RT^2_2 DOES NOT IMPLY WKL_0

JIAYI LIU

Abstract. We prove that $\mathrm{RCA}_0 + \mathrm{RT}^2_2 \not\rightarrow \mathrm{WKL}_0$ by showing that for any set C not of PA-degree and any set A, there exists an infinite subset G of A or $\overline{A}$, such that $G \oplus C$ is also not of PA-degree.

§1. Introduction. Reverse mathematics studies the proof theoretic strength of various second order arithmetic statements. Several statements are so important and fundamental that they serve as level lines. Many mathematical theorems are found to be equivalent to these statements and they are unchanged under small perturbations of themselves. The relationships between these statements and "other" statements draw large attention. WKL_0 is one of these statements. WKL_0 states that every infinite binary tree admits an infinite path. It is well known that as a second order arithmetic statement, WKL_0 is equivalent to the statement that for any set C there exists $B \gg C$, where $B \gg C$ means B is of PA-degree relative to C. A good survey of reverse mathematics is [8] or [3], [4]. One of the second order arithmetic statements close to WKL_0 is RT^2_2.

DEFINITION 1.1. Let $[X]^k$ denote $\{F \subseteq X : |F| = k\}$. A k-coloring f is a function, $[X]^n \to \{1, 2, \ldots, k\}$. A set $H \subseteq [X]^k$ is homogeneous for f iff f is constant on $[H]$. A stable coloring f is a 2-coloring of $[\mathbb{N}]^2$ such that $(\forall n \in \mathbb{N})(\exists N)(\forall m > N) \; f(\{m, n\}) = f(\{N, n\})$. For a stable coloring f, $f_1 = \{n \in \mathbb{N} : (\exists N)(\forall m > N), f(m, n) = 1\}$, $f_2 = \mathbb{N} - f_1$.

RAMSEY'S THEOREM (Ramsey [6]). *For any n and k, every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.*

Let RT^n_k denote the Ramsey's theorem for k-coloring of $[\mathbb{N}]^n$. And SRT^2_k denotes the Ramsey's theorem restricted to stable coloring of pair.

Jockusch [5] showed that for $n > 2$ RT^n_2 is equivalent to ACA_0, while Seetapun and Slaman [7] showed that RT^2_2 does not imply ACA_0. As to WKL_0, Jockusch [5] proved that WKL_0 does not imply RT^2_2. Whether RT^2_2 implies WKL_0 remained open. A more detailed survey of Ramsey's theorem in view of reverse mathematics can be found in Cholak, Jockusch and Slaman [1]. Say a set S cone avoid a class $\mathcal{M}$ iff $(\forall C \in \mathcal{M})[C \not\leq_T S]$.

The problem has been a major focus in reverse mathematics in the past twenty years. The first important progress was made by Seetapun and Slaman [7], where they showed that

THEOREM 1.2 (Seetapun and Slaman [7]). *For any countable class of sets $\{C_j\}$ $j \in \omega$, each C_i is non-computable, then any computable 2-coloring of pairs admits an infinite cone avoiding (for $\{C_j\}$) homogeneous set.*

Parallel this result, using Mathias Forcing in a different manner Dzhafarov and Jockusch [2] Lemma 3.2 proved that

THEOREM 1.3 (Dzhafarov and Jockusch [2]). *For any set A and any countable class $\mathcal{M}$, each member of $\mathcal{M}$ is non-computable, there exists an infinite set G contained in either A or its complement such that G is cone avoiding for $\mathcal{M}$.*

The main idea is to restrict the computational complexity (computability power) of the homogeneous set as much as possible, with complexity measured by various measurements. Along this line, with simplicity measured by extent of lowness, Cholak, Jockusch and Slaman [1] Theorem 3.1 showed, by a fairly ingenious argument,

THEOREM 1.4 (Cholak, Jockusch, and Slaman [1]). *For any computable coloring of the unordered pairs of natural numbers with finitely many colors, there is an infinite low_2 homogeneous set X.*

Here we adopt the same idea to prove that

THEOREM 1.5. *For any set C not of PA-degree and any set A. There exists an infinite subset G of A or $\overline{A}$, such that $G \oplus C$ is also not of PA-degree.*

COROLLARY 1.6. $\mathsf{RT}^2_2 \nrightarrow \mathsf{WKL}_0$.

PROOF. It suffices to construct a countable class $\mathcal{M}$ satisfying the following four conditions (a) $C, B \in \mathcal{M} \to C \oplus B \in \mathcal{M}$; (b) $(C \in \mathcal{M} \wedge B \leq_T C) \to B \in \mathcal{M}$; (c) $(\forall C \in \mathcal{M})[C \not\gg 0]$; (d) $\mathcal{M} \vdash \mathsf{RT}^2_2$. It is shown in [1] Lemma 7.11 that $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is equivalent to $\mathsf{RCA}_0 + \mathsf{SRT}^2_2 + \mathsf{COH}$. Moreover, it's easy to prove that for any C-uniform sequence $C_1, C_2, \ldots, C$ being non-PA-degree, there exists an infinite set G cohesive for $C_1, C_2, \ldots$ such that $G \oplus C$ is not of PA-degree. This can be proved using finite extension method as following. Here and below $\sigma \prec \rho$ means σ is an initial part of ρ; $\sigma \subseteq \rho$ means $\{n \leq |\sigma| : \sigma(n) = 1\} \subseteq \{n \leq |\rho| : \rho(n) = 1\}$. At stage s, we define

$$Z_s = \begin{cases} Z_{s-1} \cap C_s & \text{if } Z_{s-1} \cap C_s \text{ is infinite;} \\ Z_{s-1} \cap \overline{C_s} & \text{else:} \end{cases}$$

($Z_0 = C_0$ if C_0 is infinite $\overline{C_0}$ if else), $\rho_s \succ \rho_{s-1}$ with $\rho_{s-1} \subsetneq \rho_s \subseteq Z_s/\rho_{s-1}$. And whenever possible we also require $(\exists n)\left[\Phi_s^{C \oplus \rho_s}(n) = \Phi_n(n)\!\downarrow\right]$. We argue $G = \cup\rho_s$ is one of the desired sets. Clearly G is infinite since $(\forall s)\rho_{s-1} \subsetneq \rho_s$. The cohesiveness of G follows from $(\forall s)\left[G \subseteq^* Z_s\right]$ and $Z_s \subseteq^* C_s \vee Z_s \subseteq^* \overline{C_s}$. Furthermore, $(\forall s)\left[\Phi_s^{C \oplus G} \text{ is not a 2-DNR}\right]$. For else, suppose contradictory $\Phi_s^{C \oplus G}$ is a 2-DNR. Therefore $(\forall \rho \succeq \rho_{s-1}, \rho \subseteq Z_s/\rho_{s-1})\left[\Phi_s^{C \oplus \rho}(n)\!\downarrow \wedge \Phi_n(n)\!\downarrow \Rightarrow \Phi_s^{C \oplus \rho}(n) \neq \Phi_n(n)\right]$. Since $\Phi_s^{C \oplus G}$ is total so $(\forall n)(\exists \rho \succeq \rho_{s-1}, \rho \subset Z_s/\rho_{s-1})\left[\Phi_s^{C \oplus \rho}(n)\!\downarrow\right]$. Thus we could compute a 2-DNR using Z_s, but $Z_s \leq_T C$ contradict the fact that $C \ngtr 0$.

Let $B_0 = \emptyset$. Let $f \in \Delta_2^{0,B_0}$ be a stable coloring, by Theorem 1.5 there exists an infinite G_0, $G_0 \subseteq f_1 \vee G_0 \subseteq f_2$ such that $B_0 \oplus G_0 \ngtr 0$, note that such G_0 computes an infinite homogeneous set of f. Let $B_1 = B_0 \oplus G_0$, $\mathcal{M}_1 = \{X \in 2^\omega : X \leq_T B_1\}$. Clearly $\mathcal{M}_1$ satisfies (a)(b)(c). Let G_1 be cohesive for a sequence of uniformly $\mathcal{M}_1$-computable sets (where $\mathcal{M}_1$-computable means computable in some $C \in \mathcal{M}_1$), furthermore $G_1 \oplus B_1 \ngtr 0$. Let $B_2 = B_1 \oplus G_1$, $\mathcal{M}_2 = \{X \in 2^\omega : X \leq_T B_2\}$. Clearly $\mathcal{M}_2$ also satisfies (a)(b)(c). Iterate the above process in some way that ensures (1) for any uniformly $\mathcal{M}_j$-computable sequence $C_1, C_2 \ldots$, there exists $G_{i-1} \in \mathcal{M}_i$ cohesive for $C_1, C_2, \ldots$ and (2) for any $C \in \Delta_2^{0,\mathcal{M}_j}$, there exists an infinite $G_{i-1} \in \mathcal{M}_i$, $G_{i-1} \subseteq C \vee G_{i-1} \subseteq \overline{C}$, while preserving the fact that for all resulted $B_i = B_{i-1} \oplus G_{i-1}$, $B_i \ngtr 0$. It follows that $\mathcal{M} = \bigcup_{i=0}^\infty \mathcal{M}_i \vdash \mathsf{RCA}_0 + \mathsf{SRT}_2^2 \leftrightarrow \mathsf{RT}_2^2$ but clearly $\mathcal{M}$ satisfies (a)(b)(c). The conclusion so follows. $\dashv$

The organization of this paper is as following. In Section 2 we introduce some notations and the requirements we use. In Section 3 we give some intuition about the proof by demonstrating the construction of the first step. Section 4 defines the forcing conditions and shows how to use these conditions to obtain a desired set G. Section 6 is devoted to the most important construction, i.e., how to construct a successive condition to force the requirements.

§2. Preliminaries.
We say X codes an ordered k-partition of ω iff $X = X_1 \oplus X_2 \oplus \cdots \oplus X_k$ and $\bigcup_{i=1}^k X_i = \omega$, (*not necessarily* with $X_i \cap X_j = \emptyset$). A k-partition class is a non-empty collection of sets, where each set codes a k-partition of ω. A tree $T \subseteq 2^{<\omega}$ is an ordered k-partition tree of ω iff every $\sigma \in T$ codes an ordered k-partition of $\{0, 1, \ldots, |\sigma|\}$. Note that the class of all ordered k-partitions of ω is a Π_1^0 class.

DEFINITION 2.1. For n many ordered k-partitions, $X^0, \ldots, X^{n-1}$

$$Cross(X^0, X^2, \ldots, X^{n-1}; 2) = \bigoplus_{j < k, p < q \leq n-1} Y_j^{(p,q)}$$

where $Y_j^{(p,q)} = X_j^p \cap X_j^q$, i.e., $Y_j^{(p,q)}$ is the intersection of those X^p and X^q's j^{th} part, with $p \neq q$. For n classes of ordered k-partitions $S_0, S_1, \ldots, S_{n-1}$

$$Cross(S_0, S_1, \ldots, S_{n-1}; 2) = \{Y \in 2^\omega : \text{there exists } X^i \in S_i \text{ for each } i \leq n-1,$$
$$Y = Cross(X^0, \ldots, X^{n-1}; 2)\}.$$

Note that if each S_i is a Π_1^0 class, then let T_i be computable tree with $[T_i] = S_i$, operation *Cross* can be defined on strings of $\{0,1\}$ in a nature way, therefore there exists a computable tree $T \subseteq 2^{<\omega}$ such that $T = Cross(T_0, T_1, \ldots, T_{n-1}; 2)$. So $[T] = Cross(S_0, S_1, \ldots, S_{n-1}; 2)$ i.e., $Cross(S_0, S_1, \ldots, S_{n-1}; 2)$ is a Π_1^0 class.

DEFINITION 2.2.   (1) A *valuation* is a finite partial function $\omega \to 2$.
(2) A valuation p is *correct* if $p(n) \neq \Phi_n(n){\downarrow}$ for all $n \in \mathrm{dom}\, p$.
(3) Valuations p, q are *incompatible* if there is an n such that $p(n) \neq q(n)$.

We try to ensure that G satisfies the following requirements.
To ensure that $(G \cap A)$ and $(G \cap \overline{A})$ are infinite, we will satisfy the requirements

$$Q_m : |G \cap A| \geq m \wedge |G \cap \overline{A}| \geq m.$$

To ensure that $(G \cap A) \oplus C$ does not have PA-degree, we would need to satisfy the requirements

$$R_e^A : \Phi_e^{(G \cap A) \oplus C} \text{ total} \Rightarrow (\exists n)\left[\Phi_e^{(G \cap A) \oplus C}(n) = \Phi_n(n){\downarrow}\right].$$

Intuitively, R_e^A is to ensure $(G \cap A) \oplus C$ does not compute any 2-DNR via Φ_e. (Without loss of generality we assume all $\Phi_0, \Phi_1, \ldots$ in this paper are $\{0,1\}$-valued functionals.) Similarly, to ensure $(G \cap \overline{A}) \oplus C$ does not compute any 2-DNR via Φ_e, we try to make G satisfy

$$R_i^{\overline{A}} : \Phi_i^{(G \cap \overline{A}) \oplus C} \text{ total} \Rightarrow (\exists n)\left[\Phi_i^{(G \cap \overline{A}) \oplus C}(n) = \Phi_n(n){\downarrow}\right].$$

Thus we will satisfy the requirements

$$R_{e,i} : R_e^A \vee R_i^{\overline{A}}.$$

These requirements suffice to provide a desired G. Note that if there is some e that G does not satisfy R_e^A then G must satisfy all $R_i^{\overline{A}}$ since G satisfy $R_{e,i}$ for all i. This implies $G \cap \overline{A}$ is not of PA-degree. See also [1], [2].

Before we introduce the forcing condition, to get some intuition, we firstly demonstrate the construction of the first step.

§3. First step. Suppose we wish to satisfy $R_{e,i}$ that is:

either $(\exists n)\left[(\Phi_e^{(G \cap A) \oplus C}(n) = \Phi_n(n){\downarrow}) \vee \Phi_e^{(G \cap A) \oplus C} \text{ is not total}\right]$,

or $(\exists n)\left[(\Phi_i^{(G \cap \overline{A}) \oplus C}(n) = \Phi_n(n){\downarrow}) \vee \Phi_i^{(G \cap \overline{A}) \oplus C} \text{ is not total}\right]$.

CASE i.   *Try* to find a correct p such that

$$(\forall X = X_0 \oplus X_1, X_0 \cup X_1 = \omega)(\exists \rho \exists n \in \mathrm{dom}\, p)$$
$$\left[\Phi_e^{(\rho \cap X_0) \oplus C}(n){\downarrow} = \Phi_n(n){\downarrow} \neq p(n) \vee \Phi_i^{(\rho \cap X_1) \oplus C}(n){\downarrow} = \Phi_n(n){\downarrow} \neq p(n)\right] \tag{1}$$

Note that substitute $X_0 = A, X_1 = \overline{A}$ in above sentence, there is a $\rho \in 2^{<\omega}$ such that $\Phi_e^{(\rho \cap A) \oplus C}(n){\downarrow} = \Phi_n(n){\downarrow} \vee \Phi_i^{(\rho \cap \overline{A}) \oplus C}(n){\downarrow} = \Phi_n(n){\downarrow}$. Therefore finitely extend initial segment requirement to ρ and set $P_1 = \{\omega\}$. *To satisfy* $R_{e,i}$, we ensure $G \succ \rho$. Clearly all $G \succ \rho$ satisfy $R_{e,i}$.

CASE ii. *Try* to find three pairwise incompatible partial functions $p_i \colon \omega \to \{0, 1\}$, $i = 0, 1, 2$ that ensure the following Π_1^0 classes are non-empty:

$$S_i = \{X = X_0 \oplus X_1 \colon X_0 \cup X_1 = \omega$$
$$\wedge \big[(\forall Z)(\forall n \in \mathrm{dom}\, p_i) \, \neg(\Phi_e^{(Z \cap X_0) \oplus C}(n)\!\downarrow \; \neq p_i(n))$$
$$\wedge \, \neg(\Phi_i^{(Z \cap X_1) \oplus C}(n)\!\downarrow \; \neq p_i(n)) \big] \}.$$

Let

$$P_1 = Cross(S_0, S_1, S_2; 2)$$

i.e., $(\forall Y \in P_1) \; Y = Y_0 \oplus Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4 \oplus Y_5$
$(\exists X^0 \in S_0 \quad \exists X^1 \in S_1 \quad \exists X^2 \in S_2) \; X^i = X_0^i \oplus X_1^i$ for $i = 0, 1, 2$ such that

$$Y_0 = X_0^0 \cap X_0^1, \; Y_1 = X_0^1 \cap X_0^2, \; Y_2 = X_0^2 \cap X_0^0,$$
$$Y_3 = X_1^0 \cap X_1^1, \; Y_4 = X_1^1 \cap X_1^2, \; Y_5 = X_1^2 \cap X_1^0.$$

Note:

(1) S_i is a Π_1^0 class of ordered 2-partitions for all $i \leq 2$;

(2) $\Phi_e^{G \oplus C}$ is not total on any $G \subseteq Y_i$, for $i = 0, 1, 2$ and $\Phi_i^{G \oplus C}$ is not total on any $G \subseteq Y_i$, for $i = 3, 4, 5$. To see this, suppose for some $G \subseteq Y_0$, $\Phi_e^{G \oplus C}$ outputs on both $\mathrm{dom}\, p_0$, $\mathrm{dom}\, p_1$. Let $p_0(n) \neq p_1(n)$ then either $\Phi_e^{G \oplus C}(n) \neq p_0(n)$ or $\Phi_e^{G \oplus C}(n) \neq p_1(n)$. (Recall that we assume that all Φ are $\{0, 1\}$−valued.) Suppose it is the former case, but $G \subseteq Y_0 \subseteq X_0^0$, $X_0^0 \oplus X_1^0 \in S_0$, by definition of S_0 $\Phi_e^{G \oplus C}(n)\!\downarrow \; \Rightarrow \Phi_e^{G \oplus C}(n) = p_0(n)$;

(3) P_1 is a Π_1^0 class. Though seemingly not, but note that each S_i is a Π_1^0 class therefore there are computable trees T_i, $i \leq 2k$, such that $[T_i] = S_i$ for all i, furthermore *Cross* can be applied to binary strings and is computable in this sense, thus there exists some computable tree $T_1' = Cross(T_0, T_1, T_2; 2)$ with $P_1 = [T_1']$;

(4) $\bigcup_{i=0}^{5} Y_i = \omega$. (See Lemma 6.5, this is just the pigeonhole principle. This is why we choose *three* pairwise incompatible valuations at *this* step.)

To satisfy $R_{e,i}$, we ensure that for some path $Y \in P_1$, $Y = Y_0 \oplus Y_1 \oplus \cdots \oplus Y_5$, G will be contained in some Y_i. By item 2 in above note, $R_{e,i}$ is satisfied.

We will show in Lemma 6.6 that if there is no correct valuation as in case i then there must exist such three incompatible valuations i.e., either case i or case ii occurs. Now we give the framework of our construction i.e., the forcing conditions.

§4. Tree forcing. Let $\sigma \in 2^{<\omega}$ and let X be either an element of 2^ω or an element of $2^{<\omega}$ of length at least the same as that of σ. Here and below, we write X/σ for the set obtained by replacing the first $|\sigma|$ many bits of X by σ.

We will use conditions that are elaborations on Mathias forcing conditions. Here a *Mathias condition* is a pair (σ, X) with $\sigma \in 2^{<\omega}$ and $X \in 2^\omega$. The Mathias condition (τ, Y) *extends* the Mathias condition (σ, X) if $\sigma \preceq \tau$ and $Y/\tau \subseteq X/\sigma$. A set G *satisfies* the Mathias condition (σ, X) iff $\sigma \prec G$ and $G \subseteq X/\sigma$.

We will be interested in $\Pi_1^{0, C}$ k-partition classes, that is, $\Pi_1^{0, C}$ classes that are also k-partition classes.

DEFINITION 4.1. A *condition* is a tuple of the form $(k, \sigma_0, \ldots, \sigma_{k-1}, P)$, where $k > 0$, each $\sigma_i \in 2^{<\omega}$, and in this paper P is a non-empty $\Pi_1^{0,C}$ k-partition class. We think of each $X_0 \oplus \cdots \oplus X_{k-1} \in P$ as representing k many Mathias conditions (σ_i, X_i) for $i < k$.

DEFINITION 4.2. A condition

$$d = (m, \tau_0, \ldots, \tau_{m-1}, Q) \ \textit{extends} \ c = (k, \sigma_0, \ldots, \sigma_{k-1}, P),$$

also denoted by $d \leq c$, iff there is a function $f \colon m \to k$ with the following property: for each $Y_0 \oplus \cdots \oplus Y_{m-1} \in Q$ there is an $X_0 \oplus \cdots \oplus X_{k-1} \in P$ such that each Mathias condition (τ_i, Y_i) extends the Mathias condition $(\sigma_{f(i)}, X_{f(i)})$. In this case, we say that f *witnesses* this extension, and that *part i of d refines part $f(i)$ of c*. (Whenever we say that a condition extends another, we assume we have fixed a function witnessing this extension.)

DEFINITION 4.3. A set G *satisfies* the condition $(k, \sigma_0, \ldots, \sigma_{k-1}, P)$ iff there is an $X_0 \oplus \cdots \oplus X_{k-1} \in P$ such that G satisfies some Mathias condition (σ_i, X_i). In this case, we also say that G satisfies this condition *on part i*.

DEFINITION 4.4.　(1) A condition $(k, \sigma_0, \ldots, \sigma_{k-1}, P)$ *forces Q_m on part i* iff $|\sigma \cap A| \geq m \wedge |\sigma \cap \overline{A}| \geq m$. Clearly, if G satisfies such a condition on part i, then G satisfies requirement Q_m. (Note that if c forces Q_m on part i, and part j of d refines part i of c, then d forces Q_m on part j.)
(2) A condition *forces $R_{e,i}$ on part j* iff every G satisfying this condition on part j also satisfies requirement $R_{e,i}$. A condition *forces $R_{e,i}$* iff it forces $R_{e,i}$ on each of its parts. (Note that if c forces $R_{e,i}$ on part i, and part j of d refines part i of c, then d forces $R_{e,i}$ on part j. Therefore, if c forces $R_{e,i}$ and d extends c, then d forces $R_{e,i}$.)

DEFINITION 4.5. For a condition $c = (k, \sigma_0, \ldots, \sigma_{k-1}, P)$, we say that *part i of c is acceptable* if there is an $X_0 \oplus \cdots \oplus X_{k-1} \in P$ such that $X_i \cap A$ and $X_i \cap \overline{A}$ are both infinite.

For example, in the first step, $P_0 = \{\omega\}, k_0 = 1, \sigma_0 = \lambda$, and for every $Y \in P_1$ Y is of the form $Y = \bigoplus_{i=0}^5 Y_i$. Clearly $Y_i \subseteq X_{f_1(i)}$, where $X_{f_1(i)} = \omega \in P_0$. $f_1(i) = 0$ for all i witnesses this extension relation.

Note that it is *not* the case that for every $X' \in P'$ there exists a single $X \in P$ such that $(\forall i \leq k' - 1)\big[(\sigma_i', X_i') \leq (\sigma_{f(i)}, X_{f(i)})\big]$.

4.1. The general plan. The proof will consist of establishing the following two lemmas. The proof of the second lemma is the core of the argument.

LEMMA 4.6. *Every condition has an acceptable part. Therefore for every condition c and every m, there is a condition d extending c such that d forces Q_m on each of its acceptable parts.*

LEMMA 4.7. *For every condition c and every e and i, there is a condition d extending c that forces $R_{e,i}$.*

PROOF OF THEOREM 1.5. Given these lemmas, it is easy to see that we can build a sequence of conditions $c_0, c_1, \ldots$ with the following properties.

(1) Each c_{s+1} extends c_s.
(2) If $s = \langle e, i \rangle$ then c_s forces $R_{e,i}$.

(3) Each c_s has an acceptable part.

(4) If part i of c_s is acceptable, then c_s forces Q_s on part i.

Clearly, if part j of c_{s+1} refines part i of c_s and is acceptable, then part i of c_s is also acceptable. Thus we can think of the acceptable parts of our conditions as forming a tree under the refinement relation. This tree is finitely branching and infinite, so it has an infinite path. In other words, there are $i_0, i_1, \ldots$ such that for each s, part i_{s+1} of c_{s+1} refines part i_s of c_s, and part i_s of c_s is acceptable, which implies that c_s forces Q_s on part i_s. Write $c_s = (k_s, \sigma_0^s, \ldots, \sigma_{k_s-1}^s, P_s)$. Let $G = \bigcup_s \sigma_{i_s}^s$. Let U_s be the class of all Y that satisfy $(\sigma_{i_s}^s, X_{i_s})$ for some $X_0 \oplus \cdots \oplus X_{k_s-1} \in P_s$. Note that

- $U_0 \supseteq U_1 \supseteq \ldots$; Since $G \in U_{s+1} \Leftrightarrow (\exists X \in P_{s+1})\big[G$ satisfies $(\sigma_{i_{s+1}}^{s+1}, X_{i_{s+1}})\big] \Rightarrow (\exists Z \in P_s)\big[(\sigma_{i_{s+1}}^{s+1}, X_{i_{s+1}}) \leq (\sigma_{i_s}^s, Z_{i_s}) \wedge G$ satisfies $(\sigma_{i_s}^s, Z_{i_s})\big] \Leftrightarrow G \in U_s$.
- Each U_s contains an extension of $\sigma_{i_s}^s$ i.e., $U_s \neq \emptyset$;
- Each U_s is closed;

By compactness of 2^ω $\bigcap_{s=0}^\infty U_s \neq \emptyset$. But clearly $(\forall Z \in \bigcap_{s=0}^\infty U_s)\big[Z \succ \sigma_{i_s}^s\big]$ for all s.

Thus G is the unique element of $\bigcap_{s=0}^\infty U_s$. In other words, G satisfies each c_s on part i_s, and hence satisfies all of our requirements.

§5. Proof of Lemma 4.6.

PROOF OF LEMMA 4.6. It is here that we use the assumption that $A \not\leq_T C$. Let $c = (k, \sigma_0, \ldots, \sigma_{k-1}, P)$ be a condition. Write P_τ for the set of all $X \in P$ that extend τ.

CLAIM. *For each* $\tau = \tau_0 \oplus \cdots \oplus \tau_{k-1}$, *if* $P_\tau \neq \emptyset$ *then there is an* $X_0 \oplus \cdots \oplus X_{k-1} \in P_\tau$ *and an* $i < k$ *such that* X_i *contains elements* $m \in A$ *and* $n \in \overline{A}$ *such that* $m, n \geq |\tau_i|$.

Assuming the claim for now, we build a sequence of strings as follows. Let ρ^0 be the empty string. Given $\rho^s = \rho_0^s \oplus \cdots \oplus \rho_{k-1}^s$ such that P_{ρ^s} is non-empty, let $X = X_0 \oplus \cdots \oplus X_{k-1} \in P_{\rho^s}$ and $i_s < k$ be such that X_{i_s} contains elements $m \in A$ and $n \in \overline{A}$ with $m, n \geq |\rho_{i_s}^s|$. Then there is a $\rho_{s+1} = \rho_0^{s+1} \oplus \cdots \oplus \rho_{k-1}^{s+1} \prec X$ such that, thinking of strings as finite sets, $\rho_{i_s}^{s+1} \setminus \rho_{i_s}^s$ contains elements of both A and $\overline{A}$. Now let $Y = \bigcup_s \rho_s$ and let i be such that $i = i_s$ for infinitely many s. Then $Y \in P$ and Y witnesses the fact that part i of c is acceptable.

Fix m. To obtain the desired $d \leq c$ that forces Q_m on each of its acceptable part. It is enough to show that for the condition $c = (k, \sigma_0, \ldots, \sigma_{k-1}, P)$, if part i of c is acceptable, then there is a condition $d_0 = (k, \tau_0, \ldots, \tau_{k-1}, Q)$ extending c such that d_0 forces Q_m on part i, where the extension of c by d_0 is witnessed by the identity map. (Note that if part i of d_0 is acceptable, then so is part i of c.) Then we can iterate this process, forcing Q_m on each acceptable part in turn, to obtain the condition d in the statement of the lemma.

So fix an acceptable part i of c. Then there is a $\tau \succ \sigma_i$ with $|\tau \cap A| \geq m$ and $|\tau \cap \overline{A}| \geq m$, and there is an $X_0 \oplus \cdots \oplus X_{k-1} \in P$ with $\tau \prec X_i/\sigma_i$. Let $Q = \{X_0 \oplus \cdots \oplus X_{k-1} \in P : \tau \prec X_i/\sigma_i\}$. Let $d_0 = (k, \sigma_0, \ldots, \sigma_{i-1}, \tau, \sigma_{i+1}, \ldots, \sigma_{k-1}, Q)$. Then d_0 is an extension of c, with the identity function $id: \to k$ witness this extension and it clearly forces Q_m on part i.

Thus we are left with verifying the claim.

PROOF OF THE CLAIM. Assume for a contradiction that there is a $\tau = \tau_0 \oplus \cdots \oplus \tau_{k-1}$ such that $P_\tau \neq \emptyset$ and for every $X_0 \oplus \cdots \oplus X_{k-1} \in P_\tau$ and every $i < k$, either $X_i \upharpoonright_{\geq |\tau_i|} \subseteq A$ or $X_i \upharpoonright_{\geq |\tau_i|} \subseteq \overline{A}$. It is easy to see that τ has an extension $v = v_0 \oplus \cdots \oplus v_{k-1}$ such that $P_v \neq \emptyset$ and for each $i < k$, either $v_i(m_i) = 1$ for some $m_i \geq |\tau_i|$ or for every $X_0 \oplus \cdots \oplus X_{k-1} \in P_v$, we have $X_i \upharpoonright_{\geq |\tau_i|} = \emptyset$. In the latter case, let m_i be undefined. Let S_A be the set of all $i < k$ such that m_i is defined and is in A, and let $S_{\overline{A}}$ be the set of all $i < k$ such that m_i is defined and is in $\overline{A}$. If $X_0 \oplus \cdots \oplus X_{k-1} \in P_v$, then $X_i \upharpoonright_{\geq |\tau_i|} \subseteq A$ for all $i \in S_A$, and $X_i \upharpoonright_{\geq |\tau_i|} \subseteq \overline{A}$ for all $i \in S_{\overline{A}}$.

We now claim we can compute A from C, contrary to hypothesis. To see that this is the case, let T be a C-computable tree such that P_v is the set of infinite paths of T. For $\rho \in T$, write T_ρ for the tree of all strings in T compatible with ρ. Suppose we are given $n \geq |\tau|$. Let $j > |v|$ be such that for each $\rho = \rho_0 \oplus \cdots \oplus \rho_{k-1}$ of length j, we have $n < |\rho_i|$ for all $i < k$. Let L_A be the set of all $\rho \in T$ of length j such that $\rho_i(n) = 1$ for some $i \in S_A$ and let $L_{\overline{A}}$ be the set of all $\rho \in T$ of length j such that $\rho_i(n) = 1$ for some $i \in S_{\overline{A}}$. If $\rho \in L_A$ and T_ρ has an infinite path then, by the definition of S_A, we have $n \in A$. Similarly, if $\rho \in L_{\overline{A}}$ and T_ρ has an infinite path then $n \in \overline{A}$. Thus, if $\rho \in L_A$ and $\rho' \in L_{\overline{A}}$, then at least one of T_ρ and $T_{\rho'}$ must be finite. So if we C-compute T and start removing form L_A and $L_{\overline{A}}$ every ρ such that T_ρ is found to be finite, one of L_A or $L_{\overline{A}}$ will eventually be empty. They cannot both be empty because P_v is non-empty. If L_A becomes empty, then $n \in \overline{A}$. If $L_{\overline{A}}$ becomes empty, then $n \in A$. ⊣

⊣

We now turn to the proof of Lemma 4.7.

§6. Forcing $R_{e,i}$.

DEFINITION 6.1. (1) $\Phi_e^{\rho \oplus C}$ *disagrees* with a valuation p on a set X iff there is a $Y \subseteq X$ and an $n \in \operatorname{dom} p$, $\Phi_e^{Y/\rho \oplus C}(n) \neq p(n)$;

(2) Let $c = (k, \sigma_0, \ldots, \sigma_{k-1}, P)$ be a condition, p be a valuation and $U \subseteq \{0, 1, \ldots, k-1\}$. We say that c *disagrees* with p on U if for every $X_0 \oplus \cdots \oplus X_{k-1} \in P$ and every $Z_0, Z_1, \ldots, Z_{2k-1}$ with $(\forall l)[X_l = Z_{2l} \cup Z_{2l+1}]$, there is a Y, a $j \in U(c)$, and an $n \in \operatorname{dom} p$ such that either $\Phi_e^{((Y \cap Z_{2j})/\sigma_j^A) \oplus C}(n) \downarrow \neq p(n)$ or $\Phi_i^{((Y \cap Z_{2j+1})/\sigma_j^{\overline{A}}) \oplus C}(n) \downarrow \neq p(n)$.

The following facts illustrate the central idea of the construction.

FACT 6.2. For two pairwise incompatible valuations p_0, p_1, if Φ^ρ does not disagree with *both* p_0, p_1, on set X. Then for any $Y \subseteq X$, $\Phi^{Y/\rho}$ is not total on $\operatorname{dom} p_0 \cup \operatorname{dom} p_1$.

FACT 6.3. If Φ^ρ does not disagree with p on a set X then for any $Y \subseteq X$, Φ^ρ does not disagree with p on set Y.

Therefore,

FACT 6.4. For two incompatible valuations p_0, p_1. If Φ^ρ does not disagree with p_0 on a set X_0, and does not disagree with p_1 on a set X_1 then for any $Y \subseteq X_0 \cap X_1$, $\Phi^{Y/\rho}$ is not total on $\operatorname{dom} p_0 \cup \operatorname{dom} p_1$.

The following lemma tells how to ensure that the tree of each condition is an ordered partition tree.

LEMMA 6.5. *For any n many ordered $2k$-partitions of ω, namely $X^0, X^1, \ldots, X^{n-1}$, if $n > 2k$ then $Cross(X^0, X^1, \ldots, X^{n-1}; 2)$ is a $2k\binom{n}{2}$-partition. Therefore if $S_0, S_2, \ldots, S_{n-1}$ are n classes of ordered $2k$-partitions of ω then $Cross(S_0, S_1, \ldots, S_n; 2)$ is a class of $2k\binom{n}{2}$-partition of ω.*

PROOF. Straightforward by pigeonhole principle. It suffices to show that for any $x \in \omega$, there is some $i \leq 2k - 1$, some X^p, X^q, $p \neq q$, such that $X^p = \bigoplus_{i=0}^{2k-1} X_i^p$, $X^q = \bigoplus_{i=0}^{2k-1} X_i^q$, $x \in X_i^p \cap X_i^q$. For $i = 0, 2, \ldots, 2k-1$ let $F_i = \{p \leq n-1 : \in X_i^p\}$. Since each X^p is an ordered partition, therefore for each p there exists some i such that $p \in F_i$. So $\bigcup_{i=0}^{2k-1} F_i = \{0, 1, 2, \ldots, n-1\}$. But $n > 2k$ thus there is some $i \leq 2k - 1$ such that F_i contains two elements say p, q, thus $x \in X_i^p \cap X_i^q$. ⊣

6.1. Construction. Fix e, i and a condition $c = (k, \sigma_0, \ldots, \sigma_{k-1}, P)$. For any condition d, let $U(d)$ be the set of all j such that part j of d does not force $R_{e,i}$ on part j. If $U(d) = \emptyset$ then there is nothing to prove, so we assume $U(d) \neq \emptyset$. It is clearly enough to obtain a condition d extending c such that $|U(d)| < |U(c)|$. Then one could simply iterate this process. Here and below, we write σ^A for the string of the same length as σ defined by $\sigma^A(n) = 1$ iff $\sigma(n) = 1 \wedge n \in A$, and similarly for $\sigma^{\overline{A}}$.

We will use two ways to extend conditions.

Begin construction:
CASE I. c disagrees with some correct valuation p on $U(c)$.

Let $X_0 \oplus \cdots \oplus X_{k-1} \in P$. For $j = 0, 1, \ldots, k-1$ let $Z_{2j} = X_j \cap A$ and $Z_{2j+1} = X_j \cap \overline{A}$. By the definition of disagreeing with a correct valuation on $U(c)$, there exists a $j \in U(c)$, an $n \in \text{dom } p$ and a Y such that either $\Phi_e^{((Y \cap Z_{2j})/\sigma_j^A) \oplus C}(n)\downarrow = \Phi_n(n)\downarrow$ or $\Phi_i^{((Y \cap Z_{2j+1})/\sigma_j^{\overline{A}}) \oplus C}(n)\downarrow = \Phi_n(n)\downarrow$. In other words, either $\Phi_e^{(Y/\sigma_j \cap A) \oplus C}(n)\downarrow = \Phi_n(n)\downarrow$ or $\Phi_i^{(Y/\sigma_j \cap \overline{A}) \oplus C}(n)\downarrow = \Phi_n(n)\downarrow$.

If τ is a sufficiently long initial segment of Y, then for every Z extending τ, we have either $\Phi_e^{(Z \cap A) \oplus C}(n)\downarrow = \Phi_n(n)\downarrow$ or $\Phi_i^{(Z \cap \overline{A}) \oplus C}(n)\downarrow = \Phi_n(n)\downarrow$. We may assume that $\tau \succeq \sigma_j$. Let Q be the class of all $W_0 \oplus \cdots \oplus W_{k-1} \in P$ such that τ, thought of as a finite set, is a subset of W_j/σ_j and let $d = (k, \sigma_0, \ldots, \sigma_{j-1}, \tau, \sigma_{j+1}, \ldots, \sigma_{k-1}, Q)$. Note that Q is a non-empty $\Pi_1^{0,C}$ class since it contains $X_0 \oplus \cdots \oplus X_{k-1}$. Clearly d is an extension of c, with the identity function $id : k \to k$ witnessing this extension relation, and clearly d forces $R_{e,i}$ on part j, so that $|U(d)| < |U(c)|$.

CASE II. There are pairwise incompatible valuations $p_0, \ldots, p_{2k}$ such that c does not disagree with any p_l on $U(c)$. We will show in Lemma 6.6 that these are the only two cases that will occur.

For each $l < 2k$ let S_l be the class of all sets of the form $Z_0 \oplus \cdots \oplus Z_{2k-1}$ such that $(Z_0 \cup Z_1) \oplus (Z_2 \cup Z_3) \oplus \cdots \oplus (Z_{2k-2} \cup Z_{2k-1}) \in P$ and for all $j \in U(c)$, every $n \in \text{dom } p_l$, every Y we have, neither $\Phi_e^{(Y \cap Z_{2j})/\sigma_j^A \oplus C}(n)\downarrow \neq p_l(n)$ nor $\Phi_i^{(Y \cap Z_{2j+1})/\sigma_j^{\overline{A}} \oplus C}(n)\downarrow \neq p_l(n)$.

Since c does not disagree with any of the p_l on $U(c)$, all S_l are non-empty. It is then easy to see that each S_l is in fact a $\Pi_1^{0,C}$ $2k$-partition class.

Let $Q = Cross(S_0, \ldots, S_{2k}; 2)$ and let

$$d = \left(2k \binom{2k+1}{2}, \sigma_0, \ldots, \sigma_0, \sigma_1, \ldots, \sigma_1, \ldots, \sigma_{k-1}, \ldots, \sigma_{k-1}, Q \right),$$

where each σ_i appears $2\binom{2k+1}{2}$ many times. We show that d is a condition extending c, and d forces $R_{e,i}$.

(1) Since each S_i is non-empty therefore Q is non-empty. Furthermore, since each S_i is a $\Pi_1^{0,C}$ class then Q is also a $\Pi_1^{0,C}$ class. Because $Cross$, when applied to strings, is computable therefore by applying $Cross$ to the $2k + 1$ computable trees T_i with $[T_i] = S_i$ one obtains a computable tree T with $[T] = Q$.

(2) Q is a class of ordered $2k\binom{2k+1}{2}$-partitions of ω. To see this, note that S_i, $i \leq 2k$, are $2k + 1$ classes of ordered $2k$-partitions of ω, by Lemma 6.5 Q is a class of ordered $2k\binom{2k+1}{2}$-partitions of ω. Therefore combine with item 1 and recall the fact that the initial segments in d are not changed, it follows that d is a condition.

(3) For each new part i' of d and every $W_0 \oplus W_1 \oplus \cdots \oplus W_{k'-1} \in Q$, where $k' = 2k\binom{2k+1}{2}$, there exists $X_0 \oplus X_1 \oplus \cdots \oplus X_{k-1} \in P$, and $i \leq k - 1$ with $W_{i'}/\sigma_{i'} \subseteq X_i/\sigma_i$, and $\sigma_i = \sigma_{i'}$, i.e., each new part is contained in an old part of some path through P. It follows that d extends c. To see this, note that by definition of P for each $i' \leq k' - 1$ there exist $p, q \leq 2k, p \neq q$ and $j \leq 2k - 1$ determined by i', such that $(\forall W \in Q)(\exists X^p \in S_p \exists X^q \in S_q) \left[W_{i'} = X_j^p \cap X_j^q \right]$. Furthermore, by definition of S_p, $X_j^p \cup X_{j'}^p = X_i$ for some $j' \leq 2k - 1$, and some $X = X_0 \oplus X_1 \oplus \cdots \oplus X_{k-1} \in P$. Therefore

$$W_{i'} = X_j^p \cap X_j^q \subseteq X_j^p \subseteq X_j^p \cup X_{j'}^p = X_i$$

i.e., each part i' of each $W \in Q$ is contained in some part i of some $X \in P$.

(4) d forces $R_{e,i}$. To see this, let G satisfy d. Then there is some $j < k$, some $a \neq b < 2k + 1$, some $Z_0 \oplus \cdots \oplus Z_{2k-1} \in S_a$, and some $W_0 \oplus \cdots \oplus W_{2k-1} \in S_b$ such that G satisfies one of the Mathias conditions $(\sigma_j, Z_{2j} \cap W_{2j})$ or $(\sigma_j, Z_{2j+1} \cap W_{2j+1})$. Then G satisfies c on part j, so if $j \notin U(c)$, then G satisfies $R_{e,i}$. So assume $j \in U(c)$.

Let us suppose G satisfies $(\sigma_j, Z_{2j} \cap W_{2j})$, the other case being similar. Then $(G \cap A)/\sigma_j$ satisfies both of the Mathias conditions (σ_j, Z_{2j}) and (σ_j, W_{2j}). Let n be such that $p_a(n) \neq p_b(n)$. By the definitions of S_a and S_b, we have $\neg(\Phi_e^{(G \cap A) \oplus C}(n) \downarrow \neq p_a(n))$ and $\neg(\Phi_e^{(G \cap A) \oplus C}(n) \downarrow \neq p_b(n))$. Hence we must have $\Phi_e^{(G \cap A) \oplus C}(n) \uparrow$. Thus d forces $R_{e,i}$.

End of construction.

It remains to prove that

LEMMA 6.6. *For a valuation p, let S_p be the $\Pi_1^{0,C}$ class of all $Z_0 \oplus \cdots \oplus Z_{2k-1}$ with $Z_0 \cup Z_1 \oplus \cdots \oplus Z_{2k-2} \cup Z_{2k-1} \in P$ such that for every $j \in U(c)$, every $\mu \in 2^\omega$, and every $n \in \text{dom } p$,*

- *neither* $\Phi_e^{((\mu \cap Z_{2j})/\sigma_j^A) \oplus C}(n)[|\mu|]\!\downarrow \neq p(n),$
- *nor* $\Phi_i^{((\mu \cap Z_{2j+1})/\sigma_i^{\overline{A}}) \oplus C}(n)[|\mu|]\!\downarrow \neq p(n).$

One of the following must hold.

(1) *There is a correct valuation p such that S_p is empty i.e., c disagrees with the correct p on $U(c)$.*

(2) *There are pairwise incompatible valuations $p_0, \ldots, p_{2k}$ such that S_p is not empty i.e., c does not disagree with p_l on $U(c)$ for all $l \leq 2k$.*

PROOF OF LEMMA 6.6. We note that item 1 and item 2 are equivalent to case i and case ii respectively. Furthermore S_p is a $\Pi_1^{0,C}$ class uniformly in p. Consequently for each $j < k$, the set of all valuations p such that c disagrees with p on $U(c)$ is C-c.e. Let E denote this C-c.e. set of valuations.

Assume that alternative 1 above does not hold. Since C does not have PA-degree, there is no C-computable function h such that if $\Phi_n(n)\!\downarrow$ then $h(n) \neq \Phi_n(n)$.

Let S be the collection of all finite sets F such that for each $n \notin F$, either $\Phi_n(n)\!\downarrow$ or there is a $p \in E$ such that $F \cup \{n\} \subseteq \text{dom } p$ and for every $m \in \text{dom } p \setminus F \cup \{n\}$, we have $p(m) \neq \Phi_m(m)\!\downarrow$. If $F \notin S$, then there is at least one $n \notin F$ for which the above does not hold. We say that any such n *witnesses* that $F \notin S$.

First suppose that $\emptyset \in S$. Then for each n, either $\Phi_n(n)\!\downarrow$ or there is a $p \in E$ such that $n \in \text{dom } p$ and for every $m \neq n$ in $\text{dom } p$, we have $p(m) \neq \Phi_m(m)\!\downarrow$. Then we can define $h \leq_{\mathrm{T}} C$ by waiting until either $\Phi_n(n)\!\downarrow$, in which case we let $h(n) = 1 - \Phi_n(n)$, or a p as above enters E, in which case we let $h(n) = 1 - p(n)$. Since no element of E is correct, in the latter case, if $\Phi_n(n)\!\downarrow$ then $p(n) = \Phi_n(n)$, so $h(n) = \Phi_n(n)$. Since C does not have PA-degree, this case cannot occur.

Thus $\emptyset \notin S$. Let n_0 witness this fact. Given $n_0, \ldots, n_j$, if $\{n_0, \ldots, n_j\} \notin S$, then let n_{j+1} witness this fact. Note that if n_j is defined then $\Phi_{n_j}(n_j)\!\uparrow$.

Suppose that for some j, we have $\{n_0, \ldots, n_j\} \in S$. Then $\{n_0, \ldots, n_{j-1}\} \notin S$, as otherwise n_j would not be defined. We define $h \leq_{\mathrm{T}} C$ as follows. First, let $h(n_l) = 0$ for $l \leq j$. Given $n \notin \{n_0, \ldots, n_j\}$, we wait until either $\Phi_n(n)\!\downarrow$, in which case we let $h(n) = 1 - \Phi_n(n)$, or a p enters E such that $\{n_0, \ldots, n_j, n\} \subseteq \text{dom } p$ and for every $m \in \text{dom } p \setminus \{n_0, \ldots, n_j, n\}$, we have $p(m) \neq \Phi_m(m)\!\downarrow$. If $\Phi_n(n)\!\uparrow$ then the latter case must occur, since $\{n_0, \ldots, n_j\} \in S$. In this case, we cannot have $p(n) \neq \Phi_n(n)\!\downarrow$, as then p would be a counterexample to the fact that n_j witnesses that $\{n_0, \ldots, n_{j-1}\} \notin S$. Thus we can let $h(n) = 1 - p(n)$. Again, since C does not have PA-degree, this case cannot occur.

Thus $\{n_0, \ldots, n_j\} \notin S$ for all j. There are 2^{j+1} many valuations with domain $\{n_0, \ldots, n_j\}$, and they are all pairwise incompatible. None of these valuations can be in E, as that would contradict the fact that n_j witnesses that $\{n_0, \ldots, n_{j-1}\} \notin S$. Taking j large enough, we have $2k + 1$ many pairwise incompatible valuations, none of which are in E. $\dashv$

REFERENCES

[1] PETER A. CHOLAK, CARL G. JOCKUSCH, JR., and THEODORE A. SLAMAN, *On the strength of Ramsey's theorem for pairs*, this JOURNAL, vol. 66 (2001), no. 1, pp. 1–55.

[2] DAMIR D. DZHAFAROV and CARL G. JOCKUSCH, JR., *Ramsey's theorem and cone avoidance*, this JOURNAL, vol. 74 (2009), pp. 557–578.

[3] H. Friedman, *Some systems of second order arithmetic and their use*, **Proceedings of the International Congress of Mathematicians, Vancouver**, vol. 1, Canadian Mathematical Congress, 1975, pp. 235–242.

[4] ———, *Systems of second order arithmetic with restricted induction, I, II*, this Journal, vol. 41 (1976), no. 2, pp. 557–559.

[5] Carl G. Jockusch, Jr., *Ramsey's theorem and recursion theory*, this Journal, vol. 37 (1972), no. 2, pp. 268–280.

[6] F. P. Ramsey, *On a problem of formal logic*, **Proceedings of the London Mathematical Society**, vol. 2 (1930), no. 1, pp. 264–286.

[7] David Seetapun and Theodore A. Slaman, *On the strength of Ramsey's theorem*, **Notre Dame Journal of Formal Logic**, vol. 36 (1995), no. 4, pp. 570–582, Special Issue: Models of arithmetic.

[8] Stephen G. Simpson, **Subsystems of second order arithmetic**, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999.

DEPARTMENT OF MATHEMATICS
CENTRAL SOUTH UNIVERSITY
SOUTH CAMPUS
SOUTH DORMITORY NO. 6 ROOM 619
CHANGSHA, 410083, CHINA
E-mail: g.jiayi.liu@gmail.com