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p-Adic quotient sets: Linear recurrence sequences with reducible characteristic polynomials

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Abstract. Let $(x_n)_{n\geq 0}$ be a linear recurrence sequence of order $k \geq 2$ satisfying

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k}$$

for all integers $n \geq k$, where $a_1, \ldots, a_k, x_0, \ldots, x_{k-1} \in \mathbb{Z}$, with $a_k \neq 0$. In 2017, Sanna posed an open question to classify primes p for which the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$. In a recent paper, we showed that if the characteristic polynomial of the recurrence sequence has a root $\pm\alpha$, where α is a Pisot number and if p is a prime such that the characteristic polynomial of the recurrence sequence is irreducible in $\mathbb{Q}_p$, then the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$. In this article, we answer the problem for certain linear recurrence sequences whose characteristic polynomials are reducible over $\mathbb{Q}$.

1 Introduction and statement of results

For a set of integers A, the set $R(A) = \{a/b : a, b \in A, b \neq 0\}$ is called the ratio set or quotient set of A. Many authors have studied the denseness of ratio sets of different subsets of $\mathbb{N}$ in the positive real numbers. See, for example, [4–7, 12, 14–18, 24, 25, 28, 29]. An analogous study has also been done for algebraic number fields, see for example [9, 27].

For a prime p, let $\mathbb{Q}_p$ denote the field of p-adic numbers. In recent years, the denseness of ratio sets in $\mathbb{Q}_p$ have been studied by several authors, see for example [1, 3, 8, 10, 11, 19–21, 26]. Let $(F_n)_{n\geq 0}$ be the sequence of Fibonacci numbers, defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. In [11], Garcia and Luca showed that the ratio set of Fibonacci numbers is dense in $\mathbb{Q}_p$ for all primes p. Later, Sanna [26, Theorem 1.2] showed that, for any $k \geq 2$ and any prime p, the ratio set of the k-generalized Fibonacci numbers is dense in $\mathbb{Q}_p$ and made the following open question.

Question 1.1 [26, Question 1.3] *Let $(S_n)_{n\geq 0}$ be a linear recurrence sequence of order $k \geq 2$ satisfying*

$$S_n = a_1 S_{n-1} + a_2 S_{n-2} + \cdots + a_k S_{n-k},$$

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for all integers $n \geq k$, where $a_1, \ldots, a_k, S_0, \ldots, S_{k-1} \in \mathbb{Z}$, with $a_k \neq 0$. For which prime numbers p is the quotient set of $(S_n)_{n\geq 0}$ dense in $\mathbb{Q}_p$?

In [10], Garcia et al. solved the problem partially for second-order recurrences. Later, in [2], we considered kth-order recurrence sequences for which $a_k = 1$ and initial values $S_0 = \cdots = S_{k-2} = 0$, $S_{k-1} = 1$. We showed that if the characteristic polynomial of the recurrence sequence has a root $\pm\alpha$, where α is a Pisot number and if p is a prime such that the characteristic polynomial of the recurrence sequence is irreducible in $\mathbb{Q}_p$, then the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$. In this article, our objective is to study the denseness of quotient sets of linear recurrence sequences whose characteristic polynomials are reducible over $\mathbb{Q}$. Also, we extend [2, Theorem 1.9], which gives condition for the denseness of ratio sets of second order linear recurrence sequences $(x_n)_{n\geq 0}$ whose characteristic polynomials are of the form $(x-a)^2$, to kth order linear recurrence sequences with characteristic polynomials of the form $(x-a)^k$ in the case when the initial values are given as $x_0 = x_1 = \cdots = x_{k-2} = 0$, $x_{k-1} = 1$.

In our first theorem, we consider recurrence sequences having characteristic polynomials whose roots are all distinct.

Theorem 1.2 *Let $(x_n)_{n\geq 0}$ be a linear recurrence of order $k \geq 2$ satisfying*

$$x_n = b_1 x_{n-1} + b_2 x_{n-2} + \cdots + b_k x_{n-k},$$

for all integers $n \geq k$, where $b_1, \ldots, b_k, x_0, \ldots, x_{k-1} \in \mathbb{Z}$, with $b_k \neq 0$ and $x_0, x_1, \ldots, x_{k-1}$ not all zeros. Suppose that the characteristic polynomial of $(x_n)_{n\geq 0}$ is given by

$$(x-a_1)(x-a_2)\ldots(x-a_k),$$

where $a_i \in \mathbb{Z}$, $a_i \neq a_j$ for $1 \leq i \neq j \leq k$, and $\gcd(a_i, a_j) = 1$ for all $i \neq j$. Let p be a prime such that $p \nmid a_1 a_2 \cdots a_k$. If $x_0 = 0$, then the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$.

Example 1.3 Suppose that p_1, p_2, and p_3 are distinct primes. Let $(x_n)_{n\geq 0}$ be a linear recurrence sequence defined by the recurrence relation

$$x_n = (p_1 + p_2 + p_3)x_{n-1} - (p_1 p_2 + p_1 p_3 + p_2 p_3)x_{n-2} + (p_1 p_2 p_3)x_{n-3}$$

for $n \geq 3$, where $x_0 = 0$, and x_1 and x_2 are any integers not both zero. The characteristic polynomial is equal to $(x-p_1)(x-p_2)(x-p_3)$. Hence, by Theorem 1.2, the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$ for all primes $p \neq p_1, p_2, p_3$.

In the following theorem, we consider kth order linear recurrence sequences whose characteristic polynomials have exactly two equal roots.

Theorem 1.4 *Let $(x_n)_{n\geq 0}$ be a linear recurrence of order $k \geq 3$ satisfying*

$$x_n = b_1 x_{n-1} + b_2 x_{n-2} + \cdots + b_k x_{n-k},$$

for all integers $n \geq k$, where $b_1, \ldots, b_k, x_0, \ldots, x_{k-1} \in \mathbb{Z}$, with $b_k \neq 0$. Suppose that the characteristic polynomial of $(x_n)_{n\geq 0}$ is given by

$$(x-a_1)^2(x-a_2)(x-a_3)\ldots(x-a_{k-1}),$$

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where $a_i \in \mathbb{Z}, a_i \neq a_j$ for $1 \leq i \neq j \leq k-1$, and $x_0 = x_1 = \cdots = x_{k-2} = 0, x_{k-1} = 1$. Let p be a prime such that $p \nmid a_1 a_2 \cdots a_{k-1}$. If $a_i \not\equiv a_j \pmod{p}$ for all $i \neq j$, then the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$.

Example 1.5 Given an integer a, let $(x_n)_{n\geq 0}$ be a linear recurrence sequence defined by the recurrence relation

$$x_n = 4ax_{n-1} - 5a^2 x_{n-2} + 2a^3 x_{n-3}$$

for $n \geq 3$, where $x_0 = x_1 = 0$ and $x_2 = 1$. The characteristic polynomial is equal to $(x-a)^2(x-2a)$. By Theorem 1.4, the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$ for all primes $p \nmid 2a$.

Theorem 1.6 *Let $(x_n)_{n\geq 0}$ be a linear recurrence of order $k \geq 2$ satisfying*

$$x_n = b_1 x_{n-1} + b_2 x_{n-2} + \cdots + b_k x_{n-k},$$

for all integers $n \geq k$, where $b_1, \ldots, b_k, x_0, \ldots, x_{k-1} \in \mathbb{Z}$, with $b_k \neq 0$. Suppose that the characteristic polynomial of $(x_n)_{n\geq 0}$ is given by $(x-a)^k$, where $a \in \mathbb{Z}$, and $x_0 = x_1 = \cdots = x_{k-2} = 0, x_{k-1} = 1$. If p is a prime such that $p \nmid a$, then the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$.

Remark 1.7 Let $a \in \mathbb{Z}$. Consider the kth order linear recurrence sequence $(x_n)_{n\geq 0}$ generated by the recurrence relation

$$x_n = \binom{k}{1} a x_{n-1} - \binom{k}{2} a^2 x_{n-2} + \cdots + (-1)^{k-1} \binom{k}{k} a^k x_{n-k}$$

for $n \geq k$, where $x_0 = \cdots = x_{k-2} = 0, x_{k-1} = 1$. Then, the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$ for all primes p not dividing a. This generalizes [2, Theorem 1.9] for the case $k = 2$.

Note that a linear recurrence sequence generated by a relation of the above form may not always have a dense quotient set in $\mathbb{Q}_p$. For example, consider the pth order linear recurrence sequence (x_n) generated by the recurrence relation

$$x_n = \binom{p}{1} a x_{n-1} - \binom{p}{2} a^2 x_{n-2} + \cdots + (-1)^{p-1} \binom{p}{p} a^p x_{n-p}$$

for $n \geq p$, where the initial values $x_0, \ldots, x_{p-1} \in \mathbb{Z}\setminus\{0\}$ have the same p-adic valuation. Then, the quotient set of (x_n) is not dense in $\mathbb{Q}_p$ which follows from [2, Theorem 1.10].

In case of third- order recurrence sequences, we prove the following result where we do not need to fix all the initial values.

Theorem 1.8 *Let $(x_n)_{n\geq 0}$ be a third -order linear recurrence sequence given by*

$$x_n = b_1 x_{n-1} + b_2 x_{n-2} + b_3 x_{n-3},$$

for all integers $n \geq 3$, where $b_1, b_2, b_3, x_0, x_1, x_2 \in \mathbb{Z}$, with $b_3 \neq 0$. Suppose that the characteristic polynomial of $(x_n)_{n\geq 0}$ is given by $(x-a)(x-b)(x-c)$, where $a, b, c \in \mathbb{Z}$. Let p be a prime such that $p \nmid abc$. Then, the following hold.

(a) *Suppose that* $a = b = c$. *If* $p|x_0$ *and* $p \nmid 4ax_1 - x_2 - 3a^2x_0$, *then the quotient set of* $(x_n)_{n\geq 0}$ *is dense in* $\mathbb{Q}_p$. *Moreover, if* $x_0 = 0$, *then the quotient set of* $(x_n)_{n\geq 0}$ *is dense in* $\mathbb{Q}_p$ *if and only if* $4ax_1 \neq x_2$.

(b) *Suppose that* $a = c \neq b$. *If* $p|x_0$ *and* $p \nmid (a-b)(x_2 - x_1(a+b) + x_0ab)$, *then the quotient set of* $(x_n)_{n\geq 0}$ *is dense in* $\mathbb{Q}_p$.

Example 1.9 Let $a \in \mathbb{Z}$ be such that $p \nmid a$, and let $(x_n)_{n\geq 0}$ be a linear recurrence sequence defined by the recurrence relation

$$x_{n+1} = 3ax_n - 3a^2x_{n-1} + a^3x_{n-2}$$

for $n \geq 2$, where $x_0 = 0$, and x_1 and x_2 are any integers satisfying $\gcd(4a, x_2) = 1$. Then, by Theorem 1.8(a), the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$.

2 Preliminaries

Let r be a nonzero rational number. Given a prime number p, r has a unique representation of the form $r = \pm p^k a/b$, where $k \in \mathbb{Z}$, $a, b \in \mathbb{N}$ and $\gcd(a, p) = \gcd(p, b) = \gcd(a, b) = 1$. The p-adic valuation of r is defined as $v_p(r) = k$ and its p-adic absolute value is defined as $\|r\|_p = p^{-k}$. By convention, $v_p(0) = \infty$ and $\|0\|_p = 0$. The p-adic metric on $\mathbb{Q}$ is $d(x, y) = \|x - y\|_p$. The field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to the p-adic metric. The p-adic absolute value can be extended to a finite normal extension $\mathbb{K}$ over $\mathbb{Q}_p$ of degree n. For $\alpha \in \mathbb{K}$, define $\|\alpha\|_p$ as the nth root of the determinant of the matrix of linear transformation from the vector space $\mathbb{K}$ over $\mathbb{Q}_p$ to itself defined by $x \mapsto \alpha x$ for all $x \in \mathbb{K}$. Also, $v_p(\alpha)$ is the unique rational number satisfying $\|\alpha\|_p = p^{-v_p(\alpha)}$. The ring of integers of $\mathbb{K}$, denoted by $\mathcal{O}$, is defined as the set of all elements in $\mathbb{K}$ with p-adic absolute value less than or equal to one. A function $f : \mathcal{O} \to \mathcal{O}$ is called analytic if there exists a sequence $(a_n)_{n\geq 0}$ in $\mathcal{O}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for all $z \in \mathcal{O}$.

We recall definitions of p-adic exponential and logarithmic function. For $a \in \mathbb{K}$ and $r > 0$, we denote $\mathcal{D}(a, r) := \{z \in \mathbb{K} : \|z - a\|_p < r\}$. Let $\rho = p^{-1/(p-1)}$.

For $z \in \mathcal{D}(0, \rho)$, the p-adic exponential function is defined as

$$\exp_p(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The derivative is given by $\exp_p'(z) = \exp_p(z)$. For $\mathcal{D}(1, \rho)$, the p-adic logarithmic function is defined as

$$\log_p(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(z-1)^n}{n}.$$

For $z \in \mathcal{D}(1, \rho)$, we have $\exp_p(\log_p(z)) = z$. If $\mathbb{K}$ is unramified and $p \neq 2$, then $\mathcal{D}(0, \rho) = \mathcal{D}(0, 1)$ and $\mathcal{D}(1, \rho) = \mathcal{D}(1, 1)$. More properties of these functions can be found in [13].

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Next, we state a result for analytic functions which will be used in the proofs of our theorems.

Theorem 2.1 [13, Hensel's lemma] *Let $f : \mathcal{O} \to \mathcal{O}$ be analytic. Let $b_0 \in \mathcal{O}$ be such that $\|f(b_0)\|_p < 1$ and $\|f'(b_0)\|_p = 1$. Then there exists a unique $b \in \mathcal{O}$ such that $f(b) = 0$ and $\|b - b_0\|_p < \|f(b_0)\|_p$.*

Note that in [13], Gouvêa states Hensel's lemma for polynomials with coefficients in $\mathcal{O}$. However, Hensel's lemma is also true and follows similarly for functions given by power series with coefficients in the ring $\mathcal{O}$. We will only be considering $\mathbb{K} = \mathbb{Q}_p$ throughout this article. The following results are useful in proving denseness of quotient sets.

Theorem 2.2 [20, Corollary 1.3] *Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be an analytic function with a simple zero in $\mathbb{Z}_p$. Then, $R(f(\mathbb{N}))$ is dense in $\mathbb{Q}_p$.*

Lemma 2.3 [10, Lemma 2.1] *If S is dense in $\mathbb{Q}_p$, then for each finite value of the p-adic valuation, there is an element of S with that valuation.*

3 Proof of the theorems

In the proofs, we will use certain representation of the nth term of linear recurrence sequence in terms of the roots of the characteristic polynomial. More details on such representations can be found in [23].

Proof of Theorem 1.2 For $n \geq 0$, the nth term of the sequence (x_n) is given by

$$x_n = c_0 a_1^n + c_1 a_2^n + \cdots + c_{k-1} a_k^n,$$

where

$$C = \begin{bmatrix} c_0 & c_1 & \dots & c_{k-1} \end{bmatrix}^t$$

is given by $C = \frac{1}{\det(A)} \operatorname{adj}(A) \cdot X_0$, where

$$X_0 = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{k-1} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_k \\ a_1^2 & a_2^2 & \dots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{k-1} & a_2^{k-1} & \dots & a_k^{k-1} \end{bmatrix}.$$

We define a function f as

$$f(z) := \det(A) \left[c_0 \exp_p(z \log_p a_1^{p-1}) + \cdots + c_{k-1} \exp_p(z \log_p a_k^{p-1}) \right].$$

Since $p \nmid a_1 a_2 \dots a_k$, f is defined for all $z \in \mathbb{Z}_p$ and $f(n) = \det(A) x_{n(p-1)}$ for all $n \in \mathbb{Z}_{\geq 0}$. Moreover, $\mathbb{Z}_{\geq 0}$ is dense in $\mathbb{Z}_p$. Therefore, f is an analytic function from $\mathbb{Z}_p$ to $\mathbb{Z}_p$. We have,

$$f(0) = \det(A)(c_0 + c_1 + \cdots + c_{k-1}) = \det(A) x_0 = 0$$

and

$$f'(0) = \det(A)(c_0 \log_p a_1^{p-1} + c_1 \log_p a_2^{p-1} + \cdots + c_{k-1} \log_p a_k^{p-1}).$$

Suppose that $f'(0) = 0$. Since $\gcd(a_i, a_j) = 1$ for all $i \neq j$, therefore, $a_1^{p-1}, \ldots, a_k^{p-1}$ are multiplicatively independent i.e, $(a_1^{p-1})^{u_1}(a_2^{p-1})^{u_2} \ldots (a_k^{p-1})^{u_k} = 1$ for some integers $u_1, u_2, \ldots, u_k$ only if $u_1 = u_2 = \cdots = u_k = 0$. Hence,

$$\log_p a_1^{p-1}, \log_p a_2^{p-1}, \ldots, \log_p a_k^{p-1}$$

are linearly independent over $\mathbb{Z}$. Thus, if $f'(0) = 0$ then $c_0 = c_1 = \cdots = c_{k-1} = 0$ which is not possible. Hence, $f'(0)$ is nonzero. Therefore, 0 is a simple zero of f in $\mathbb{Z}_p$. By Theorem 2.2, $R(f(\mathbb{N})) = R((x_{n(p-1)}))$ is dense in $\mathbb{Q}_p$. Hence, the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$. ∎

Proof of Theorem 1.4 The nth term of the sequence is given by

$$\begin{aligned} x_n &= a_1^n(c_0 + c_1 n) + c_2 a_2^n + c_3 a_3^n + \cdots + c_{k-1} a_{k-1}^n \\ &= a_1^n(c_0 + c_1 n + c_2(a_2 a_1^{-1})^n + c_3(a_3 a_1^{-1})^n + \cdots + c_{k-1}(a_{k-1} a_1^{-1})^n), \end{aligned}$$

where

$$C = \begin{bmatrix} c_0 & c_1 & \ldots & c_{k-1} \end{bmatrix}^t$$

is given by $C = \frac{1}{\det(A)} \operatorname{adj}(A) \cdot X_0$, where

$$X_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 1 & \ldots & 1 \\ a_1 & a_1 & a_2 & \ldots & a_{k-1} \\ a_1^2 & 2a_1^2 & a_2^2 & \ldots & a_{k-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{k-1} & (k-1)a_1^{k-1} & a_2^{k-1} & \ldots & a_{k-1}^{k-1} \end{bmatrix}.$$

We define an analytic function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ as

$$\begin{aligned} f(z) := \det(A) \exp_p(z \log_p(a_1)^{p-1})(c_0 + c_1 z(p-1) + c_2 \exp_p(z \log_p(a_2 a_1^{-1})^{p-1}) \\ + \cdots + c_{k-1} \exp_p(z \log_p(a_{k-1} a_1^{-1})^{p-1})). \end{aligned}$$

Then, $f(n) = \det(A) x_{n(p-1)}$ for all $n \in \mathbb{Z}_{\geq 0}$. Also, we have

$$f(0) = \det(A)(c_0 + c_2 + \cdots + c_{k-1}) = \det(A) x_0 = 0$$

and

$$\begin{aligned} f'(0) &= \det(A)(c_1(p-1) + c_2 \log_p(a_2 a_1^{-1})^{p-1} + \cdots + c_{k-1} \log_p(a_{k-1} a_1^{-1})^{p-1} \\ &\quad + (c_0 + c_2 + \cdots + c_{k-1}) \log_p(a_1)^{p-1}) \\ &= \det(A)(c_1(p-1) + c_2 \log_p(a_2 a_1^{-1})^{p-1} + \cdots + c_{k-1} \log_p(a_{k-1} a_1^{-1})^{p-1}). \end{aligned}$$

We find that $\det(A) c_1 = (-1)^{k+1} \prod_{1 \leq i < j \leq (k-1)} (a_i - a_j)$. By the hypothesis, we have $p \nmid \det(A) c_1$. Using the definition of $\log_p(z)$, we obtain that p divides $\log_p(a_i a_1^{-1})^{p-1}$ for $2 \leq i \leq k-1$. Therefore, $p \nmid f'(0)$ which implies $f'(0)$ is nonzero. Hence, 0 is a

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simple zero of f in $\mathbb{Z}_p$. By Theorem 2.2, $R(f(\mathbb{N})) = R(x_{n(p-1)})$ is dense in $\mathbb{Q}_p$. Hence, the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$. ■

Proof of Theorem 1.6 The nth term of the sequence is given by

$$x_n = a^n(c_0 + c_1 n + \cdots + c_{k-1}n^{k-1}),$$

where

$$C = \begin{bmatrix} c_0 & c_1 & \ldots & c_{k-1} \end{bmatrix}^t$$

is given by $C = \frac{1}{\det(A)}\operatorname{adj}(A)\cdot X_0$, where

$$X_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ a & a & a & \ldots & a \\ a^2 & 2a^2 & 2^2a^2 & \ldots & 2^{k-1}a^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{k-1} & (k-1)a^{k-1} & (k-1)^2a^{k-1} & \ldots & (k-1)^{k-1}a^{k-1} \end{bmatrix}.$$

We simplify $C = \frac{1}{\det(A)}\operatorname{adj}(A)\cdot X_0$ and obtain

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & 2^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (k-1) & \ldots & (k-1)^{k-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1/a^{k-1} \end{bmatrix}.$$

Next, we consider an analytic function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ defined as

$$f(z) := \det(A)\exp_p(z\log_p(a^{p-1}))(c_0 + c_1(p-1)z + c_2(p-1)^2z^2 + + \cdots + c_{k-1}(p-1)^{k-1}z^{k-1}).$$

Let

$$h(z) := \exp_p(z\log_p(a^{p-1}))$$

and

$$g(z) := \det(A)(c_0 + c_1(p-1)z + c_2(p-1)^2z^2 + \cdots + c_{k-1}(p-1)^{k-1}z^{k-1}).$$

We have $\|a^n\|_p = 1$ and $h(n) = a^{n(p-1)}$ for all positive integers n. Hence, $\|h(z)\|_p = 1$ for all $z \in \mathbb{Z}_p$. Therefore, $f(z) = 0$ if and only if $g(z) = 0$ for some $z \in \mathbb{Z}_p$. We have, $g(0) = \det(A)c_0 = \det(A)x_0 = 0$ and $g'(0) = \det(A)c_1(p-1)$. Using [22, Lemma 2.2], we find that

$$c_1 = \frac{(-1)^k}{a^{k-1}(k-1)}.$$

Thus, $c_1 \neq 0$ for all $k \geq 2$. Therefore, 0 is a simple zero of f in $\mathbb{Z}_p$. By Theorem 2.2, $R(f(\mathbb{N})) = R(x_{n(p-1)})$ is dense in $\mathbb{Q}_p$, which yields that the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$. ■

Proof of Theorem 1.8 We first prove part (a) of the theorem. For $n \geq 0$, the nth term of the sequence is given by the formula

$$x_n = a^n(c_0 + c_1 n + c_2 n^2),$$

where

$$c_0 = x_0,$$
$$c_1 = \frac{4ax_1 - x_2 - 3a^2 x_0}{2a^2},$$
$$c_2 = \frac{x_2 - 2ax_1 + a^2 x_0}{2a^2}.$$

We define a function f as

$$f(z) := 2a^2 \exp_p(z \log_p a^{p-1})(c_0 + c_1(p-1)z + c_2(p-1)^2 z^2).$$

Since $p \nmid a$, f is defined for all $z \in \mathbb{Z}_p$ and $f(n) = 2a^2 x_{n(p-1)}$ for all $n \in \mathbb{Z}_{\geq 0}$. Moreover, $\mathbb{Z}_{\geq 0}$ is dense in $\mathbb{Z}_p$. Therefore, f is an analytic function from $\mathbb{Z}_p$ to $\mathbb{Z}_p$. We have,

$$f(0) = 2a^2 c_0 = 2a^2 x_0 \equiv 0 \pmod{p}$$

and

$$f'(0) \equiv 2a^2 c_1(p-1) = 2a^2(4ax_1 - x_2 - 3a^2 x_0)(p-1) \not\equiv 0 \pmod{p}.$$

Therefore, by Hensel's lemma, f has a zero z_0 in $\mathbb{Z}_p$ such that $z_0 \equiv 0 \pmod{p}$. Since f has a power series expansion with p-adic integral coefficients, we have $f'(z_0) \equiv f'(0) \pmod{p}$. Hence, z_0 is a simple zero of f in $\mathbb{Z}_p$. Therefore, by Theorem 2.2, $R(f(\mathbb{N})) = R((x_{n(p-1)}))$ is dense in $\mathbb{Q}_p$. Hence, the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$.

Next, if $x_0 = 0$, then $c_0 = 0$ and $c_1 = \frac{4ax_1 - x_2}{2a^2}$. We have $f(0) = 0$. Suppose that $4ax_1 \neq x_2$. Then, $f'(0) \neq 0$ which implies that 0 is a simple zero of f. Therefore, by Theorem 2.2, the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$. Conversely, suppose that $4ax_1 = x_2$. This gives $c_1 = 0$, and hence $x_n = a^n c_2 n^2$. If $c_2 = 0$, then $x_n = 0$ for all n. If $c_2 \neq 0$, then the quotient set of $(x_n)_{n\geq 0}$ is equal to the quotient set of $\{a^n n^2 : n \in \mathbb{Z}_{\geq 0}\}$. Since $v_p(a^n n^2) = 2v_p(n)$, the p-adic valuation of these elements is even for all $n \in \mathbb{Z}_{>0}$. Therefore, by Lemma 2.3, the quotient set of $(x_n)_{n\geq 0}$ is not dense in $\mathbb{Q}_p$. This completes the proof of part (a) of the theorem.

Next, we prove part (b) of the theorem. For $n \geq 0$, the nth term of the sequence is given by

$$x_n = c_0 a^n + c_1 n a^n + c_2 b^n = a^n(c_0 + c_1 n + c_2(ba^{-1})^n),$$

where

$$c_0 = \frac{b^2 x_0 - 2abx_0 - x_2 + 2ax_1}{(b-a)^2},$$
$$c_1 = \frac{x_2 - x_1(a+b) + x_0 ab}{a(a-b)},$$
$$c_2 = \frac{x_2 - 2ax_1 + a^2 x_0}{(b-a)^2}.$$

Since $p \nmid ab(a-b)$, we can define an analytic function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ as

$$f(z) := \exp_p(z \log_p a^{p-1})(c_0 + c_1 z(p-1) + c_2 \exp_p(z \log_p (ba^{-1})^{p-1})),$$

which satisfies the equation $f(n) = x_{n(p-1)}$ for all $n \geq 0$. Now, we have

$$f(0) = c_0 + c_2 = x_0 \equiv 0 \pmod{p}$$

and

$$f'(0) = c_1(p-1) + c_2 \log_p (ba^{-1})^{p-1} + (c_0 + c_2) \log_p a^{p-1} \not\equiv 0 \pmod{p}.$$

Therefore, by Hensel's lemma, f has a zero z_0 in $\mathbb{Z}_p$ such that $z_0 \equiv 0 \pmod{p}$. Since f has a power series expansion with p-adic integral coefficients, we have $f'(z_0) \equiv f'(0) \pmod{p}$. Hence, z_0 is a simple zero of f in $\mathbb{Z}_p$. Therefore, by Theorem 2.2, $R(f(\mathbb{N})) = R(x_{n(p-1)})$ is dense in $\mathbb{Q}_p$. Hence, the quotient set of $(x_n)_{n \geq 0}$ is dense in $\mathbb{Q}_p$. ■

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