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Height Uniformity for Algebraic Points on Curves

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Abstract. We recall the main result of L. Caporaso, J. Harris, and B. Mazur's 1997 paper of 'Uniformity of rational points.' It says that the Lang conjecture on the distribution of rational points on varieties of general type implies the uniformity for the *numbers* of rational points on curves of genus at least 2. In this paper we will investigate its analogue for their heights under the assumption of the Vojta conjecture. Basically, we will show that the Vojta conjecture gives a naturally expected simple uniformity for their heights.

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Key words. ample divisor, big divisor, canonical divisor, fiber product, height, height zeta function, symmetric product, variety of general type, Vojta conjecture.

1. Introduction

According to G. Faltings' theorem on the Mordell conjecture ([11]), every nonsingular projective curve of genus at least 2 defined over a number field has only finitely many rational points. Then there arise two natural questions: how many there are and how 'big' (in the sense of height) they are. To the former there is a known answer. In fact, P. Vojta's subsequent proof ([36]) (and also E. Bombieri's ([5])) give(s) an *effective* upper bound for the *number* of rational points. To the latter, however, there has been no known answer yet.

Let us talk a little bit more about the first question above. In 1997 there appeared a remarkable paper by L. Caporaso, J. Harris and B. Mazur ([7]). They proved, assuming the (weak) Lang conjecture, that there exists a uniform upper bound (depending on g and k) for the *number* of k-rational points of curves of genus $g \geqslant 2$ defined over a number field k. The Lang conjecture says that the set of rational points of a variety of *general type* is not Zariski dense. And by a (projective) variety of general type we mean that (one, hence every, desingularization of) the variety has *regular pluri-canonical forms* enough to give rise to a birational map *into* a projective space. Later, D. Abramovich ([1]) and P. Pacelli ([25]) strengthened their result by proving that their uniform upper bound can be chosen to be independent of a field of definition of curves, but dependent only on its degree.

Besides Faltings' finiteness theorem above, it is well known that there are only finitely many points of bounded degree over an arbitrary number field on some

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curves of genus $\geqslant 2$, too. It is due to D. Abramovich, J. Harris, M. Hindry, J. Silverman, P. Vojta and others with the aid of Faltings' result on another conjecture of Lang ([12]). See [3, 14, 16, 17, 35].

Now we come to the point that we should mention Vojta's work. In the early 1980's he discovered an uncanny similarity between Diophantine approximation theory and Nevanlinna theory (value distribution theory for complex analytic functions)–the fact is that it should be noted that C. Osgood ([23]) has also, previously, noticed a Nevanlinna–Roth connection. This great insight of his led him to his main conjecture and his independent proof of the Mordell conjecture ([36]).

What we will prove in this paper may be regarded as a height version of the works of [7] and [25] above. Our result may also be thought of as a *conditional* strengthened and generalized version of the work of T. de Diego ([10]). In the case $d = e = 1$ below, she gives an *unconditional* height upper bound for all but finitely many rational points on each fiber together with an explicit description of the number of exceptional points. Our result *conditionally* eliminates the need of this exceptional subset as follows.

Let d and e be integers $\geqslant 1$. Let $\pi: X \to Y$ be a family of curves of genus $\geqslant 2$, i.e., both X and Y are nonsingular projective varieties (of arbitrary dimension) and the generic fiber is a nonsingular projective curve of genus $\geqslant 2$. And, in addition, X, Y and π are all assumed to be defined over a number field k. Finally choose an arbitrary height h on X and a height h_Y on Y associated to an ample divisor satisfying $h_Y \geqslant 1$.

THEOREM 1.0.1. *Assume, in addition, that all the one-dimensional nonsingular fibers of π have only finitely many algebraic points of degree $\leqslant e$ over an arbitrary number field. Then, assuming the Vojta conjecture for varieties of dimension $\leqslant d \cdot (e + \dim Y)$, there is a constant $c > 0$ such that*

$$h(P) \leqslant c \cdot h_Y(\pi(P)),$$

whenever P is an algebraic point of degree $\leqslant e$ over $k(\pi(P))$, $\pi(P)$ has degree $\leqslant d$ over k and the fiber of π over the point $\pi(P)$ is a nonsingular projective curve of genus $\geqslant 2$. Note that c is independent of P.

The hypothesis of Theorem 1.0.1 implies, in particular, that none of the nonsingular fibers are e'-gonal or e'-elliptic ($e' = 1, 2, \ldots, e$) and hence that they have genus $\geqslant 2e - 1$. And to my best knowledge, conversely it has been proved the finiteness of algebraic points of degree bounded by e (over an arbitrary number field) on nonsingular projective curves of genus $\geqslant 2$, non-e'-gonal and non-e'-elliptic ($1 \leqslant e' \leqslant e$), for the following cases:

- $e = 1$ (the Mordell conjecture first proved by G. Faltings),
- $e = 2$ (by J. Harris and J. Silverman),
- $e = 3$, or $e = 4$ and genus $\geqslant 8$ (both by D. Abramovich and J. Harris), and
- $e = 6$ and genus $\geqslant 17$.

(There may be more scattered results known. Indeed, there is a very nice result of the same sort concerning plane curves. It requires, however, a different setting, so we will add it as a separate theorem at the end of the paper. Anyway, the curves treated there will also satisfy the above hypothesis for e which will be specified there.)

Later we will add some more cases for which the *height uniformity* can be proved. By the way, we will use the phrase *the height uniformity* for the type of results on the comparison of heights appearing in the theorem above. Indeed, contrary to the case of the number of points, when it comes to height we clearly cannot expect *genuine* uniformity in general. So there should be no objection to the choice of our term.

As an application of our results we will introduce in the last section the so-called *height zeta function* associated to families studied in Theorem 1.0.1. We will see that it is closely related to the Riemann zeta function in some case.

Finally, as a matter of fact, we have also proven similar results concerning integral points on elliptic and rational curves. The results will appear in a separate paper.

2. Preliminaries

2.1. NOTATION, DEFINITIONS, AND CONVENTIONS

In this section we will make explicit part of the tools we will use in our subsequent proofs. Unless otherwise stated, by heights we will always mean *logarithmic* ones. For the general theory of heights we refer to Lang [18], Silverman [33] and Vojta [34]. In particular, we have:

FACT [DHA: *Dominance of a height associated to an ample divisor*] ([[18], Chap. 4, Prop. 5.4]). Let h_c be a height associated to an ample divisor class c with $h_c \geqslant 1$, and h an arbitrary height. Then we have $h = \mathrm{O}(h_c)$. Let $d \geqslant 1$ be an integer. Then, for an arbitrary variety V defined over a number field k, let $V(k, d)$ (resp. $V(k, = d)$) be the set of algebraic points on V that have degree $\leqslant d$ (resp. $= d$) over k. Now let π: $X \to Y$ be a surjective morphism between projective varieties. Assume that X, Y and the morphism π are all defined over a number field k. Then we define

$$Y^\circ := \{t \in Y(\bar{k})\colon \text{ The fiber } X_t \text{ over } t \text{ is nonsingular and has the same dimension as that of the generic fiber}\},$$

and

$$X^\circ := \bigcup_{t \in Y^\circ} X_t = \text{The union of nonsingular fibers having the same dimension as that of the generic fiber.}$$

DEFINITION. For an integer $d \geqslant 1$ (but essentially $d \geqslant 2$), a nonsingular projective curve of genus $\geqslant 2$ is said to be *d-gonal* (resp. *d-elliptic*) if it is a d-cover of the projective line $\mathbb{P}^1$ (resp. of an elliptic curve). In particular, it is called *hyperelliptic* (resp. *bielliptic*) for 2-gonal (resp. 2-elliptic), and *trigonal* (resp. *trielliptic*) for 3-gonal (resp. 3-elliptic) for their old use.

Let $\pi: X \to Y$ be a surjective morphism between projective varieties. For an integer $d \geqslant 1$, let

$$X^{(d)} = \mathrm{Sym}_d\, X \; (= X^d/S_d) = \text{the } d\text{th symmetric product of } X$$

where S_d is the symmetric group of order $d!$, and let

$$X_Y^d = \overbrace{X \times_Y \cdots \times_Y X}^{d \text{ times}} = \text{ the } d\text{th fiber product of } X \text{ over } Y.$$

Then we have the natural surjective finite morphism

$$X^d \longrightarrow X^{(d)}, \qquad (P_1, \dots, P_d) \mapsto \sum_{n=1}^{d} (P_n),$$

and we denote by $X_Y^{(d)}$ the image of X_Y^d under the morphism. And we have the nth projection $(1 \leqslant n \leqslant d)$ $X^d \to X$. We also have another important induced fibration

$$\pi^{(d)}: X^{(d)} \longrightarrow Y^{(d)}, \qquad \sum_{n=1}^{d} (P_n) \mapsto \sum_{n=1}^{d} (\pi(P_n)).$$

EXAMPLE 2.1.1. Under the same notation just as above we assume, furthermore, that X is nonsingular. Fix integers d and $e \geqslant 1$. We then have the natural induced surjective finite morphisms $\alpha: X^e \to X^{(e)}$ and

$$\beta: (X^{(e)})^d \longrightarrow X^{(e,d)} := (X^{(e)})^{(d)}, \quad \left(\sum_{j=1}^{e} (P_{i,j}) \right)_{1 \leqslant i \leqslant d} \mapsto \sum_{i=1}^{d} \left(\sum_{j=1}^{e} (P_{i,j}) \right),$$

where $P_{i,j} \in X$ for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant e$.

Let W be a closed subvariety of $X^{(e,d)}$, e.g., in particular, a subvariety of $(X_Y^{(e)})^{(d)}$ for some of our later applications. Then we look at the composition $f: \widetilde{W} \to X^{(e,d)}$ of v and the inclusion $W \hookrightarrow X^{(e,d)}$ where $v: \widetilde{W} \to W$ is a desingularization. Note that both $X^{(e)}$ and $X^{(e,d)}$ are normal and projective. Let H be a hyperplane section of X (or an arbitrary ample divisor of X) and h the height of X which is associated to H (not necessarily with the assumption that h be nonnegative). Write

$$H^{(e)} = \alpha_* \sum_{j=1}^{e} \alpha_j^* H \in \mathrm{Div}\, X^{(e)} \quad \text{and} \quad H^{(e,d)} = \beta_* \sum_{i=1}^{d} \beta_i^* H^{(e)} \in \mathrm{Div}\, X^{(e,d)},$$

where

$$\alpha_j: X^e \to X \quad (1 \leqslant j \leqslant e) \quad \text{and} \quad \beta_i: (X^{(e)})^d \to X^{(e)} \quad (1 \leqslant i \leqslant d)$$

are the jth and ith projections, respectively. These are locally principal (Weil) divisors, hence to them we can associate heights. We also note that

$$\alpha^* H^{(e)} = c_1 \cdot \sum_{j=1}^{e} \alpha_j^* H \quad \text{and} \quad \beta^* H^{(e,d)} = c_2 \cdot \sum_{i=1}^{d} \beta_i^* H^{(e)}$$

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for some constants $c_1, c_2 \geqslant 1$. Furthermore, it also follows that both $H^{(e)}$ and $H^{(e,d)}$ are ample divisors, since both α and β are surjective finite morphisms.

And consider the composition

$$\sigma\colon X^{ed} = (X^e)^d \xrightarrow{\phi} (X^{(e)})^d \xrightarrow{\beta} (X^{(e)})^{(d)} = X^{(e,d)},$$

and the ith projection $(1 \leqslant i \leqslant d)$ $\phi_i\colon (X^e)^d \to X^e$. Then we have

$$\begin{aligned} h_{X^{(e,d)},H^{(e,d)}} \circ \sigma &= h_{(X^{(e)})^d,\beta^* H^{(e,d)}} \circ \phi = c_2 \cdot h_{(X^{(e)})^d,\sum_{i=1}^d \beta_i^* H^{(e)}} \circ \phi \\ &= c_2 \cdot \sum_{i=1}^d h_{(X^{(e)})^d,\beta_i^* H^{(e)}} \circ \phi = c_2 \cdot \sum_{i=1}^d h_{X^{(e)},H^{(e)}} \circ \beta_i \circ \phi \\ &= c_2 \cdot \sum_{i=1}^d h_{X^{(e)},H^{(e)}} \circ \alpha \circ \phi_i = c_2 \cdot \sum_{i=1}^d h_{X^e,\alpha^* H^{(e)}} \circ \phi_i \\ &= c_1 c_2 \cdot \sum_{i=1}^d \sum_{j=1}^e h_{X^e,\alpha_j^* H} \circ \phi_i = c_1 c_2 \cdot \sum_{i=1}^d \sum_{j=1}^e h_{X,H} \circ \alpha_j \circ \phi_i \\ &= c \cdot \sum_{i=1}^d \sum_{j=1}^e h \circ \alpha_j \circ \phi_i, \end{aligned}$$

where $c := c_1 c_2$. Let $\mathcal{P}_i = \sum_{j=1}^e (P_{i,j})$ where $P_{i,j} \in X$ for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant e$. We then note:

$$h_{X^{(e,d)},H^{(e,d)}}\left(\sum_{i=i}^d (\mathcal{P}_i)\right) = c \cdot \sum_{i,j} h(P_{i,j}) \quad (\geqslant 1, \text{ if } h \geqslant 1), \tag{1}$$

and

$$\begin{aligned} h_{\widetilde{W}}(\mathfrak{p}) &:= h_{\widetilde{W},f^*(H^{(e,d)})}(\mathfrak{p}) = h_{X^{(e,d)},H^{(e,d)}}(f(\mathfrak{p})) \\ &= h_{X^{(e,d)},H^{(e,d)}}(v(\mathfrak{p})) \\ &\left(= c \cdot \sum_{i,j} h(P_{i,j}), \text{ if } v(\mathfrak{p}) = \sum_{i=1}^d \left(\sum_{i=1}^e (P_{i,j})\right)\right), \end{aligned}$$

where i and j run over $\{1, 2, \ldots, d\}$ and $\{1, 2, \ldots, e\}$, respectively, in the summation $\sum_{i,j}$, and $\mathfrak{p} \in \widetilde{W}$. Therefore, if we assume, in addition, that $h \geqslant 0$, then we have

$$h(P) \leqslant \mathrm{O}(h_{\widetilde{W}}(\mathfrak{p})), \tag{2}$$

where $P = P_{1,1}$ and $v(\mathfrak{p}) = \sum_{i=1}^d \left(\sum_{j=1}^e (P_{i,j})\right)$. On the other hand, if we assume, in addition, that $h \geqslant 1$ (by noticing H is an (indeed, very) ample divisor) and that $P_{i,j}$'s $(1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant e)$ are Galois conjugates of $P = P_{1,1}$, then we have

$$h_{X^{(e,d)}}\left(\sum_{i=1}^d (\mathcal{P}_i)\right) \leqslant \mathrm{O}(h(P)), \tag{3}$$

where $h_{X^{(e,d)}} := h_{X^{(e,d)},H^{(e,d)}}$ (which is then $\geqslant 1$). (In particular, to Y of a family $X \to Y$ we will apply what we stated above. And the fact is that (1) and (3) are interesting, in particular, for $e = 1$ in view of our purpose.)

EXAMPLE 2.1.2. We assume X, Y and π are all defined over a number field k and let $t^{(1)} = t, t^{(2)}, \ldots, t^{(d)}$ be the d distinct $\mathrm{Gal}(\bar{k}/k)$-conjugates of $t \in Y(k, = d)$. Let $P \in X_t(k(t))$. (Here X_t denotes the fiber over t under the morphism π.) Then for all $\sigma \in \mathrm{Gal}\,(\bar{k}/k)$, $\pi(P^\sigma) = (\pi(P))^\sigma = t^\sigma$. Thus, by the hypothesis that $P \in X_t(k(t))$, we know P has exactly d distinct Gal $(\bar{k}/k)$-conjugates. So $P \in X(k, = d)$. $P \in X(k, = d)$. (Indeed, conversely, if $P \in X(k, d) \cap X_t$, then $P \in X_t(k(t))$ (and, hence, P also has exact degree d over k) where t is as above. This is immediate from a simple computation of degrees of field extensions. Note π is defined over k.)

EXAMPLE 2.1.3. As above, we assume X, Y and π are all defined over a number field k and let $(t^{(1)} = t, t^{(2)}, \ldots, t^{(d)}$ be the d distinct $\mathrm{Gal}(\bar{k}/k)$-conjugates of) $t \in Y(k, = d)$. For an integer $e \geqslant 1$, let $P^{(1)} = P, P^{(2)}, \ldots, P^{(e)}$ be the e distinct $\mathrm{Gal}(\bar{k}/k(t))$-conjugates of $P \in X_t(k(t), = e)$. Then notice that $\sum_{j=1}^{e}(P^{(j)}) \in X_t^{(e)}(k(t))$.

The morphism β in Example 2.1.1 restricts to $(X_Y^{(e)})^d \to (X_Y^{(e)})^{(d)}$. In particular, for $d = 1$, we can extend it to Y via $\pi_Y^{(e)}$: $X_Y^{(e)} \to Y$, the natural morphism induced by π – in fact, we see that the composition $X_Y^e \to X_Y^{(e)} \xrightarrow{\pi_Y^{(e)}} Y$ is the previous natural morphism. Later in our practical application we will use this extension and the following argument in a more general setting with $d \geqslant 2$, too, i.e., we will also look at the composition $(X_Y^{(e)})^d \to (X_Y^{(e)})^{(d)} \to Y^{(d)}$.

We may assume, by enlarging k if necessary, that both $X_Y^{(e)}$ and $\pi_Y^{(e)}$ are also defined over k. (As a matter of fact, fiber products and symmetric products are already defined over k.) Then we have

$$\pi_Y^{(e)}\left(\sum_{j=1}^{e}(P^{(j)})\right) = t \in Y(k, = d), \tag{4}$$

and, by applying Example 2.1.2 to $\pi_Y^{(e)}$, we then have

$$\sum_{j=1}^{e}(P^{(j)}) \in X^{(e)}(k, = d). \tag{5}$$

2.2. THE VOJTA MAIN CONJECTURE

This will play a crucial role in our work.

CONJECTURE 2.2.1 (The Vojta (Main) Conjecture). *Let X be a nonsingular complete variety with canonical divisor K and let D be a divisor with normal crossings of X, all defined over a number field k. Let S be a finite set of primes of k containing all the infinite ones. Then, if $\epsilon > 0$ and A is a big divisor of X, then there exists a proper Zariski closed subset $Z = Z(X, D, k, S, A, \epsilon)$ of X such that for all $P \in X(k) - Z$,*

$$\sum_{v\in S}\lambda_{v,D}(P)+h_K(P)\leqslant \epsilon h_A(P)+\mathrm{O}(1).$$

(Refer to [18] or [34] for the definition of local heights $\lambda_{v,D}$. And refer to [[34], 3.4.3] for the details of the conjecture. For our purpose we are primarily interested in the case where $D=0$, so the first term will disappear in our applications.)

2.3. PRELIMINARY WORK

In this section we will introduce some preliminary results which will be used later and which may also be interesting in their own right.

PROPOSITION 2.3.1. *Let π: $X\to Y$ be a surjective morphism between projective varieties. Then there exist desingularizations $\widetilde{X}\to X$ and $\widetilde{Y}\to Y$, and an induced morphism $\widetilde{\pi}$: $\widetilde{X}\to\widetilde{Y}$ such that $\widetilde{X}\xrightarrow{\pi}\widetilde{Y}\to Y$ is equal to $\widetilde{X}\to X\xrightarrow{\pi} Y$. Furthermore, when X, Y and π are all assumed to be defined over a number field k, we may assume, by enlarging k if necessary, that all the varieties and the morphisms here are also defined over k.*

Proof. Let $\widetilde{Y}=$ a desingularization of Y which is the blow-up of Y with respect to a coherent sheaf $\mathcal{I}$ of ideals on Y, $X_{\pi^{-1}\mathcal{I}\cdot\mathcal{O}_X}=$ the blow-up of X with respect to the inverse image ideal sheaf $\pi^{-1}\mathcal{I}\cdot\mathcal{O}_X$ on X, and $\widetilde{X}=$ a desingularization of $X_{\pi^{-1}\mathcal{I}\cdot\mathcal{O}_X}$. Then use [[15], II 7.15] and we immediately have the desired commutative diagram (with induced morphisms). And the last statement is trivial. □

Here we want to see an example that should be recalled later – indeed, the inequality of what we are going to prove later is the opposite direction to that of the following example that is immediate from basic properties of heights.

EXAMPLE 2.3.2. Let $\pi: X\to Y$ be a morphism between nonsingular projective varieties. And let h_Y be an arbitrary height on Y and $h\geqslant 1$ a height on X which is associated to an ample divisor. Let D be a divisor of Y such that $h_Y=h_{Y,D}$. Then for all $P\in X$, we have

$$h_Y(\pi(P))=h_{X,\pi^*D}(P)+\mathrm{O}(1)\leqslant \mathrm{O}(h(P))\quad\text{by DHA.}$$

I.e., $\mathrm{O}(h_Y\circ\pi)\leqslant h$ on X. Notice this is true of *all* points of X.

THEOREM 2.3.3. *Let X, Y, and X_0 be projective varieties where X is nonsingular and Y is normal. Let $f: X_0\to X$ be a morphism. Suppose that π: $X\to Y$ be a morphism such that $\pi\circ f$ is a generically finite (but not necessarily dominant) morphism. And, let h and h_Y be heights on X and Y, respectively, with $h_Y\geqslant 1$ associated to an ample divisor of Y. Then, $h\circ f\leqslant \mathrm{O}(h_Y\circ\pi\circ f)$ outside a proper Zariski closed subset of X_0.*

Proof. Consider $\alpha: \widetilde{X}_0\xrightarrow{\mu} X_0\xrightarrow{f} X\xrightarrow{\pi} Y$ where μ is a desingularization of X_0. Choose an ample divisor D of Y to which the given height h_Y is associated. It will pull back to a big divisor of $\widetilde{X}_0$ under α. Then, by [[34], 1.2.9, (h)], $h_{\widetilde{X}_0} \leqslant \mathrm{O}(h_{\widetilde{X}_0,\alpha^* D})$ outside a proper Zariski closed subset Z of $\widetilde{X}_0$ where $h_{\widetilde{X}_0} \geqslant 1$ is a height on $\widetilde{X}_0$ associated to an ample divisor. Now apply Example 2.3.2 to see $h \circ f \circ \mu \leqslant \mathrm{O}(h_{\widetilde{X}_0})$. Combine these two inequalities and notice that μ is a birational morphism to get the desired result outside the proper Zariski closed subset $\mu(Z)$ of X_0.

This is slightly more general than we will use.

2.4. MAIN PROPOSITION

This will play a significant role later.

PROPOSITION 2.4.1. *Let $\pi\colon X \to Y$ be a family of varieties of general type, i.e., both X and Y are nonsingular projective varieties (of arbitrary dimension) and all the nonsingular fibers having the same dimension as that of the generic fiber are projective varieties of general type – the generic fiber is a nonsingular projective variety of general type. And, in addition, assume that X, Y and π are all defined over a number field k.*

Now we choose an arbitrary height h on X and a height h_Y on Y associated to an ample divisor satisfying $h_Y \geqslant 1$. Then, assuming the Vojta conjecture for varieties of dimension $\leqslant \dim X$, we have $h(P) \leqslant \mathrm{O}(h_Y(\pi(P)))$ for all $P \in X(k) - Z$ where Z is a proper Zariski closed subset of X and the implied constant is independent of P.

In fact, we may allow the following: Fibers may be possibly reducible varieties (not necessarily and yet, for convenience, assumed to have pure dimension) and the geometric generic fiber may be possibly reducible and then has only (nonsingular) irreducible components of general type. For almost the same proof will work because the generic fiber of X over Y is still of general type. However, we will talk about the above case only.

Proof. Let η be the generic point of Y, K the canonical divisor (class) of X, K_η the canonical divisor (class) of the generic fiber X_η and A an arbitrary ample divisor (class) of X (with $\mathbb{Q}$-coefficients). We may assume, by enlarging k if necessary, that K is also defined over k. For a ($\mathbb{Q}$-)divisor (class) D of X we denote by $D|_{X_\eta}$ its restriction to the generic fiber X_η.

Then, firstly, notice that $K|_{X_\eta} = K_\eta$. For we observe that

$$\omega_X|_{X_t} = \omega_{X/Y}|_{X_t} = \omega_{X_t} \quad \text{and} \quad \omega_X|_{X_\eta} = \omega_{X/Y}|_{X_\eta} = \omega_{X_\eta} \quad (t \in Y^\circ).$$

Secondly, apply the Kodaira criterion for bigness to the divisor K_η. Hence we see that there exist some (big) positive integer n, some ample divisor (class) $\mathcal{A}$ of X_η and some effective divisor (class) $\mathcal{E}$ of X_η such that $nK_\eta = \mathcal{A} + \mathcal{E}$.

Thirdly, we notice the Vojta conjecture for X

$$h_K \leqslant \tfrac{1}{2}\epsilon h_A + \mathrm{O}(1) \tag{6}$$

in $X(k) - Z_0$ where Z_0 is a proper Zariski closed subset of X. Here we choose h_A to be $\geqslant 1$. And we also choose a rational $\epsilon > 0$ to be so (sufficiently) small

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(relative to $1/n$) that $1/n\ \mathcal{A} - \epsilon A|_{X_\eta}$ is ample. (Both are just for our later use.) Then we observe that

$$\begin{aligned}(K - \epsilon A)|_{X_\eta} &= K|_{X_\eta} - \epsilon A|_{X_\eta} = K_\eta - \epsilon A|_{X_\eta}\\ &= \left(\frac{1}{n}\mathcal{A} + \frac{1}{n}\mathcal{E}\right) - \epsilon A|_{X_\eta} = \left(\frac{1}{n}\mathcal{A} - \epsilon A|_{X_\eta}\right) + \frac{1}{n}\mathcal{E}.\end{aligned}$$

We know, from our choice of ϵ above, that for some big positive integer m, we have $m \cdot ((1/n)\mathcal{A} - \epsilon A|_{X_\eta} + (1/n)\mathcal{E})$ is linearly equivalent to some effective divisor (class) of X_η. Hence (by taking the Zariski closure in X) we see, for some fibral ($\mathbb{Q}$-)divisor F of X, that $K - \epsilon A + F$ is linearly equivalent to $1/m \cdot$ (some effective ($\mathbb{Q}$-)divisor E of X) in X. (Here by 'fibral' we mean 'support not surjecting onto Y by π'.) Then, since F is fibral, write $F \leqslant \pi^* B$ for some effective divisor B of Y. Then we see

$$\begin{aligned}h_F &\leqslant h_{X,\pi^* B} + \mathrm{O}(1) \quad \text{outside Supp } \pi^* B\\ &= h_{Y,B} \circ \pi + \mathrm{O}(1) \leqslant \mathrm{O}(h_Y \circ \pi) + \mathrm{O}(1) \quad \text{by DHA.}\end{aligned}$$

Thus we have

$$\begin{aligned}-\mathrm{O}(1) &\leqslant h_{K-\epsilon A+F} \quad \text{outside Supp } E\\ &= \left(h_K - \frac{1}{2}\epsilon h_A\right) - \frac{1}{2}\epsilon h_A + h_F + \mathrm{O}(1)\\ &\leqslant \mathrm{O}(1) - \frac{1}{2}\epsilon h_A + \mathrm{O}(h_Y(t)) + \mathrm{O}(1) \quad \text{with the aid of (6)}\end{aligned}$$

in $X_t(k) - Z_1$ where $Z_1 = Z_0 \cup \text{Supp } \pi^* B$. By adjusting the implied constants appearing and putting $Z = Z_1 \cup$ Supp E, we then have $h_A \leqslant \mathrm{O}(h_Y(t))$ in $X_t(k) - Z$ for all $t \in Y(k)$. Therefore we immediately get the desired result by DHA, since A is an ample divisor (class) and $h_A \geqslant 1$ by our hypothesis. □

3. Proof of Theorem 1.0.1

3.1. LEMMA FOR THEOREM 1.0.1

LEMMA 3.1.1. *Fix integers d and $e \geqslant 1$. Let $X \to Y$ be a surjective morphism between projective varieties such that its generic fiber is a nonsingular projective curve of genus $\geqslant 2$. Assume that all the one-dimensional nonsingular fibers have only finitely many algebraic points of degree bounded by e over an arbitrary number field. Let f: $\mathrm{Sym}_d\, X_Y^{(e)} = (X_Y^{(e)})^{(d)} \to Y^{(d)}$ be the natural morphism induced by $\pi_Y^{(e)}$: $X_Y^{(e)} \to Y$. Let $t_1, \ldots, t_d$ be d distinct points of Y° and $\mathcal{T} = \sum_{i=1}^{d}(t_i) \in Y^{(d)}$. Then every positive-dimensional closed subvariety of the fiber over $\mathcal{T}$ under the morphism f is of general type.*

Proof. Observe that the fiber over $\mathcal{T}$ under f is

$$\left\{\sum_{i=1}^{d}\left(\sum_{j=1}^{e}(P_{i,j})\right) \in \mathrm{Sym}_d\, X_Y^{(e)}\colon P_{i,j} \in X_{t_i},\ \text{for } 1 \leqslant i \leqslant d \text{ and } 1 \leqslant j \leqslant e\right\}.$$

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So we may as well be tempted to write formally $\sum_{i=1}^{d} X_{t_i}^{(e)}$ for it. We have an obvious isomorphism $\prod_{i=1}^{d} X_{t_i}^{(e)} \xrightarrow{\cong} \sum_{i=1}^{d} X_{t_i}^{(e)}$. It suffices to show that every positive-dimensional closed subvariety of $\prod_{i=1}^{d} X_{t_i}^{(e)}$ is of general type. Notice that $X_{t_i}^{(e)} \subset \mathrm{Jac}(X_{t_i})$ for $i = 1, \ldots, d$, since the X_{t_i}'s have no pencils of degree $\leqslant e$. (See also the paragraph just below Theorem 1.0.1 of the introduction.) We then have $\prod_{i=1}^{d} X_{t_i}^{(e)} \subset \prod_{i=1}^{d} \mathrm{Jac}(X_{t_i})$, i.e., $\prod_{i=1}^{d} X_{t_i}^{(e)}$ is a subvariety of an abelian variety $\prod_{i=1}^{d} \mathrm{Jac}(X_{t_i})$. And $\prod_{i=1}^{d} X_{t_i}^{(e)}$ has only finitely many rational points over an arbitrary number field. (Recall Examples 2.1.2 and 2.1.3 and note X_{t_i}'s have, by hypothesis, only finitely many points of degree bounded by e over an arbitrary number field.) Hence, it cannot contain a translation of a positive-dimensional Abelian subvariety of $\prod_{i=1}^{d} \mathrm{Jac}(X_{t_i})$. Notice that a closed subvariety of an abelian variety which contains no abelian subvariety is of general type. So the desired result follows. □

For a better understanding of the nature of 'conjugate fibers' let us mention another fact, though it is not directly related to the main work.

Fact. Let $X \to Y$ be a morphism between projective varieties. Assume X, Y and the morphism are all defined over a number field k. And let $\sigma \in \mathrm{Gal}\,(\bar{k}/k)$ and $t \in Y^{\circ}$. Then, also $t^{\sigma} \in Y^{\circ}$. (We can actually say something more: Their (geometric) fibers (called conjugate) are *isomorphic*, hence, in particular, also $t^{\sigma} \in Y^{\circ}$. However, it is important to notice that this *isomorphism* is an isomorphism of abstract schemes, but *not* of schemes over $\bar{k}$ or of varieties over $\bar{k}$.)

EXAMPLE 3.1.2 (due to Silverman). To make the above become clearer, let us consider the elliptic surface $\mathcal{E}_T$: $y^2 = x^3 + Tx + 1$. We view it as a surface fibered over $\mathbb{P}^1$. The fibers over the points $\pm\sqrt{2}$ are nonsingular, since their discriminants are nonzero. But, they have distinct j-invariants, hence they are not isomorphic over $\bar{\mathbb{Q}}$. I.e., The two (geometric) fibers $\mathcal{E}_{\pm\sqrt{2}}$ over $\pm\sqrt{2}$ are *isomorphic as abstract schemes*, but *not as varieties*.

Anyway, for example, assume, in addition, that X, Y and the morphism $X \to Y$ are all defined over a number field k in Lemma 3.1.1. And let t be a point of Y° which has exact degree $d \geqslant 1$ over k (i.e., $t \in Y^{\circ}(k, = d)$) and let $t^{(1)} = t, t^{(2)}, \ldots, t^{(d)}$ be the d distinct $\mathrm{Gal}\,(\bar{k}/k)$-conjugates of t, which are supposed to belong to Y° by the above fact. So every positive-dimensional closed subvariety of the fiber of the induced morphism $(X_Y^{(e)})^{(d)} \to Y^{(d)}$ over the point $\sum_{i=1}^{d}(t^{(i)}) \in Y^{(d)}$ is of general type.

In what follows, by Y° we always mean the set of t's $\in Y(\bar{k})$ such that the fiber X_t over t under the morphism $X \to Y$ is nonsingular and has the same dimension as that of the generic fiber of $X \to Y$, unless otherwise stated. In other words, it will be the case, even when we may have other various kinds of fibrations over Y, unless otherwise stated.

3.2. PROOF OF THEOREM 1.0.1: CASE $d = 1$

We separate the two cases Case 1: $d = 1$ and Case 2: $d \geqslant 2$. We will use induction on d.

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By applying DHA to h, we may assume, whenever necessary in steps of our proof, (mainly and enough) that the height h on X is associated to an ample divisor of X with $h \geqslant 0$ or 1. (Of course, we may start by assuming that $h \geqslant 1$ is associated to an ample divisor of X. However, to see exact requirement to apply and to understand previous results better, we will indicate $h \geqslant 0$ or $h \geqslant 1$ in specific applications.)

Proof of Theorem 1.0.1: *Case* $d = 1$. Write

$$X_t(k, e) = \bigcup_{1 \leqslant j \leqslant e} X_t(k, = j)$$

for $t \in Y^\circ(k)$ where the union is a disjoint one. We then have only to deal with $X_t(k, = j)$ separately for $1 \leqslant j \leqslant e$. So we will do only for $j = e$.

Let $t \in Y^\circ(k)$. And, let $P \in X_t(k, = e)$ and let $P^{(j)} \in X_t(k, = e)$ be its e distinct $\mathrm{Gal}(\bar{k}/k)$-conjugates for $1 \leqslant j \leqslant e$ with $P^{(1)} = P$. And we introduce the notation

$$\mathcal{P} := \sum_{j=1}^{e} (P^{(j)}) \in X_t^{(e)}(k) \subset X^{(e)}(k)$$

for convenience. Recall that we have the natural surjective finite morphism $\beta\colon X^e \to X^{(e)}$ and let $W = \beta(X_Y^e) = X_Y^{(e)}$.Then we have the induced fibration $\pi_Y^{(e)}$: $W \to Y$. (See Example 2.1.3.) Lemma 3.1.1 (with $d = 1$) says that W (via $\pi_Y^{(e)}$) is a family of varieties of general type (of dimension e). Let $\widetilde{W}$ be a desingularization of W, which will then be a nonsingular projective $(e + \dim Y)$-fold and family of varieties of general type. We may assume (by enlarging k, if necessary) that W, $\widetilde{W}$ and the morphism $\pi_0\colon \widetilde{W} \to Y$ (= the composition of the natural morphism $\mu\colon \widetilde{W} \to W$ and $\pi_Y^{(e)}$) are all defined over k, too.

Now apply Proposition 2.4.1 to $\pi_0\colon \widetilde{W} \to Y$ and we get: With any choice of a height $h_{\widetilde{W}}$ on $\widetilde{W}$ (which will actually be chosen below),

$$h_{\widetilde{W}} \leqslant \mathrm{O}\big(h_Y(t)\big) \tag{1}$$

on $\widetilde{W}_t(k) - Z_0$ for all $t \in Y(k)$ where Z_0 is a proper Zariski closed subset of $\widetilde{W}$. We may assume, by enlarging k if necessary, that both the isomorphism and its inverse

$$\widetilde{W} - \mu^{-1}(W_0) \cong W - W_0 \tag{2}$$

are also defined over k where W_0 is a proper Zariski closed subset of W. We may assume that $h \geqslant 0$ is associated to a hyperplane section of X. Then choose $h_{X^{(e)}}$ to be the one introduced in Example 2.1.1 and $h_{\widetilde{W}}$ the height that is associated to the pull-back under the composition $\widetilde{W} \xrightarrow{\mu} W \hookrightarrow X^{(e)}$ of the divisor of $X^{(e)}$ defining the height $h_{X^{(e)}}$ on $X^{(e)}$ – we call it the height defined (from $h_{X^{(e)}}$) by the pull-back under the morphism or the pull-back of h under the morphism, for brevity.

With the aid of (2), let $\mathfrak{p} \in \widetilde{W}(k)$ with $\mu(\mathfrak{p}) = \mathcal{P} \in W(k) \subset X^{(e)}(k)$. Then we have (recalling Example 2.1.1): $h(P) \leqslant \mathrm{O}(h_{\widetilde{W}}(\mathfrak{p})) \leqslant \mathrm{O}(h_Y(t))$, i.e.,

$$h(P) \leqslant \mathrm{O}(h_Y(t))$$

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where $\mathcal{P} \in W(k) - W_0 \cup \mu(Z_0)$. Note that $Z := W_0 \cup \mu(Z_0)$ is a proper Zariski closed subset of W, i.e., we have

$$h(P) \leqslant \mathrm{O}\big(h_Y(t)\big), \tag{3}$$

whenever $P \in X_t(k, = e)$, $t \in Y(k)$ satisfies $\mathcal{P} \in W(k) - Z$.

So we are reduced to dealing with $Z \subset W = X_Y^{(e)} \subset X^{(e)}$ which has lower dimension than W. (Some of the notation above like $\mathfrak{p}$ will be used here, but they are not supposed to have the same meaning as above and there should be no confusion.)

Suppose that Z may have several irreducible components, and pick any one of them, say B. Then the morphism $\pi_Y^{(e)}$: $W \to Y$ restricts to $B \to Y_0 := \pi_Y^{(e)}(B)$. Here B or Y_0 may be singular, in which case we apply Proposition 2.3.1 (together with its last statement) to see that $\widetilde{B} \xrightarrow{\pi_1} \widetilde{Y}_0 \xrightarrow{\sigma} Y_0$ is equal to $\widetilde{B} \xrightarrow{\rho} B \to Y_0$ where $\widetilde{B}$ and $\widetilde{Y}_0$ are desingularizations of B and Y_0, respectively and all the morphisms are obviously (and appropriately) induced ones, respectively.

Then Lemma 3.1.1 shows π_1: $\widetilde{B} \to \widetilde{Y}_0$ must be a family of varieties of general type whose generic fiber is a nonsingular projective variety of general type. And, as before, we assume that both the isomorphism and its inverse

$$\widetilde{B} - \rho^{-1}(B_0) \cong B - B_0 \tag{4}$$

are also defined over k where B_0 is a proper Zariski closed subset of B.

Keep in mind that we want to take care of only the points $P \in X_t(k, = e)$, $t \in Y^{\circ}(k)$ with $\mathcal{P} \in B$ (indeed, $\mathcal{P} \in B(k)$ and $t \in Y_0(k)$), i.e., to show that $h(P) \leqslant \mathrm{O}(h_Y(t))$ for those points. Now apply Proposition 2.4.1 to π_1: $\widetilde{B} \to \widetilde{Y}_0$ and we get: With any choice of both the heights $h_{\widetilde{B}}$ on $\widetilde{B}$ and $h_{\widetilde{Y}_0} \geqslant 1$ on $\widetilde{Y}_0$ associated to an ample divisor of $\widetilde{Y}_0$,

$$h_{\widetilde{B}} \leqslant \mathrm{O}(h_{\widetilde{Y}_0} \circ \pi_1) \tag{5}$$

on $\widetilde{B}(k) - Z_1$ where Z_1 is a proper Zariski closed subset of $\widetilde{B}$. (We will choose $h_{\widetilde{B}}$ appropriately later, while we will keep $h_{\widetilde{Y}_0}$ here without any additional restriction.)

With the aid of (4), let $\mathfrak{p} \in \widetilde{B}(k)$ with $\rho(\mathfrak{p}) = \mathcal{P} \in B(k) \subset X^{(e)}(k)$. Under the assumption that $\mathcal{P} \notin B_0$. Consider the composition morphism f: $\widetilde{B} \xrightarrow{\mu} B \hookrightarrow W \hookrightarrow X^{(e)}$. And, in particular, take $h_{\widetilde{B}}$ to be the height on $\widetilde{B}$ that is the pull-back of $h_{X^{(e)}}$ (introduced before with the hypothesis that $h \geqslant 0$) under the morphism f. Then we have (recalling Example 2.1.1):

$$\begin{aligned} h(P) &\leqslant \mathrm{O}(h_{\widetilde{B}}(\mathfrak{p})) \leqslant \mathrm{O}(h_{\widetilde{Y}_0} \circ \pi_1(\mathfrak{p})) \\ &\leqslant \mathrm{O}(h_Y \circ g_0(\pi_1(\mathfrak{p}))) \quad \text{by Theorem 2.3.3 with } g_0\text{: } \widetilde{Y}_0 \xrightarrow{\sigma} Y_0 \hookrightarrow Y \\ &= \mathrm{O}(h_Y(t)) \\ \text{i.e., } h(P) &\leqslant \mathrm{O}(h_Y(t)), \end{aligned} \tag{6}$$

where

$$\mathcal{P} \in B(k) - B_0 \cup \rho\big(Z_1 \cup \pi_1^{-1}(E_1)\big)$$

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and E_1 is a proper Zariski closed subset of $\widetilde{Y}_0$ appearing because of (6). (For (6) we may also be able to use Proposition 2.4.1 with a little more work.) Note that $Z_2 := B_0 \cup \rho\big(Z_1 \cup \pi_1^{-1}(E_1)\big)$ is a proper Zariski closed subset of B. So we reduce the problem of dealing with B to that of doing the proper Zariski closed subset Z_2 of B (which has lower dimension than B).

Repeat this process as often as needed. Finally, take the maximum of all the appearing implied constants and we get the desired result for Case $d = 1$. □

3.3. PROOF OF THEOREM 1.0.1: CASE $d \geqslant 2$

Inevitably, many arguments will be very similar to those in the proof for Case 1. And, for convenience, we will use many of the same letters as there for their (though not always) corresponding meanings. So let us take care not to be disturbed by the old notation too much.

Proof of Theorem 1.0.1: *Case* $d \geqslant 2$. We assume, by induction, that the desired result holds up to $d-1$. Write

$$Y^\circ(k, d) = \bigcup_{1 \leqslant i \leqslant d} Y^\circ(k, = i) \quad \text{and} \quad X_t(k(t), e) = \bigcup_{1 \leqslant j \leqslant e} X_t(k(t), = j)$$

for $t \in Y^\circ(k, d)$ where the unions are disjoint ones. We have only to deal with $Y^\circ(k, = i)$ (resp. $X_t(k(t), = j)$) separately for $1 \leqslant i \leqslant d$ (resp. $1 \leqslant j \leqslant e$). So we will do only for $i = d$ (resp. $j = e$).

First, we fix some more notation that will be used through this section. Let

$$t \in Y^\circ(k, = d), \quad \text{and let} \quad t^{(i)} \in Y^\circ(k, = d)$$

be its d distinct $\mathrm{Gal}(\bar{k}/k)$-conjugates for $1 \leqslant i \leqslant d$ with $t^{(1)} = t$. So $\sum_{i=1}^d (t^{(i)}) \in Y^{(d)}$ (indeed, $\sum_{i=1}^d (t^{(i)}) \in Y^{(d)}(k)$). And, let

$$P \in X_t(k(t), = e) \quad \text{and let} \quad P^{(j)} \in X_t(k(t), = e)$$

be its e distinct $\mathrm{Gal}(\bar{k}/k(t))$-conjugates for $1 \leqslant j \leqslant e$ with $P^{(1)} = P$. Then recall that $\sum_{j=1}^e (P^{(j)}) \in X_t^{(e)}(k(t))$ and also that $\sum_{j=1}^e (P^{(j)}) \in X^{(e)}(k, = d)$. (Cf. Example 2.1.3.) Let

$$\mathcal{P}^{(i)} := \sum_{j=1}^{e} (P^{(i,j)}) \in X_{t^{(i)}}^{(e)}(k(t^{(i)}))$$

(indeed, then also $\mathcal{P}^{(i)} \in X^{(e)}(k, = d)$), $1 \leqslant i \leqslant d$ with $P^{(i,j)} \in X_{t^{(i)}}\big(k(t^{(i)}), = e\big)$ (independently of j) be the d distinct $\mathrm{Gal}(\bar{k}/k)$-conjugates of $\sum_{j=1}^e (P^{(j)})$ with $P^{(1,1)} = P$. (cf. Example 2.1.3 again.) And finally, we introduce the notation

$$\mathcal{P} := \sum_{i=1}^{d} (\mathcal{P}^{(i)}) = \sum_{i=1}^{d} \left(\sum_{j=1}^{e} (P^{(i,j)}) \right) \in X^{(e,d)}(k),$$

which will be a point of the fiber of λ over the point $\sum_{i=1}^d (t^{(i)}) \in Y^{(d)}(k)$ according to the notation below.

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Recall we have the surjective finite morphism

$$\lambda\colon W := (X_Y^{(e)})^{(d)} \longrightarrow Y^{(d)}, \quad \left(\sum_{j=1}^{e}(P_{1,j})\right) + \cdots + \left(\sum_{j=1}^{e}(P_{d,j})\right) \mapsto \sum_{i=1}^{d}(t_i)$$

where $P_{i,j} \in X_{t_i}$, i.e., $t_i = \pi(P_{i,j})$ depending only on $i) \in Y$, $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant e$. (cf. Example 2.1.3.) Note that $W \subset X^{(e,d)}$. Apply Proposition 2.3.1 (together with its last statement) to λ so that we see $\widetilde{W} \xrightarrow{\pi_0} \widetilde{Y^{(d)}} \xrightarrow{\nu} Y^{(d)}$ is equal to $\widetilde{W} \xrightarrow{\mu} W \xrightarrow{\lambda} Y^{(d)}$ where $\widetilde{W}$ and $\widetilde{Y^{(d)}}$ are desingularizations of W and $Y^{(d)}$, respectively and all the morphisms are obviously (and appropriately) induced ones, respectively. Then Lemma 3.1.1 implies that $\widetilde{W}$ will be a nonsingular projective variety of dimension $d \cdot (e + \dim\, Y)$ and (via π_0) family of varieties of general type.

Then apply Proposition 2.4.1 to π_0: $\widetilde{W} \rightarrow \widetilde{Y^{(d)}}$ and we get: With any choice of both the heights $h_{\widetilde{W}}$ on $\widetilde{W}$ (which will actually be chosen below) and $h_{\widetilde{Y^{(d)}}} \geqslant 1$ on $\widetilde{Y^{(d)}}$ which is associated to an ample divisor of $\widetilde{Y^{(d)}}$,

$$h_{\widetilde{W}} \leqslant \mathrm{O}(h_{\widetilde{Y^{(d)}}} \circ \pi_0) \tag{7}$$

on $\widetilde{W}(k) - Z_0$ where Z_0 is a proper Zariski closed subset of $\widetilde{W}$. We assume that both the isomorphism and its inverse

$$\widetilde{W} - \mu^{-1}(W_0) \cong W - W_0 \tag{8}$$

are defined over k where W_0 is a proper Zariski closed subset of W.

With the aid of (8), let $\mathfrak{p} \in \widetilde{W}(k)$ with $\mu(\mathfrak{p}) = \mathcal{P} \in W(k)$ under the assumption that $\mathcal{P} \notin W_0$. (Recall that $W \subset X^{(e,d)}$.) We may assume that $h \geqslant 0$ is associated to a hyperplane section of X. Then choose $h_{\widetilde{W}}$ to be the one introduced in Example 2.1.1. And, choose $h_{Y^{(d)}}$ to be the one introduced in Example 2.1.1 (using the ample divisor defining $h_Y \geqslant 1$). So notice, in particular, that $h_{Y^{(d)}}$ is also $\geqslant 1$ and associated to an ample divisor. Then observe (by recalling Example 2.1.1):

$$\begin{aligned} h(P) \leqslant \mathrm{O}(h_{\widetilde{W}}(\mathfrak{p})) \quad &\leqslant \quad \mathrm{O}(h_{\widetilde{Y^{(d)}}}(\pi_0(\mathfrak{p}))) \\ &\leqslant \mathrm{O}(h_{Y^{(d)}} \circ \nu(\pi_0(\mathfrak{p}))) \quad \text{by Theorem 2.3.3 with } \nu, \\ &= \mathrm{O}(h_{Y^{(d)}} \circ \lambda(\mu(\mathfrak{p}))) = \mathrm{O}\left(h_{Y^{(d)}}\left(\sum_{i=1}^{d}(t^{(i)})\right)\right) \leqslant \mathrm{O}(h_Y(t)) \end{aligned} \tag{9}$$

where

$$\mu(\mathfrak{p}) \in W(k) - W_0 \cup \mu(Z_0) \cup \mu(\pi_0^{-1}(E_0))$$

and E_0 is a proper Zariski closed subset of $\widetilde{Y^{(d)}}$ appearing because of (9). I.e., We have $h(P) \leqslant \mathrm{O}(h_Y(t))$, whenever $P \in X_t(k(t), = e)$, $t \in Y(k, = d)$ satisfies that $\mathcal{P} \in W(k) - Z$, where Z: $W_0 \cup \mu\big(Z_0 \cup \pi_0^{-1}(E_0)\big)$ is a proper Zariski closed subset of W.

Therefore we are reduced to dealing with $Z \subset W$ which has lower dimension than W. (As before, some of the notation above like $\mathfrak{p}$ will be used here, but they are not supposed to have the same meaning as above and there should be no confusion.)

As before, suppose that Z may have several irreducible components, and pick any one of them, say B. Then the morphism λ: $W \to Y^{(d)}$ restricts to $B \to Y_0 := \lambda(B)$. Here B or Y_0 may be singular, in which case we apply Proposition 2.3.1 (together with its last statement) to see that $\widetilde{B} \xrightarrow{\pi_1} \widetilde{Y}_0 \xrightarrow{\sigma} Y_0$ is equal to $\widetilde{B} \xrightarrow{\rho} B \to Y_0$ where $\widetilde{B}$ and $\widetilde{Y}_0$ are desingularizations of B and Y_0, respectively and all the morphisms are obviously (and appropriately) induced ones, respectively.

Then Lemma 3.1.1 shows that π_1: $\widetilde{B} \to \widetilde{Y}_0$ must be a family of varieties of general type. We assume that both the isomorphism and its inverse

$$\widetilde{B} - \rho^{-1}(B_0) \cong B - B_0 \tag{10}$$

are also defined over k where B_0 is a proper Zariski closed subset of B.

Notice we want to deal with only the points $P \in X_t(k(t), = e)$, $t \in Y^{\circ}(k, = d)$ with $\mathcal{P} \in B$ (indeed, $\mathcal{P} \in B(k)$ and $\sum_{i=1}^{d}(t^{(i)}) \in Y_0(k)$), i.e., to show that $h(P) \leqslant \mathrm{O}(h_Y(t))$ for those points. So apply Proposition 2.4.1 to π_1: $\widetilde{B} \to \widetilde{Y}_0$ to get: With any choice of both the heights $h_{\widetilde{B}}$ on $\widetilde{B}$ and $h_{\widetilde{Y}_0} \geqslant 1$ on $\widetilde{Y}_0$ associated to an ample divisor of $\widetilde{Y}_0$,

$$h_{\widetilde{B}} \leqslant \mathrm{O}(h_{\widetilde{Y}_0} \circ \pi_1) \tag{11}$$

on $\widetilde{B}(k) - Z_1$ where Z_1 is a proper Zariski closed subset of $\widetilde{B}$. (We will choose $h_{\widetilde{B}}$ appropriately for our purpose in what follows, while we keep the same $h_{\widetilde{Y}_0}$ as here without any additional restriction.) Note (11) is *formally* the same as (5) despite the different meanings of the symbols. And, for convenience, we repeat it here.

With the aid of (10), let $\mathfrak{p} \in \widetilde{B}(k)$ with $\rho(\mathfrak{p}) = \mathcal{P} \in B(k) \subset X^{(e,d)}(k)$ under the assumption that $\mathcal{P} \notin B_0$. As above, we assume that $h \geqslant 0$ is associated to a hyperplane section of X. Then choose $h_{\widetilde{B}}$ to be the one introduced in Example 2.1.1 (via the composition morphism $\widetilde{B} \xrightarrow{\rho} B \hookrightarrow X^{(e,d)}$). And, choose $h_{Y^{(d)}}$ to be exactly the same as just above. (Again note then, of course, that $h_{Y^{(d)}} \geqslant 1$ is associated to an ample divisor.) As before, we then have (recalling Example 2.1.1):

$$\begin{aligned} h(P) &\leqslant \mathrm{O}(h_{\widetilde{B}}(\mathfrak{p})) \leqslant \mathrm{O}(h_{\widetilde{Y}_0} \circ \pi_1(\mathfrak{p})) \\ &\leqslant \mathrm{O}(h_{Y^{(d)}} \circ g_0(\pi_1(\mathfrak{p}))) \text{ by Theorem 2.3.3 with } g_0\colon \widetilde{Y}_0 \xrightarrow{\sigma} Y_0 \hookrightarrow Y^{(d)}, \quad (12) \\ &= \mathrm{O}(h_{Y^{(d)}} \circ \sigma(\pi_1(\mathfrak{p}))) = \mathrm{O}(h_{Y^{(d)}} \circ \lambda(\rho(\mathfrak{p}))) \leqslant \mathrm{O}(h_Y(t)) \end{aligned}$$

i.e., $h(P) \leqslant \mathrm{O}(h_Y(t))$

where $\rho(\mathfrak{p}) \in B(k) - B_0 \cup \rho\big(Z_1 \cup \pi_1^{-1}(E_1)\big)$ and E_1 is a proper Zariski closed subset of $\widetilde{Y}_0$ appearing because of (12). I.e., We have

$$h(P) \leqslant \mathrm{O}(h_Y(t)), \tag{13}$$

whenever $P \in X_t(k(t), = e)$, $t \in Y(k, = d)$ satisfies that $\mathcal{P} \in B(k) - Z_2$, (hence, indeed, $\sum_{i=1}^{d}(t^{(i)}) \in Y_0(k)$) where $Z_2 := B_0 \cup \rho\big(Z_1 \cup \pi_1^{-1}(E_1)\big)$ which is a proper Zariski closed subset of B.

Thus we reduce the problem of dealing with $B \subset W\,(\subset X^{(e,d)})$ to that of doing the proper Zariski closed subset Z_2 of B of which every irreducible component has lower dimension than B.

Repeat this process as often as needed and take the maximum of all the appearing implied constants to get the desired result for Case $d \geqslant 2$ and Theorem 1.0.1. $\square$

Remark 3.3.1 (On Theorem 1.0.1). Notice the exceptional subset of X consists of the singular fibers as well as the fibers of dimension $\geqslant 2$. But we can actually remove the singular one-dimensional fibers having geometric genus $\geqslant 2$ from the exceptional subset, which is obvious from the proof above, i.e., we may include those fibers in Y°.

4. Miscellanea

4.1. SLIGHTLY STRENGTHENED VERSION OF THEOREM 1.0

We combine Example 2.3.2 and Theorem 1.0.1 to get

THEOREM 4.1.1. *Keep the hypotheses of Theorem* 1.0.1 *and let* $h \geqslant 1$ (*resp.* $h_Y \geqslant 1$) *be a height on* X (*resp.* Y) *associated to an ample divisor of* X (*resp.* Y). *Then, assuming the Vojta conjecture for varieties of dimension* $\leqslant d\cdot(e+\dim Y)$, *we have:*

$$h(P) \gg\ll h_Y(\pi(P)),$$

whenever $P \in X_{\pi(P)}(k(\pi(P)), e)$, *i.e., P is an algebraic point of degree* $\leqslant e$ *over* $k(\pi(P))$, *and* $\pi(P) \in Y^{\circ}(k, d)$, *where the implied constants are independent of* P.

4.2. AN EXAMPLE

Let us see an example of the use of Theorem 1.0.1.

EXAMPLE 4.2.1. For an integer $n \geqslant 4$, consider the family of equations $x^n + y^n = tz^n$, where $x, y, z \in \mathbb{Z}$ have no common factor > 1 and $t \in \mathbb{Z}$, too. Homogenize it with respect to t so that this defines an algebraic surface $X \subset \mathbb{P}^2 \times \mathbb{P}^1$ and we have the natural fibration $X \to \mathbb{P}^1$. We desingularize it to get a new surface $\widetilde{X}$ together with an induced morphism $\mu\colon \widetilde{X} \to X$. And let $f\colon \widetilde{X} \to \mathbb{P}^2 \times \mathbb{P}^1$ be the composition of μ and the inclusion $X \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1$. Let

$$h_{\mathbb{P}^1}([t_1, t_2]) = \log(\max\{|t_1|,\ |t_2|\} + 1),$$

where $t_1, t_2 \in \mathbb{Z}$ are relatively prime. And let $h_{\widetilde{X}}$ be the height on $\widetilde{X}$ which is defined by the pull-back of the height $h_{\mathbb{P}^2\times\mathbb{P}^1}$ (on $\mathbb{P}^2 \times \mathbb{P}^1$) under the morphism f where

$$h_{\mathbb{P}^2\times\mathbb{P}^1}([x, y, z], [t_1, t_2]) := \log(\max\{|x|, |y|, |z|\}) + \log(\max\{|t_1|,\ |t_2|\}),$$

where x, y, z and t_1, t_2 are as above.

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Now let $\mathfrak{p} \in \widetilde{X}$ map onto the point $([x, y, z], t)$ via μ. Since we are concerned with only nonsingular fibers of X, we know that $t \neq 0$ (and $t \neq \infty$). By Theorem 1.0.1,

$$\begin{aligned}
&\log(\max\{|x|, |y|, |z|\}) \\
&\quad \leqslant \log(\max\{|x|, |y|, |z|\}) + \log|t| = h_{\mathbb{P}^2 \times \mathbb{P}^1}([x, y, z], [t, 1]) \\
&\quad = h_{\widetilde{X}}(\mathfrak{p}) \leqslant \mathrm{O}\big(h_{\mathbb{P}^1}([t, 1])\big) = \mathrm{O}(\log(|t| + 1)),
\end{aligned}$$

for all $t \in \mathbb{Z}$, $t \neq 0$. It is then immediate to have:

$$\max\{|x|, |y|, |z|\} \leqslant (|t| + 1)^c$$

for all $t \in \mathbb{Z}$, $t \neq 0$ where c is a positive constant (independent of x, y, z and t).

4.3. AN APPLICATION

EXAMPLE 4.3.1 (Due to Silverman). Let $\mathcal{E} \to C$ be an elliptic surface over a curve C. We assume that it is defined over a number field k. Furthermore, let us assume that there are infinitely many t's $\in C^{\circ}(k)$ such that rank $\mathcal{E}_t(k) \geqslant 1$. We can thus choose a sequence of points $P_t \in \mathcal{E}_t(k)$ $(t \in C^{\circ}(k))$ such that $h_{\mathcal{E}}(P_t) \geqslant \exp h_C(t)$ for all such t's $\in C^{\circ}(k)$ where $h_{\mathcal{E}}$ and h_C are heights on $\mathcal{E}$ and C, respectively, which are associated to ample divisors of $\mathcal{E}$ and C, respectively. Then let $h_C(t) \to \infty$.N.B. The notation P does not stand for a section here. Then it is immediate that

$$h_{\mathcal{E}}(P_t) \not\leqslant \mathrm{O}(h_C(t)).$$

This will also account for our choice of the term *height uniformity* introduced in the title of the paper and the introduction. And the fact is that, conversely, if rank $\mathcal{E}_t(k) = 0$ for all $t \in C^{\circ}(k)$, then the desired height uniformity turns out to be true. Let us talk about it.

The following basically comes from results of [6]. From now on we keep the notation of [6]: $X \to Y$ is a family of varieties, $\varphi\colon X \to X$ is a Y-morphism, and all of them are assumed to be defined over a number field. Let $\eta \in \mathrm{Pic}(X) \otimes \mathbb{R}$ such that $\varphi^*\eta = \alpha\eta$ for some $\alpha > 1$. We use the subscript t to denote the restrictions of the corresponding objects to the fiber over $t \in Y^{\circ}$ and let $\hat{h}_{X_t, \eta_t, \varphi_t}$ be the canonical height on X_t associated to the divisor (class) η_t with respect to the morphism φ_t. We then have ([[6], Theorem 3.1]): There exist two positive constants c_1 and c_2 such that

$$\left|\hat{h}_{X_t, \eta_t, \varphi_t} - h_{X, \eta}\right| \leqslant c_1 h_Y(t) + c_2$$

on X_t for all $t \in Y^{\circ}$. Furthermore, we assume that $h_Y \geqslant 1$. (Note h_Y was already assumed to be associated to an ample divisor there.) In particular, we then see:

$$\left|\hat{h}_{X_t, \eta_t, \varphi_t} - h_{X, \eta}\right| \ll h_Y(t) \tag{1}$$

on X_t for all $t \in Y^{\circ}$. For $t \in Y^{\circ}$, we introduce the notation

$$\begin{aligned}
&\mathrm{Preper}\ \varphi_t \\
&\quad := \{P \in X_t \mid \text{Only finitely many of } \varphi_t^n(P) \text{ are distinct for integers } n \geqslant 0\}.
\end{aligned}$$

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Its points are called preperiodic (with respect to φ_t). Then we know that $\hat{h}_{X_t,\eta_t,\varphi_t}$ vanishes on Preper φ_t. Thus, assuming η is an ample divisor class of X with $h_{X,\eta} \geqslant 1$ and applying DHA, we immediately get (from (1)): With any choice of the height h on X, $h \leqslant \mathrm{O}(h_Y(t))$ on Preper φ_t for all $t \in Y^\circ$. This case includes as a special case the families of abelian varieties and of $K3$ surfaces studied by Silverman ([32]) and others.

EXAMPLE 4.3.2. Now let us take a little different point of view. Under the same hypotheses of Theorem 1.0.1 (for convenience, with $d = e = 1$), let J/Y be the associated family of Jacobian varieties. So we see that π: $X \to Y$ is the composition $X \xrightarrow{\alpha} J \to Y$ and let P be a section of J/Y. (For simplicity, we assume, by enlarging k if necessary, that all the varieties and the morphisms here and the section P are defined over k.) Let n be an integer. And let h_J be a height on J which is associated to an ample divisor η of J, and $\hat{h}_t := \hat{h}_{J_t,\eta_t,\varphi_t}$, e.g., we may put φ to be the multiplication by 2. (See [6].)

Then apply Theorem 1.0.1 to π: $X \to Y$ with the height $h_J \circ \alpha$ on X. Then (1) for J (under the same hypothesis on h_Y as there) implies that $\hat{h}_t \leqslant \mathrm{O}(h_Y(t))$ on $X_t(k)$ $(t \in Y^\circ(k))$. Hence, we have $n^2\hat{h}_t(P_t) = \hat{h}_t(nP_t) \leqslant \mathrm{O}(h_Y(t))$, if $P_t \in X$ $(t \in Y^\circ(k))$. Thus we get

$$|n| \leqslant \mathrm{O}\left(\sqrt{\frac{h_Y(t)}{\hat{h}_t(P_t)}}\right) = \mathrm{O}(1),$$

if P_t is, in addition, a nontorsion point. (The (last) equality is an easy consequence of [6].) Therefore we conclude that there are only finitely many such n's. I.e., There are only finitely many $n \in \mathbb{Z}$ such that nP meets X over some $t \in Y^\circ(k)$ for which P_t is non-torsion. As an immediate consequence, we see: There exists a uniform upper bound for

$$|\mathbb{Z}P \cap X_t(k)|$$

when t runs over $Y^\circ(k)$ for which P_t is nontorsion. (Of course, this is completely superseded by [7], though.)

4.4. WHAT CAN BE SAID MORE

As should be already noticed, essentially the same corresponding proof will give the following.

In this section we agree: Whatever X and Y are, h is an arbitrary height on X and h_Y is a height on Y associated to an ample divisor satisfying $h_Y \geqslant 1$.

LEMMA 4.4.1. *Fix an integer* $d \geqslant 1$. *Let* $X \to Y$ *be a* (*surjective*) *morphism between nonsingular projective varieties such that all the nonsingular fibers are closed subvarieties of Abelian varieties.* (*i.e.* X *is a family of* (*closed*) *subvarieties of Abelian*

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varieties.) *Assume none of the nonsingular fibers contain any translations of nontrivial Abelian subvarieties. Let* $f\colon X^{(d)} \to Y^{(d)}$ *be the natural morphism. Let* $t_1, \ldots, t_d$ *be* d *distinct points of* Y° *and* $\mathcal{T} = \sum_{i=1}^{d}(t_i) \in Y^{(d)}$. *Then every positive-dimensional closed subvariety of the fiber over* $\mathcal{T}$ *under the morphism* f *is of general type.*

Proof. It is immediate from essentially the same proof as that of Lemma 3.1.1 with $e = 1$. □

Now we invoke Faltings' theorem ([12]) on another conjecture of Lang. It says that every nonsingular fiber in the lemma above has only finitely many rational points.

THEOREM 4.4.2. *Let* d *be an integer* $\geqslant 1$. *Let* $\pi\colon X \to Y$ *be a family of* (*closed*) *subvarieties of Abelian varieties, i.e., both* X *and* Y *are nonsingular projective varieties and all the nonsingular fibers are* (*closed*) *subvarieties of Abelian varieties. Assume none of the nonsingular fibers contain any translations of nontrivial Abelian subvarieties, and, in addition, that* X, Y *and* π *are all defined over a number field* k.

Then, assuming the Vojta conjecture for varieties of dimension $\leqslant d \cdot \dim X$, *we have:*

$$h(P) \leqslant \mathrm{O}(h_Y(\pi(P))),$$

whenever $P \in X_{\pi(P)}(k(\pi(P)))$, *i.e.,* P *is a* $k(\pi(P))$*-rational point, and* $\pi(P) \in Y^\circ(k, d)$, *where the implied constant is independent of* P.

Proof. It is immediate from essentially the same proof as that of Theorem 1.0.1 with $e = 1$. □

Here is a simple remark on Lemma 4.4.1 and Theorem 4.4.2. Notice that contrary to the results up to now their hypotheses are *already* about *all* the nonsingular fibers (regardless of the dimension restriction). Thus, also with essentially the same proof as mentioned above we may simply drop the dimension restriction in the definition of Y° in these two cases. (Or we may even repeat exactly the same argument to deal with the *exceptional* varieties satisfying *all the other requirements in their hypotheses* except only the dimension restriction. This would require some additional (but usual) height comparison we have often seen so far.) Another alternative is to keep the usual Y° in the consequence and, instead, a weakened hypothesis with the usual dimension restriction up to now. Anyway, this would make the results slightly better. This observation was motivated by the referee's suggestion.

Now recall the result of O. Debarre and M. J. Klassen ([9]), i.e., that every nonsingular projective plane curve of degree $e \geqslant 7$ defined over a number field has only finitely many points of degree $\leqslant e - 2$.

THEOREM 4.4.3. *Let* $d \geqslant 1$ *and* $e \geqslant 7$ *be integers. Let* $X \subset \mathbb{P}^2_Y$ *be a nonsingular projective variety which is* (*via* π *below*) *a family of plane curves of degree* e. *More precisely speaking, both* $X \subset \mathbb{P}^2_Y$ *and* Y *are nonsingular projective varieties* (*of arbitrary dimension*)*, and all the one-dimensional nonsingular fibers* (*of* π) *are plane curves*

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of degree e — the generic fiber of $\pi\colon X \to Y$ is a nonsingular projective plane curve of degree e. In addition, X, Y and π are all assumed to be defined over a number field k.

Then, assuming the Vojta conjecture for varieties of dimension $\leqslant d\cdot(e-2+\dim Y)$, we have: $h(P) \leqslant O(h_Y(\pi(P)))$, whenever $P \in X_{\pi(P)}(k(\pi(P)), e-2)$, i.e., P is an algebraic point of degree $\leqslant e-2$ over $k(\pi(P))$, and $\pi(P) \in Y^\circ(k,d)$, where the implied constant is independent of P.

Proof. Essentially the same proof as that of Theorem 1.0.1 also works here and we omit its details. □

4.5. HEIGHT ZETA FUNCTION

Finally, we define *height zeta function* associated to families which we have considered, as follows:

DEFINITION. Under the situation of Theorem 1.0.1 (for convenience, with the assumption of $d = e = 1$) we define a *height zeta function*

$$\zeta_X(s) = \sum_{P\in X^\circ(k)} \frac{H(P)}{H_Y(\pi(P))^s}$$

for $s \in \mathbb{C}$ with $\operatorname{Re} s \gg 0$ where H and H_Y are the exponential heights corresponding to h and h_Y, respectively. (Notice, despite the abbreviated notation of ζ_X, that ζ_X actually depends on the choices of appearing heights as well as on X, Y, π and k.)

CLAIM 4.5.1. *ζ_X defines an analytic function for $s \in \mathbb{C}$ with* $\operatorname{Re} s \gg 0$.

Proof. We may assume that $h_Y \geqslant 1$ is associated to a very ample divisor, by means of which we embed $Y \hookrightarrow \mathbb{P}^N$. We enlarge k (if necessary) so that the embedding is also defined over k. Then we see that the height h_Y is just the pull-back of the standard (logarithmic) height of $\mathbb{P}^N$ up to constant multiplication.

Regard $t := \pi(P) \in Y^\circ(k)$ as a point of $\mathbb{P}^N(k)$ and apply Schanuel's formula for $\mathbb{P}^N(k)$. Then we have: For all integers $n \geqslant 1$,

$$|\{t \in Y(k)\colon n \leqslant H_Y(t) \leqslant n+1\}| \ll (n+1)^{N+1[k:\mathbb{Q}]} \ll n^{N+1[k:\mathbb{Q}]}. \tag{2}$$

Let $\sigma = \operatorname{Re} s$ and let c be the positive constant appearing in Theorem 1.0.1. Then: □

$$\begin{aligned}\sum_{P\in X^\circ(k)} \left|\frac{H(P)}{H_Y(\pi(P))^s}\right| &= \sum_{P\in X^\circ(k)} \frac{H(P)}{H_Y(\pi(P))^\sigma} = \sum_{P\in X^\circ(k)} \frac{1}{H_Y(\pi(P))^{\sigma-c}}\cdot\frac{H(P)}{H_Y(\pi(P))^c}\\ &\leqslant \sum_{P\in X^\circ(k)} \frac{1}{H_Y(\pi(P))^{\sigma-c}} \ll \sum_{t\in Y^\circ(k)} \frac{1}{H_Y(t)^{\sigma-c}} \quad \text{by [7]}\\ &\ll \sum_{n=1}^{\infty} \frac{n^{N+1[k:\mathbb{Q}]}}{n^{\sigma-c}} \quad \text{(by (2))} \quad = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma-c-(N+1)[k:\mathbb{Q}]}}.\end{aligned}$$

Therefore $\zeta_X(s)$ is an analytic function for $\sigma = \operatorname{Re} s > c + (N+1)[k:\mathbb{Q}] + 1$. □

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EXAMPLE 4.5.2. Let C be a nonsingular projective curve of genus at least 2 defined over $\mathbb{Q}$, and let $\pi: C \times \mathbb{P}^1 \to \mathbb{P}^1$ be the second projection. Choose $H_{\mathbb{P}^1} := 4\times$(the usual (exponential) height of $\mathbb{P}^1$) and $H_{C\times\mathbb{P}^1} := H_C\times$(the usual (exponential) height of $\mathbb{P}^1$). Then, for $X := C \times \mathbb{P}^1$ and $k := \mathbb{Q}$, an elementary computation yields that:

$$\zeta_X(s) = \frac{\alpha}{4^{s-2}} \cdot \frac{\zeta(s-2)}{\zeta(s-1)} \left(= \frac{\alpha}{4^{s-2}} \cdot \prod_{p \text{ prime}} \frac{1-p^{1-s}}{1-p^{2-s}} \right) \tag{12}$$

where ζ is the Riemann zeta function and α is the sum of (exponential) heights of $\mathbb{Q}$-rational points of C with respect to any (exponential) height H_C on C. A possible question at this stage may be whether, conversely, we can derive some information on the O-constant in Theorem 1.0.1 from knowing a 'specific' boundary $\sigma = \sigma_0$ (e.g., $\sigma_0 = 3$ in the example) to which the height zeta function can be analytically extended.

For reference, we would like to make the following remark. V. V. Batyrev, J. Franke, Yu. I. Manin and Yu. Tschinkel ([4] and [13]) have studied their height zeta *function* to relate it to the Manin conjecture on the rational points of *Fano* varieties. Their significant observation there is that they can identify their zeta function with one of the Langlands–Eisenstein series. Their height zeta function is *essentially* the same as ours when $\pi: X \to X$ is the identity. One may expect the same sort of observation for our zeta function, too.

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