\mathbb{Z}/p) = E(u_i, y, y') \otimes \mathbb{F}_p[v_i, x, x'], \quad \beta_i(u_i) = v_i,$$

where generators are sorted as components. Then, for $n \geqslant 4$, there is a tower of extensions:

$$P(p,n) \xrightarrow{\pi_{n-3}} \mathbb{Z}/p^{n-3} \times \mathbb{Z}/p \times \mathbb{Z}/p \xrightarrow{\pi_{n-4}} \mathbb{Z}/p^{n-4} \times \mathbb{Z}/p \times \mathbb{Z}/p \to \cdots \xrightarrow{\pi_1} \mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p,$$

where each extension π_i, for $1 \leqslant i < n-3$, is classified by $\beta_i(u_i)$, π_{n-3} is classified by $\beta_{n-3}(u_{n-3}) - yy'$, and where π_{n-3} is the abelianization morphism $P(p,n) \to P(p,n)_{\mathrm{ab}} \cong \mathbb{Z}/p^{n-3} \times \mathbb{Z}/p \times \mathbb{Z}/p$.

Proof. According to Lemma 3.11, the extension π_i is classified (up to isomorphism) by a generator of $\ker \pi_i^*|_{H^2}$. Note that $\pi_i^*|_{H^1}$ is always an isomorphism, then $\ker \pi_i^*|_{H^2}$ can easily be calculated by comparison of the Bockstein spectral sequences of the groups involved. □

https://doi.org/10.1017/S0013091512000247 Published online by Cambridge University Press

4. Cohomological uniqueness

Let p be an odd prime, let $n \geqslant 3$ and let $P(p,n)$ be the group defined in (3.1). In this section we prove that the homotopy type of the classifying space of $P(p,n)$ is determined by its cohomology (Definition 2.2). The initial step towards that result is to study the behaviour of some endomorphisms of the mod p cohomology ring of $BP(p,n)$.

First we consider the case $n \leqslant 4$. In this case we do not need to use higher Bocksteins and it is enough to consider the structure of unstable algebra.

Theorem 4.1. *Let $\varphi\colon H^*(BP(3,3)) \to H^*(BP(3,3))$ be a homomorphism of $\mathcal{A}_3$-algebras which restricts to the identity in H^1. Then φ is an isomorphism.*

Proof. In this proof we follow the notation in Theorem 3.1 for generators and relations in cohomology.

By hypothesis, $\varphi(y) = y$ and $\varphi(y') = y'$. Now, since $\beta(y) = x$ and $\beta(y') = x'$, we have $\varphi(x) = \varphi(\beta(y)) = \beta(\varphi(y)) = \beta(y) = x$ and, analogously, $\varphi(x') = x'$. Moreover, by Remark 3.3,

$$\varphi(Y) = aY + bY' + cx + dx'$$

for some $a, b, c, d \in \mathbb{F}_3$. Because $yY = xy'$, we obtain

$$xy' = \varphi(xy') = \varphi(yY) = y\varphi(Y) = ayY + byY' + cyx + dyx'$$

and, by regrouping terms,

$$xy' = (a+d)xy' + byY' + cyx.$$

From Remark 3.3 we obtain $a+d = 1$ and $b = c = 0$, and $\varphi(Y) = aY + dx'$ with $a+d = 1$. Analogously $\varphi(Y') = bY' + cx$ with $b, c \in \mathbb{F}_3$ and $b + c = 1$. Now, as $Y^2 = xY'$, we have

$$\begin{gathered} \varphi(Y)^2 = x\varphi(Y'), \\ a^2Y^2 + d^2x'^2 + 2adYx' = bxY' + cx^2. \end{gathered}$$

Remark 3.3 now implies that $c = d = 0$ and $a^2 = a = b = 1$. So $\varphi(Y) = Y$ and $\varphi(Y') = Y'$, and, applying Bockstein again, $\varphi(X) = X$ and $\varphi(X') = X'$ too. So φ is the identity up to dimension 5 and it remains to check where it maps z.

Using the first Steenrod power of X,

$$\begin{aligned} \varphi(\mathcal{P}^1(X)) &= \mathcal{P}^1(\varphi(X)), \\ \varphi(x^2X + zy) &= \mathcal{P}^1(X), \\ x^2X + \varphi(z)y &= x^2X + zy, \\ \varphi(z)y &= zy. \end{aligned}$$

Thus, $\varphi(z) = z + \alpha$ where $\alpha y = 0$ and $\alpha \in \langle y, y', x, x', Y, Y', X, X'\rangle$. So $\varphi(\alpha) = \alpha$, $z = \varphi(z - \alpha)$ and φ is an epimorphism. In fact, because $H^*(BP(3,3))$ is a finite-dimensional $\mathbb{F}_3$-vector space in each dimension, φ is an isomorphism dimension-wise and thus φ is an isomorphism. □

Theorem 4.2. *Let $p \geqslant 5$ be a prime. If $\varphi\colon H^*(BP(p,3)) \to H^*(BP(p,3))$ is a homomorphism of $\mathcal{A}_p$-algebras that restricts to the identity in H^1, then φ is an isomorphism.*

Proof. Consider the notation of generators and relations in cohomology given in Theorem 3.2. We calculate the image under φ of every generator of $H^*(BP(p,3))$.

As φ is the identity on y and y', by applying Bockstein operations we get that $\varphi(x) = x$ and $\varphi(x') = x'$.

As Y is of degree 2, there exist coefficients a, b, c and d such that

$$\varphi(Y) = ax + bx' + cY + dY'.$$

Using the relation $Y^2 = 0$, we get $\varphi(Y)^2 = 0$, which implies via Remark 3.3 that $a = b = 0$, and so $\varphi(Y) = cY + dY'$. The relation $yY = 0$ implies $0 = y\varphi(Y) = dyY'$, so $d = 0$, yielding that there exists c such that $\varphi(Y) = cY$. Using the same arguments, there exists d such that $\varphi(Y') = dY'$.

According to Remark 3.5, there are $a_n \in \mathbb{F}_p$ and $f_{n,i}$ polynomials in x and x' such that, for $4 \leqslant n \leqslant p$,

$$\varphi(d_n) = a_n d_n + f_{n,1} y + f_{n,2} y' + f_{n,3} X + f_{n,4} X',$$

and, by applying the Bockstein operation, we get that, for $4 \leqslant n \leqslant p-1$,

$$\varphi(c_n) = a_n c_n + f_{n,1} x + f_{n,2} x'.$$

The relation $c_{p-1}x = -x^p$ gives rise to the following equalities:

$$\begin{aligned} -x^p &= \varphi(-x^p) \\ &= \varphi(c_{p-1}x) \\ &= \varphi(c_{p-1})\varphi(x) \\ &= \varphi(c_{p-1})x \\ &= a_{p-1}c_{p-1}x + f_{p-1,1}x^2 + f_{p-1,2}xx' \\ &= -a_{p-1}x^p + f_{p-1,1}x^2 + f_{p-1,2}xx', \end{aligned}$$

so $(a_{p-1} - 1)x^p = f_{p-1,1}x^2 + f_{p-1,2}xx'$. By Remark 3.3 we can simplify to

$$f_{p-1,1}x + f_{p-1,2}x' = (a_{p-1} - 1)x^{p-1}. \tag{4.1}$$

Doing the same computations using the relation $c_{p-1}x' = -x'^p$, we get

$$f_{p-1,1}x + f_{p-1,2}x' = (a_{p-1} - 1)x'^{p-1}. \tag{4.2}$$

Now, comparing (4.1) and (4.2) and again using Remark 3.3, we get $a_{p-1} = 1$, $\varphi(c_{p-1}) = c_{p-1}$.

We now show that $\varphi(c_n) = a_n c_n$, for $4 \leqslant n < p-1$: using the relation $c_n x = 0$ and applying φ, we get $f_{n,1}x + f_{n,2}x' = 0$, so

$$\varphi(c_n) = a_n c_n. \tag{4.3}$$

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In order to calculate $\varphi(z)$, we apply φ to the following equality:

$$\mathcal{P}^1(c_{p-1}) = -zc_{p-2} + x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1}.$$

Since $\varphi(c_{p-1}) = c_{p-1}$, $\varphi(x) = x$ and $\varphi(x') = x'$, we get

$$zc_{p-2} = \varphi(z)a_{p-2}c_{p-2}. \tag{4.4}$$

The generator z is free in $H^*BP(p,3)$, i.e. $H^*BP(p,3) = \mathbb{F}_p[z] \otimes (H^*BP(p,3)/\langle z \rangle)$. Hence, (4.4) implies that $a_{p-2} \neq 0$ and $\varphi(z) = a_{p-2}^{-1}z + g$, where g is an expression not involving z (hence g is decomposable), and such that $gc_{p-2} = 0$.

We use the knowledge that $a_{p-2} \neq 0$ to check that $a_n \neq 0$ for $4 \leqslant n < p-2$ with an induction argument: assume $\varphi(c_n) = a_n c_n$ with $a_n \neq 0$ and $5 \leqslant n \leqslant p-2$, and compute $\varphi(c_{n-1})$:

$$nzc_{n-1} = \mathcal{P}^1(c_n) = \mathcal{P}^1(\varphi(a_n^{-1}c_n)) = \varphi(a_n^{-1}\mathcal{P}^1(c_n)) = a_n^{-1}n\varphi(z)a_{n-1}c_{n-1}.$$

This implies $zc_{n-1} = a_n^{-1}a_{n-1}\varphi(z)c_{n-1}$, and this can only happen if $a_{n-1} \neq 0$ and $\varphi(z) = a_n a_{n-1}^{-1}z + g$ (g not involving z).

From the expression $c_3 = \mu XX'$ we deduce that $\varphi(c_3) = a_3c_3$ with $a_3 = cd$, where c and d were introduced at the beginning of the proof and are such that $\varphi(Y) = cY$ and $\varphi(Y') = dY'$. Again

$$\begin{aligned} 4\mu XX'z &= 4zc_3 \\ &= \mathcal{P}^1(c_4) \\ &= \mathcal{P}^1(\varphi(a_4^{-1}c_4)) \\ &= \varphi(a_4^{-1}\mathcal{P}^1(c_4)) \\ &= a_4^{-1}4\varphi(z)a_3c_3 \\ &= 4a_4^{-1}\mu cd\varphi(z)XX'; \end{aligned}$$

hence, $a_3 = cd$ is also non-zero. Therefore, c, d and a_n for all $n \in \{3,\dots,p-1\}$ are non-zero.

We now check that the coefficients c and d are equal: recall that c_2 was defined as $\lambda(xY' + x'Y)$ with λ non-zero. Then, applying $\mathcal{P}^1$ to c_3 we get

$$\begin{aligned} 3zc_2 &= \mathcal{P}^1(c_3) \\ &= \mathcal{P}^1(\varphi(a_3^{-1}c_3)) \\ &= a_3^{-1}\varphi(\mathcal{P}^1(c_3)) \\ &= a_3^{-1}(\varphi(3zc_2)) \\ &= a_3^{-1}3\varphi(z)\varphi(c_2) \\ &= a_3^{-1}3\varphi(z)\lambda(dxY' + cx'Y), \end{aligned}$$

which implies $\lambda z(xY' + x'Y) = a_3^{-1}\lambda\varphi(z)(dxY' + cx'Y)$ and can be simplified to

$$zxY' + zx'Y = da_3^{-1}\varphi(z)xY' + ca_3^{-1}\varphi(z)x'Y. \tag{4.5}$$

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Again, as z does not appear in any relation and $\varphi(z) = a_{p-2}^{-1}z + g$, (4.5) can be true only if $c = d$. In particular, $\varphi(c_2) = a_2 c_2$ with $a_2 = c\lambda \neq 0$.

Now we can assume that all the coefficients a_n for $2 \leqslant n \leqslant p-1$ and c and d are equal to 1: as all are different from zero and $r^{p-1} = 1$ if $r \in \mathbb{F}_p \setminus \{0\}$, φ^{p-1} is the identity on Y, Y' and c_n. Use now that φ is an isomorphism if and only if φ^{p-1} is so. Therefore, at this point we have that

$$\begin{array}{llll} \varphi(y) = y, & \varphi(y') = y', & \varphi(x) = x, & \varphi(x') = x', \\ \varphi(Y) = Y, & \varphi(Y') = Y', & \varphi(X) = X, & \varphi(X') = X', \end{array}$$

and

$$\begin{aligned} \varphi(c_i) &= c_i && \text{for } 2 \leqslant i \leqslant p-1, \\ \varphi(d_i) &= d_i + g_i && \text{for } 4 \leqslant i \leqslant p-1, \\ \varphi(z) &= z + g, \end{aligned}$$

where g and all g_i are expressions in x, x', y, y', X, X', Y and Y' (Remarks 3.5 and 3.6). This implies that all generators but d_p are in the image of φ.

The image of d_p, as it is in odd degree greater than 6, must be

$$\varphi(d_p) = a_p d_p + f_{p,1} y + f_{p,2} y' + f_{p,3} X + f_{p,4} X'$$

with $a_p \in \mathbb{F}_p$, and $f_{p,i}$ polynomials in x and x'. As $\beta(d_p) = 0$, the Bockstein operation on $\varphi(d_p)$ must vanish, and this means that

$$0 = \beta(\varphi(d_p)) = f_{p,1} x + f_{p,2} x'.$$

So this is a polynomial of degree $2p$ in x, x' which must be zero. By Remark 3.3 there exists a polynomial f_p in x and x' such that $f_{p,1} = f_p x'$ and $f_{p,2} = -f_p x$.

This implies that (recall $xy' = x'y$),

$$f_{p,1} y + f_{p,2} y' = f_p(x'y - xy') = 0,$$

and then

$$\varphi(d_p) = a_p d_p + f_{p,3} X + f_{p,4} X'.$$

As any expression on x, x', X and X' is in the image, we have only to check that $a_p \neq 0$.

Applying $\mathcal{P}^1$ to the above expression for $\varphi(d_p)$ and using that $\mathcal{P}^1(d_p) = 0$, we get

$$\begin{aligned} 0 &= a_p \cdot 0 + \mathcal{P}^1(f_{p,3})X + f_{p,3}\mathcal{P}^1(X) + \mathcal{P}^1(f_{p,4})X' + f_{p,4}\mathcal{P}^1(X') \\ &= \mathcal{P}^1(f_{p,3})X + \mathcal{P}^1(f_{p,4})X' + f_{p,3}(x^{p-1}X + zy) + f_{p,4}(x'^{p-1}X' - zy') \\ &= \mathcal{P}^1(f_{p,3})X + \mathcal{P}^1(f_{p,4})X' + f_{p,3}x^{p-1}X + f_{p,4}x'^{p-1}X' + z(f_{p,3}y - f_{p,4}y'). \end{aligned}$$

Again using the fact that z is a free generator, we obtain $f_{p,3}y - f_{p,4}y' = 0$, and then, applying the Bockstein homomorphism, we get $f_{p,3}x - f_{p,4}x' = 0$, i.e. $f_{p,3}x = f_{p,4}x'$.

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From this we deduce that there exists a polynomial $f \in \mathbb{F}_p[x, x']$ such that $f_{p,3} = x'f$ and $f_{p,4} = xf$.

Going back to the description of $\varphi(d_p)$ we find that

$$\varphi(d_p) = a_p d_p + f_{p,3}X + f_{p,4}X' = a_p d_p + x'fX + xfX' = a_p d_p + f(x'X + xX') = a_p d_p,$$

where the last equality holds because $x'X + xX' = 0$. Hence, we learn that $\varphi(d_p) = a_p d_p$.

To finish the proof we recall that $d_p x = x^{p-1}X$ and apply the homomorphism φ:

$$\varphi(d_p x) = \varphi(d_p)x = a_p d_p x = a_p x^{p-1}X = \varphi(x^{p-1}X) = x^{p-1}X.$$

Then we deduce that $a_p \neq 0$ and φ is an isomorphism. □

We now consider the case of $n > 3$. Here the use of Bockstein operators is needed.

Theorem 4.3. *Let p be an odd prime and consider the notation of the generators and relations in $H^*_\beta(BP(p,n))$ as in Theorem 3.7.*

(a) *If $\varphi\colon H^*(BP(p,4)) \to H^*(BP(p,4))$ is a homomorphism of unstable algebras that fixes y and y', then φ is an isomorphism.*

(b) *If $n \geqslant 5$ and $\varphi\colon H^*_\beta(BP(p,n)) \to H^*_\beta(BP(p,n))$ is a homomorphism in $\mathcal{K}_\beta$ which fixes y and y'. Then φ is an isomorphism.*

Proof. We prove both results at the same time. Just observe that the Bockstein used in the proof is β_{n-3}, which is part of the mod p Steenrod algebra when $n = 4$.

Starting from $\varphi(y) = y$ and $\varphi(y') = y'$ and using the Bockstein operator we reach $\varphi(x) = x$ and $\varphi(x') = x'$. On the other hand, there exist $a, b, c \in \mathbb{F}_p$ such that $\varphi(u) = au + by + cy'$. From Remark 3.9 we know that $\beta_{n-3}(u) = y'y$ and $\beta_i(u) = 0$ for $i = 1, \dots, n-4$. For the case $n = 4$ we have

$$\varphi(\beta(u)) = \varphi(yy') = yy' = \beta(\varphi(u)) = ay'y + bx + cx'.$$

Hence, $a = 1$, $b = c = 0$ and $u \in \operatorname{Im}\varphi$. For $n > 4$ we have in particular that $\beta(u) = 0$,

$$\varphi(\beta(u)) = 0 = \beta(\varphi(u)) = bx + cx'$$

and hence $b = c = 0$. Applying now β_{n-3} we find that

$$\varphi(\beta_{n-3}(u)) = \varphi(yy') = yy' = \beta_{n-3}(\varphi(u)) = ay'y$$

and that $a = 1$. In either case ($n = 4$ or $n > 4$) we get $\langle u, y, y', x, x' \rangle \leqslant \operatorname{Im}\varphi$.

Now consider the generator c_{p-1}. We can write

$$\varphi(c_{p-1}) = a_{p-1}c_{p-1} + bx^{p-1} + cx'^{p-1} + g_{p-1}$$

with $a_{p-1}, b, c \in \mathbb{F}_p$ and g_{p-1} not containing scalar multiples of the monomials c_{p-1}, x^{p-1} and x'^{p-1}. Applying φ to the equation $c_{p-1}x' = -x'^p$, we obtain

$$\begin{aligned} -x'^p &= a_{p-1}c_{p-1}x' + bx^{p-1}x' + cx'^p + g_{p-1}x' \\ &= -a_{p-1}x'^p + bx^{p-1}x' + cx'^p + g_{p-1}x'. \end{aligned}$$

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Then from Remark 3.8 we get $-1 = -a_{p-1} + c$ and $b = 0$. The same argument with $c_{p-1}x = -x^p$ instead gives

$$\begin{aligned} -x^p &= a_{p-1}c_{p-1}x + bx^p + cx'^{p-1}x + g_{p-1}x \\ &= -a_{p-1}x^p + bx^p + cx'^{p-1}x + g_{p-1}x. \end{aligned}$$

Again by Remark 3.8 we get $-1 = -a_{p-1} + b$ and $c = 0$. We conclude that $b = c = 0$, $a_{p-1} = 1$ and $\varphi(c_{p-1}) = c_{p-1} + g_{p-1}$.

Next we deal with c_{p-2} of degree $2(p-2)$ and z of degree $2p$. Their images are $\varphi(c_{p-2}) = a_{p-2}c_{p-2} + g_{p-2}$ and $\varphi(z) = a_z z + g_z$, with $a_{p-2}, a_z \in \mathbb{F}_p$, and g_{p-2} and g_z not involving the monomials c_{p-2} and z, respectively. Write the Steenrod power

$$\mathcal{P}^1(c_{p-1}) = -zc_{p-2} + x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1}$$

as $\mathcal{P}^1(c_{p-1}) = -zc_{p-2} + f$, with $f = x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1}$. Applying φ, we get

$$\begin{aligned} \varphi(\mathcal{P}^1(c_{p-1})) &= \mathcal{P}^1(\varphi(c_{p-1})), \\ \varphi(-zc_{p-2} + f) &= \mathcal{P}^1(c_{p-1} + g_{p-1}), \\ -(a_z z + g_z)(a_{p-2}c_{p-2} + g_{p-2}) + f &= -zc_{p-2} + f + \mathcal{P}^1(g_{p-1}), \\ -a_z a_{p-2} z c_{p-2} - a_z z g_{p-2} - a_{p-2} g_z c_{p-2} - g_z g_{p-2} &= -zc_{p-2} + \mathcal{P}^1(g_{p-1}). \end{aligned}$$

Because g_{p-1} does not involve c_{p-1} and the action of $\mathcal{P}^1$ on u, y, y', x, x' is determined by the axioms, we deduce that $\mathcal{P}^1(g_{p-1})$ does not involve zc_{p-2}. Then from Remark 3.8 we have that $a_z a_{p-2} = 1$ and both a_z and a_{p-2} are non-zero.

For the rest of the generators c_i for $i = 2, 3, \dots, p-3$ we can write $\varphi(c_i) = a_i c_i + g_i$, with $a_i \in \mathbb{F}_p$ and g_i not involving c_i. The Steenrod power $\mathcal{P}^1(c_{i+1}) = (i+1)zc_i$ then yields

$$\begin{aligned} \varphi(\mathcal{P}^1(c_{i+1})) &= \mathcal{P}^1(\varphi(c_{i+1})), \\ \varphi((i+1)zc_i) &= \mathcal{P}^1(\alpha_{i+1}c_{i+1} + g_{i+1}), \\ (i+1)(a_z z + g_z)(a_i c_i + g_i) &= (i+1)a_{i+1}zc_i + \mathcal{P}^1(g_{i+1}), \\ (i+1)(a_z a_i z c_i + a_z z g_i + a_i g_z c_i + g_z g_i) &= (i+1)a_{i+1}zc_i + \mathcal{P}^1(g_{i+1}). \end{aligned}$$

Note again that there is no relation involving the generator z and the relations involving c_i are $c_i y = c_i y' = c_i x = c_i x' = c_i c_j = 0$ for $j < 2p-2-i$. Also, the monomial zc_i cannot appear in zg_i, $g_z c_i$ and $g_z g_i$ because g_i does not contain c_i and g_z does not contain z. Moreover, $\mathcal{P}^1(g_{i+1})$ does not involve zc_i as g_{i+1} does not involve c_{i+1}. We deduce that $(i+1)a_z a_i = (i+1)a_{i+1}$. As $a_z \neq 0$ and $a_{p-2} \neq 0$, an inductive argument shows that $a_i \neq 0$ for $i = 2, 3, \dots, p-3$, and hence for all $i = 2, 3, \dots, p-1$.

To finish we show that all the generators $c_2, c_3, \dots, c_{p-1}, z$ are in the image of φ. We start with $c_2 = (\varphi(c_2) - g_2)/\alpha_2$. As $g_2 \in \langle u, x, x, y, y' \rangle \leqslant \operatorname{Im}\varphi$, c_2 is also in the image of φ. An inductive argument shows that $c_i = (\varphi(c_i) - g_i)/\alpha_i$ is in the image of φ as g_i belongs to $\langle u, x, x', y, y', c_2, c_3, \dots, c_{i-1} \rangle$. This argument also applies to show that $z \in \operatorname{Im}\varphi$.

Hence, φ is an epimorphism. Because $H^*_\beta(BP(p,n))$ is finite in each dimension, φ is an isomorphism. □

Then, the following corollary is straightforward.

Corollary 4.4. *$H^*_\beta(BP(p,n))$ for odd p and $n \geqslant 3$ is weakly generated (Definition 2.3) by y and y'.*

Proof. Let φ be an endomorphism of $H^*_\beta(BP(p,n))$ which is an isomorphism on $\langle y, y'\rangle$. Using the outer automorphism group of $P(p,n)$ that is described in [**5**, Lemma A.5], there is a morphism $f\colon BP(p,n) \to BP(p,n)$ such that the composition $f^* \circ \varphi$ fixes y and y'. Now use Theorems 4.1, 4.2 and 4.3 to get the result. □

Note that for any finite p-group there is a natural isomorphism $H^1P \cong P/\Phi(P)$, where $\Phi(P)$ stands for the Frattini subgroup of P [**6**, p. 173]. Therefore, Theorems 4.1–4.3 can be seen as a cohomological counterpart of the following group theoretical result.

Proposition 4.5. *Let P be a finite p-group and let $f\colon P \to P$ be a group morphism such that the induced morphism at the level of Frattini quotients $\tilde{f}\colon P/\Phi(P) \to P/\Phi(P)$ is an isomorphism. Then f is an isomorphism.*

Proof. Let n be such that $P/\Phi(P) = (\mathbb{Z}/p)^n$ [**6**, Theorem 5.1.3]. Assume f is not an isomorphism. Then $f(P) \leqslant H < P$ for some maximal subgroup $H < P$, and therefore $\tilde{f}(P/\Phi(P)) < H/\Phi(P) = (\mathbb{Z}/p)^{n-1} < P/\Phi(P)$, that is, $\tilde{f}$ is not an isomorphism. □

Now, we apply the results above to obtain the cohomology uniqueness of the classifying space $BP(p,n)$. We split this result into two corollaries because the structure of $P(p,3)$ is essentially different from that of $P(p,n)$, $n > 4$.

Corollary 4.6. *Let p be an odd prime and let X be a p-complete space such that $H^*(X) \cong H^*(BP(p,3))$ as unstable algebras. Then $X \simeq BP(p,3)$.*

Proof. Consider the central extension

$$0 \to \mathbb{Z}/p \to P(p,3) \xrightarrow{\pi} \mathbb{Z}/p \times \mathbb{Z}/p \to 0$$

and denote by y and y' the two generators of $H^1(\mathbb{Z}/p \times \mathbb{Z}/p)$ that are mapped by π to the generators of the same name in $H^1(P(p,3))$ (see Remark 3.4).

By the same argument used in the proof of Proposition 3.12 or by a direct computation using the cochains in Remark 3.4, we find that this central extension is classified by $yy' \in H^2(\mathbb{Z}/p \times \mathbb{Z}/p)$, and it gives rise to the principal fibration

$$BP(p,3) \xrightarrow{B\pi} B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{yy'} B^2\mathbb{Z}/p.$$

Consider the map $\pi_X: X \to B\mathbb{Z}/p \times B\mathbb{Z}/p$ that classifies the classes $y, y' \in H^1(X)$. Then the composite

$$X \xrightarrow{\pi_X} B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{yy'} B^2\mathbb{Z}/p$$

is null-homotopic because of Theorems 3.1 and 3.2, and so π_X lifts to $\varphi\colon X \to BP(p,3)$, giving the commutative diagram

$$\begin{array}{ccc} & & BP(p,3) \\ & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle B\pi} \\ X & \xrightarrow{\pi_X} & B\mathbb{Z}/p \times B\mathbb{Z}/p \end{array}$$

which implies that φ^* fixes y and y'. Now apply Theorems 4.1 and 4.2 to φ^*. □

Corollary 4.7. *Let p be an odd prime and let X be a p-complete space.*

(a) *If $H^*(X) \cong H^*(BP(p,4))$ as unstable algebras, then $X \simeq BP(p,4)$.*

(b) *If $n \geqslant 5$ and $H^*_\beta(X) \cong H^*_\beta(BP(p,n))$ as objects in $\mathcal{K}_\beta$, then $X \simeq BP(p,n)$.*

Proof. Consider the central extensions and notation in Proposition 3.12. For $i = 1, \dots, n-4$ we have the short exact sequences

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^{i+1} \times \mathbb{Z}/p \times \mathbb{Z}/p \xrightarrow{\pi_i} \mathbb{Z}/p^i \times \mathbb{Z}/p \times \mathbb{Z}/p \to 0,$$

which are classified by $\beta_i(u_i) \in H^2(\mathbb{Z}/p^i \times \mathbb{Z}/p \times \mathbb{Z}/p)$ with $u_i \in H^1(\mathbb{Z}/p^i)$.

Now let $\pi_{1,X}$ be the map $\pi_{1,X}\colon X \to B\mathbb{Z}/p \times B\mathbb{Z}/p \times \mathbb{Z}/p$ that classifies the classes $u, y, y' \in H^1(X)$, i.e. such that, in cohomology, $\pi^*_{1,X}$ maps u_1, y and y' from $H^1(B\mathbb{Z}/p \times B\mathbb{Z}/p \times \mathbb{Z}/p)$ to u, y and y' from $H^1(X)$ respectively.

The composite

$$X \xrightarrow{\pi_{1,X}} B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{\beta(u_1)} B^2\mathbb{Z}/p$$

is null-homotopic because $\beta(u) = 0$ in $H^*(X)$ according to Remark 3.9. Hence, the map $\pi_{1,X}$ extends to a map $\pi_{2,X}$ which fits into the following commutative diagram:

$$\begin{array}{ccc} & & B\mathbb{Z}/p^2 \times B\mathbb{Z}/p \times B\mathbb{Z}/p \\ & \overset{\pi_{2,X}}{\nearrow} & \downarrow{\scriptstyle B\pi_1} \\ X & \xrightarrow{\pi_{1,X}} & B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p \end{array}$$

Note that in cohomology $B\pi_1$ maps u_1, y and y' to u_2, y and y' respectively. Hence, $\pi_{2,X}$ maps u_2, y and y' to u, y and y' respectively. Using inductively that all the higher Bockstein operators $\beta_i(u)$ vanish for $i = 2, \dots, n-4$, we build step by step a map

$$\pi_{n-3,X}\colon X \to B\mathbb{Z}/p^{n-3} \times B\mathbb{Z}/p \times B\mathbb{Z}/p,$$

which in cohomology maps u_{n-3}, y and y' to u, y and y' respectively. To finish the proof we use the abelianization morphism from Proposition 3.12:

$$0 \to \mathbb{Z}/p \to P(p,n) \xrightarrow{\pi_{n-3}} \mathbb{Z}/p^{n-3} \times \mathbb{Z}/p \times \mathbb{Z}/p \to 0,$$

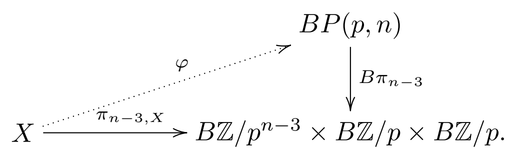
https://doi.org/10.1017/S0013091512000247 Published online by Cambridge University Press

which is classified by $\beta_{n-3}(u_{n-3}) - yy' \in H^2(\mathbb{Z}/p^{n-3} \times \mathbb{Z}/p \times \mathbb{Z}/p)$, where u_{n-3}, y and y' are generators of $H^1(\mathbb{Z}/p^{n-3} \times \mathbb{Z}/p \times \mathbb{Z}/p)$ that are mapped by π_{n-3} to the generators u, y and y' in $H^1(P(p,n))$.

Because $\beta_{n-3}(u) - yy' = 0$ in $H^*(X)$, the composite

$$X \xrightarrow{\pi_{n-3,X}} B\mathbb{Z}/p^{n-3} \times B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{\beta_{n-3}(u_{n-3})-yy'} B^2\mathbb{Z}/p$$

is null-homotopic and we can lift $\pi_{n-3,X}$ to a map φ that makes the following diagram commutative:



This shows that φ^* fixes y and y', and hence Theorem 4.3 gives the result. □

5. Some applications to group theory

The techniques used in the proof of Corollaries 4.6 and 4.7 can be used to obtain a cohomological characterization of $P(p,n)$ as a complement for some $N \trianglelefteq G$, for a super group $P(p,n) \leqslant G$. Recall that, given a group G and a normal subgroup $N \trianglelefteq G$, $K \leqslant G$ is a complement for N if $G = NK$ and $N \cap K = 1$, that is, if $G = N \rtimes K$.

Again, we consider the case $n = 3$ separately.

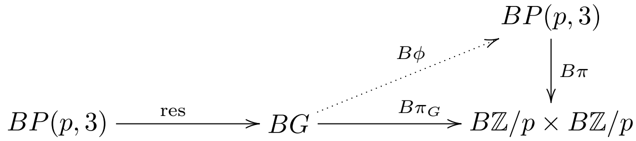
Proposition 5.1. *Let p be an odd prime and let G be a finite group such that $P(p,3) \leqslant G$. Assume also that there exists $\psi\colon H^*(BP(p,3)) \to H^*(BG)$ as unstable algebras such that $(\mathrm{res} \circ \psi)|_{H^1_\beta(BP(p,3))}$ is the identity. Then $P(p,3)$ is a complement for some $N \trianglelefteq G$.*

Proof. As stated above, we work along the same lines as in the proof of Corollary 4.6. We begin by considering the map $B\pi_G\colon BG \to B\mathbb{Z}/p \times B\mathbb{Z}/p$ that classifies the classes $\psi(y), \psi(y') \in H^1(BG)$. This means that if we denote (as we did in Corollary 4.6) by y and y' the two generators of $H^1(B\mathbb{Z}/p \times B\mathbb{Z}/p)$ that are mapped by $B\pi : BP(p,3) \to B\mathbb{Z}/p \times B\mathbb{Z}/p$ to the generators of the same name in $H^1(BP(p,3))$ (see also Remark 3.4), then $B\pi_G^*(y) = \psi(y)$ and $B\pi_G^*(y') = \psi(y')$.

Moreover, $B\pi_G^*(yy') = B\pi_G^*(y)B\pi_G^*(y') = \psi(y)\psi(y') = \psi(yy') = \psi(0) = 0$ (Theorems 3.1 and 3.2), and the composite

$$BG \xrightarrow{B\pi_G} B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{yy'} B^2\mathbb{Z}/p$$

is null-homotopic. Therefore, $B\pi_G$ lifts to $B\phi\colon BG \to BP(p,3)$, giving the commutative diagram



which implies that $B\phi^*(y) = \psi(y)$ and $B\phi^*(y') = \psi(y')$, and

$$(\mathrm{res} \circ B\phi)^*(y) = (\mathrm{res}^* \circ \psi)(y) = y \quad \text{and} \quad (\mathrm{res} \circ B\phi)^*(y') = (\mathrm{res}^* \circ \psi)(y) = y'.$$

Now, applying Theorems 4.1 and 4.2 or Proposition 4.5, we obtain that $\phi|_{P(p,3)}$ is an automorphism of $P(p,3)$, that is, $P(p,3)$ is a complement for $N = \ker \phi \trianglelefteq G$. □

We now proceed with the case $n > 3$.

Proposition 5.2. *Let p be an odd prime and let G be a finite group such that $P(p,n) \leqslant G$.*

(a) *If $n = 4$ and there exists $\psi\colon H^*(BP(p,4)) \to H^*(BG)$ as unstable algebras such that $(\mathrm{res} \circ \psi)|_{H^1_\beta(BP(p,4n))}$ is the identity, then $P(p,4)$ is a complement for some $N \trianglelefteq G$.*

(b) *If $n \geqslant 5$ and there exists $\psi\colon H^*_\beta(BP(p,n)) \to H^*_\beta(BG)$ a morphism in $\mathcal{K}_\beta$ such that $(\mathrm{res} \circ \psi)|_{H^1_\beta(BP(p,n))}$ is the identity, then $P(p,n)$ is a complement for some $N \trianglelefteq G$.*

Proof. We now follow the lines of the proof of Corollary 4.7 but start with the map $B\pi_{1,G}\colon BG \to B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p$ that classifies the classes $\psi(u), \psi(y), \psi(y') \in H^1(BG)$. This means that in cohomology this map carries the elements u_1, y and y' from $H^1(B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p)$ (defined in Proposition 3.12) to $\psi(u)$, $\psi(y)$ and $\psi(y')$, respectively.

The arguments in Corollary 4.7 together with the fact that ψ preserves relations and higher Bockstein operators show that there exists a map

$$BG \overset{B\phi}{\to} BP(p,n)$$

which satisfies $B\phi^*(y) = \psi(y)$, $B\phi^*(y') = \psi(y')$ and $B\phi^*(u) = \psi(u)$. Hence, we also get the following:

$$\begin{aligned}
(\mathrm{res} \circ B\phi)^*(y) &= (\mathrm{res}^* \circ \psi)(y) = y, \\
(\mathrm{res} \circ B\phi)^*(y') &= (\mathrm{res}^* \circ \psi)(y) = y', \\
(\mathrm{res} \circ B\phi)^*(u) &= (\mathrm{res}^* \circ \psi)(u) = u.
\end{aligned}$$

Again, applying Proposition 4.5 or Theorem 4.3, we obtain that $\phi|_{P(p,n)}$ is an automorphism of $P(p,n)$, that is, $P(p,n)$ is a complement for $N = \ker \phi \trianglelefteq G$. □

Acknowledgements. A.D. and A.V. are partly supported by FEDER-MCI Grant MTM2010-18089 and Junta de Andalucía Grants FQM-213 and P07-FQM-2863. A.R. is partly supported by FEDER-MCI Grant MTM2010-20692. A.R. and A.V. are partly supported by Generalitat de Catalunya Grant 2009SGR-1092.

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