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On the Generalized Marcinkiewicz Integral Operators with Rough Kernels

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Abstract. A class of generalized Marcinkiewicz integral operators is introduced, and, under rather weak conditions on the integral kernels, the boundedness of such operators on L^p and Triebel–Lizorkin spaces is established.

1 Introduction

As is well known, the Marcinkiewicz integral is an important special case of the Littlewood–Paley–Stein functions and plays a key role in harmonic analysis. One can consult [1–3, 6, 12, 14, 16, 17], among numerous references, for its development and applications.

In this note, we will study a class of generalized Marcinkiewicz integral operators and we shall be primarily concerned with the two-parameter case. As for the one-parameter and multiple-parameter cases, we shall only present the corresponding results, since they can be handled similarly (see Section 5).

Let $\mathbb{R}^N (N = m \text{ or } n)$, $N \geq 2$, be the N-dimensional Euclidean space and S^{N-1} be the unit sphere in $\mathbb{R}^N$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. For nonzero points $x \in \mathbb{R}^N$, we denote $x' = x/|x|$. For $m \geq 2$, $n \geq 2$, let Ω be homogeneous of degree zero, integrable on $S^{m-1} \times S^{n-1}$ and satisfy

$$\int_{S^{m-1}} \Omega(x', y')d\sigma(x') = \int_{S^{n-1}} \Omega(x', y')d\sigma(y') = 0. \tag{1.1}$$

We define the convolution operator on $\mathbb{R}^m \times \mathbb{R}^n$

$$\phi_{s,t} * f(x, y) = \frac{1}{2^{s+t}} \iint_{|u| \leq 2^s,\, |v| \leq 2^t} \frac{\Omega(u', v')}{|u|^{m-1}|v|^{n-1}} f(x - u, y - v)dudv.$$

Then for $1 < q \leq \infty$, the integral operators of Marciniewicz type $\mu_{\Omega,q}$ are defined by

$$\mu_{\Omega,q}(f)(x, y) := \Big(\iint_{\mathbb{R}^2} |\phi_{s,t} * f(x, y)|^q dsdt \Big)^{1/q}.$$

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It is well known that $\mu_{\Omega,2}$ is the Marcinkiewicz integral operator on the product space $\mathbb{R}^m \times \mathbb{R}^n$, which has been studied by many authors. For a sampling of past studies, see [1,3,4,6,7,16,17] et al. In particular, Al-Qassem, Al-Salman, Cheng, and Pan [1] established the following theorem.

Theorem A *Let $1 < p < \infty$. If $\Omega \in L(\log^+ L)(S^{m-1} \times S^{n-1})$, then $\mu_{\Omega,2}$ is bounded on $L^p(S^{m-1} \times S^{n-1})$.*

It is natural to ask whether $\Omega \in L(\log^+ L)^{2/q}(S^{m-1} \times S^{n-1})$ is sufficient to imply the L^p-boundedness of $\mu_{\Omega,q}$ for $1 < q \le \infty$, $1 < p < \infty$. The main purpose of this paper is to address this question. Before stating our main results, we recall the definitions of Triebel–Lizorkin spaces on $\mathbb{R}^m \times \mathbb{R}^m$ (see [5] or [15]).

Let $U \in C^\infty(\mathbb{R}^m)$ and $V \in C^\infty(\mathbb{R}^n)$ satisfy

$$\operatorname{supp}(U) \subset \{x \in \mathbb{R}^m : 1/2 < |x| \le 2\}, \quad \operatorname{supp}(V) \subset \{y \in \mathbb{R}^n : 1/2 < |y| \le 2\}$$
$$\text{and} \quad U(x) > c > 0, \quad V(y) > c > 0$$

if $3/5 \le |x|, |y| \le 5/3$. Let Φ, Ψ be the Fourier transform of U and V, respectively. For $1 < p, q < \infty$, the Triebel–Lizorkin space $\dot{F}_p^{0,q}(\mathbb{R}^m \times \mathbb{R}^n)$ is the set of all distribution f on $\mathbb{R}^m \times \mathbb{R}^n$ such that

$$(1.2) \qquad \|f\|_{\dot{F}_p^{0,q}(\mathbb{R}^m\times\mathbb{R}^n)} := \Big\| \Big(\iint_{\mathbb{R}^2} |(\Phi_s \otimes \Psi_t) * f|^q ds dt \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} < \infty,$$

where $\Phi_s(x) = 2^{-ms}\Phi(2^{-s}x)$, $\Psi_t(y) = 2^{-nt}\Psi(2^{-t}y)$. Employing the ideas in [13], we know that (also see [15]):

$$\big(\dot{F}_p^{0,q}(\mathbb{R}^m \times \mathbb{R}^n)\big)^* = \dot{F}_{p'}^{0,q'}(\mathbb{R}^m \times \mathbb{R}^n), \quad 1/q' + 1/q = 1 = 1/p' + 1/p,$$
$$\dot{F}_p^{0,2}(\mathbb{R}^m \times \mathbb{R}^n) = L^p(\mathbb{R}^m \times \mathbb{R}^n).$$

Now we can formulate our main results as follows.

Theorem 1.1 *Let $2 \le q \le \infty$, $1 < p < \infty$. If $\Omega \in L(\log^+ L)^{2/q}(S^{m-1} \times S^{n-1})$, then*

$$\big\|\mu_{\Omega,q}(f)\big\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} \le C_{p,q}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)}.$$

Theorem 1.2 *Let $1 < q < 2$, $1 < p < \infty$. If $\Omega \in L(\log^+ L)^{2/q+\varepsilon}(S^{m-1} \times S^{n-1})$ for any $\varepsilon > 0$, then*

$$\big\|\mu_{\Omega,q}(f)\big\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} \le C_{p,q}\|f\|_{\dot{F}_p^{0,q}(\mathbb{R}^m\times\mathbb{R}^n)}.$$

To prove Theorem 1.2, we will use the following theorem, which is itself interesting.

Theorem 1.3 *Let $1 < q \le \infty$, $1 < p < \infty$. If $\Omega \in L(\log^+ L)^2(S^{m-1} \times S^{n-1})$, then*

$$\big\|\mu_{\Omega,q}(f)\big\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} \le C_{p,q}\|f\|_{\dot{F}_p^{0,q}(\mathbb{R}^m\times\mathbb{R}^n)}.$$

Remark 1.4 Obviously, Theorem A is the special case of Theorem 1.1 for $q = 2$. It is an interesting problem whether the ε in Theorem 1.2 can be removed, or the condition $\Omega \in L(\log^+ L)^2(S^{m-1} \times S^{n-1})$ in Theorem 1.3 can be replaced by $\Omega \in L(\log^+ L)^{2/q}(S^{m-1} \times S^{n-1})$.

This paper is organized as follows. The proof of Theorem 1.1 will be given in Section 2. After proving Theorem 1.3 in Section 3, we will prove Theorem 1.2 in Section 4. Finally, some concluding remarks will be given in Section 5.

Throughout this paper, we always use the letter C to denote positive constants that may vary at each occurrence but are independent of the essential variables.

2 Proof of Theorem 1.1

Following the proof in [1], for $k \in \mathbb{N}$, let

$$E_k := \{(x, y) \in S^{m-1} \times S^{n-1} : 2^{k-1} \leq |\Omega(x', y')| \leq 2^k\},$$

and

$$\Omega_k(x', y') = \Omega(x', y')\chi_{E_k}(x', y') - \int_{S^{m-1}} \Omega(x', y')\chi_{E_k}(x', y')d\sigma(x') - \int_{S^{n-1}} \Omega(x', y')\chi_{E_k}(x', y')d\sigma(y') + \iint_{E_k} \Omega(x', y')d\sigma(x')d\sigma(y').$$

Denote $D = \{k \in \mathbb{N} : |E_k| > 2^{-4k}\}$, where $|E_k|$ is the Lebesgue measure of E_k for $k \in \mathbb{N}$. Set

$$\Omega_0(x', y') = \Omega(x', y') - \sum_{k\in D} \Omega_k(x', y').$$

It is easy to see that for each $k \in D \cup \{0\}$, Ω_k satisfies (1.1) and $\Omega_0 \in L^2(S^{m-1} \times S^{n-1})$. Thus

$$\phi_{s,t} * f(x, y) = \phi_{s,t,0} * f(x, y) + \sum_{k\in D} \phi_{s,t,k} * f(x, y),$$

where

$$\phi_{s,t,k} * f(x, y) = \frac{1}{2^{s+t}} \iint_{|u|\leq 2^s,\ |v|\leq 2^t} \frac{\Omega_k(x', y')}{|u|^{m-1}|v|^{n-1}} f(x - u, y - v)dudv$$

for $k \in D \cup \{0\}$. Consequently,

$$\big\| \mu_{\Omega,q}(f)\big\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} \leq \big\| \|\phi_{s,t,0}*f\|_{L^q(\mathbb{R}^2)}\big\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} + \sum_{k\in D}\big\| \|\phi_{s,t,k}*f\|_{L^q(\mathbb{R}^2)}\big\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)}.$$

Note that $\Omega_0 \in L^2(S^{m-1} \times S^{n-1})$, it is easy to treat $\phi_{s,t,0} * f$. Thus, without loss of generality, we write

$$\big\| \mu_{\Omega,q}(f)\big\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} \leq \sum_{k\in D}\big\| \|\phi_{s,t,k} * f\|_{L^q(\mathbb{R}^2)}\big\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{2.1}$$

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Let $A_k = \|\Omega\chi_{E_k}\|_{L^1(S^{m-1}\times S^{n-1})}$. Then

$$\sum_{k\in D} k^{2/q} A_k \leq \|\Omega\|_{L(\log^+ L)^{2/q}(S^{m-1}\times S^{n-1})}. \tag{2.2}$$

By [1, (3.11)], we have

$$\big\| \|\phi_{s,t,k} * f\|_{L^2(\mathbb{R}^2)} \big\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} \leq CkA_k \|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)}. \tag{2.3}$$

On the other hand, we have

$$\big|\phi_{s,t,k} * f(x,y)\big| \leq \iint_{S^{m-1}\times S^{n-1}} |\Omega_k(u',v')| M_{u',v'}(f)(x,y)\, d\sigma(u')\, d\sigma(v'),$$

where

$$M_{u',v'}(f)(x,y) = \sup_{R_1,R_2>0} \frac{1}{R_1R_2}\int_0^{R_1}\int_0^{R_2} |f(x-r_1u',\, y-r_2v')|\, dr_1\, dr_2.$$

Applying [11, Proposition 2, pp. 477-478] iteratively (also see [3, 9]), we know that

$$\big\|M_{u',v'}(f)\big\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)}, \quad 1 < p \leq \infty,$$

where C is independent of u' and v'. Then it is easy to see that

$$\big\| \|\phi_{s,t,k} * f\|_{L^\infty(\mathbb{R}^2)} \big\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} \leq CA_k \|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)}. \tag{2.4}$$

Thus Theorem 1.1 follows from (2.1), (2.2), and an interpolation between (2.3) and (2.4).

3 Proof of Theorem 1.3

Choose two radial functions $\Phi \in \mathcal{S}(\mathcal{R}^m)$, $\Psi \in \mathcal{S}(\mathcal{R}^n)$ as in the definition of the Triebel–Lizorkin spaces such that the values of their Fourier transforms $\widehat{\Phi}$ and $\widehat{\Psi}$ are between 0 and 1 and satisfy

$$\int_{\mathcal{R}} \widehat{\Phi}(2^s)\,ds = 1 = \int_{\mathcal{R}} \widehat{\Psi}(2^t)\,dt, \quad \widehat{\Phi}(x), \quad \widehat{\Psi}(y) > c > 0 \text{ if } 5/3 \leq |x|, \quad |y| \leq 5/3,$$

$$\operatorname{supp}(\widehat{\Phi}) \subseteq \{x \in \mathcal{R}^m : 1/2 < |x| \leq 2\}, \quad \operatorname{supp}(\widehat{\Psi}) \subseteq \{y \in \mathcal{R}^n : 1/2 < |y| \leq 2\}.$$

It is easy to check that for any test function $f \in \mathcal{S}(\mathcal{R}^m \times \mathcal{R}^n)$,

$$f \cong \iint_{\mathcal{R}^2} (\Phi_s \otimes \Psi_t) * f\,ds\,dt = k^2 \iint_{\mathcal{R}^2} (\Phi_{ks} \otimes \Psi_{kt}) * f\,ds\,dt, \quad \forall\, k \in \mathcal{N}, \tag{3.1}$$

where $\Phi_s(x) = 2^{-ms}\Phi(2^{-s}x)$, $\Psi_t(y) = 2^{-nt}\Psi(2^{-t}y)$. And then by (1.2), we have

$$\|f\|_{\dot{F}_p^{0,q}(\mathcal{R}^m\times\mathcal{R}^n)} = k^{2/q} \Big\| \Big(\iint_{\mathcal{R}^2} |(\Phi_{ks}\otimes\Psi_{kt}) * f|^q\,ds\,dt \Big)^{1/q} \Big\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)}. \tag{3.2}$$

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Let E_k, Ω_k, A_k, D be the same as in the proof of Theorem 1.1. For each $k \in D$, define the family of measures $\nu^{(k)} = \{\nu_{s,t,k} : s,\, t \in \mathcal{R}\}$ by

$$\iint_{\mathcal{R}^m \times \mathcal{R}^n} f d\nu_{s,t,k} := \frac{1}{2^{k(s+t)}} \iint_{|x| \le 2^{ks},\, |y| \le 2^{kt}} \frac{\Omega_k(x', y')}{|x|^{m-1}|y|^{n-1}} f(x, y) dx dy,$$

which implies

$$\nu_{s,t,k} * f(x, y) = \frac{1}{2^{k(s+t)}} \iint_{|u| \le 2^{ks},\, |v| \le 2^{kt}} \frac{\Omega_k(u', v')}{|u|^{m-1}|v|^{n-1}} f(x-u, y-v) du dv. \tag{3.3}$$

Following the same arguments as in [1], it is not difficult to show that $\|\nu_{s,t,k}\| \le CA_k$, and

$$\begin{aligned} \big|\widehat{\nu}_{s,t,k}(\xi, \eta)\big| \le CA_k \min\big\{ &|2^{ks}\xi|^{1/6k}|2^{kt}\eta|^{1/6k}, |2^{ks}\xi|^{1/6k}|2^{kt}\eta|^{-1/6k}, \\ &|2^{ks}\xi|^{-1/6k}|2^{kt}\eta|^{1/6k}, |2^{ks}\xi|^{-1/6k}|2^{kt}\eta|^{-1/6k}\big\}, \end{aligned} \tag{3.4}$$

$$\big\| (\nu^{(k)})^*(f) \big\|_{L^p(\mathcal{R}^m \times \mathcal{R}^n)} \le CA_k \|f\|_{L^p(\mathcal{R}^m \times \mathcal{R}^n)}, \qquad 1 < p \le \infty, \tag{3.5}$$

where $\|\nu_{s,t,k}\|$ denotes the total variation of $\nu_{s,t,k}$, and

$$(\nu^{(k)})^*(f)(x, y) = \sup_{s,t \in \mathcal{R}} |\nu_{s,t,k}| * f(x, y).$$

Also, by (3.3) and the Minkowski inequality, we have

$$\begin{aligned} \mu_{\Omega,q}(f)(x, y) \le &\Big(\iint_{\mathcal{R}^2} |\nu_{s,t,0} * f(x, y)|^q \, ds dt \Big)^{1/q} \\ &+ C \sum_{k \in D} k^{2/q} \Big(\iint_{\mathcal{R}^2} |\nu_{s,t,k} * f(x, y)|^q \, ds dt \Big)^{1/q}, \end{aligned}$$

where

$$\nu_{s,t,0} * f(x, y) = \frac{1}{2^{s+t}} \iint_{|u| \le 2^s,\, |v| \le 2^t} \frac{\Omega_0(u', v')}{|u|^{m-1}|v|^{n-1}} f(x-u, y-v) du dv.$$

Set $F_k(f)(x, y, s, t) = \nu_{s,t,k} * f(x, y)$ for $k \in D \cup \{0\}$. Then, we have

$$\begin{aligned} \mu_{\Omega,q}(f)(x, y) \le &\big\| F_0(f)(x, y, \cdot\,, \cdot\,) \big\|_{L^q(\mathcal{R}^2)} \\ &+ C \sum_{k \in D} k^{2/q} \big\| F_k(f)(x, y, \cdot\,, \cdot\,) \big\|_{L^q(\mathcal{R}^2)}. \end{aligned} \tag{3.6}$$

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It is easy to treat $F_0(f)(x,y,s,t)$. Thus, we first consider $F_k(f)(x,y,s,t)$ for $k \in D$. By (3.1), we can write

$$F_k(f)(x,y,s,t) = k^2 \iint_{\mathcal{R}^2} \left(\Phi_{k(s+s')} \otimes \Psi_{k(t+t')}\right) * \nu_{s,t,k} * f(x,y) ds' dt'.$$

Consequently,

$$\left\| F_k(f)(x,y,\cdot,\cdot)\right\|_{L^q(\mathcal{R}^2)} \le k^2 \iint_{\mathcal{R}^2} I_{s',t'}(f)(x,y) ds' dt',$$

where

$$I_{s',t'}(f)(x,y) = \Big(\iint_{\mathcal{R}^2} \left|(\Phi_{k(s+s')} \otimes \Psi_{k(t+t')}) * \nu_{s,t,k} * f(x,y)\right|^q ds dt\Big)^{1/q}.$$

Let $L_{s',t'}(f)(x,y,s,t) = (\Phi_{k(s+s')} \otimes \Psi_{k(t+t')}) * \nu_{s,t,k} * f(x,y)$. Then

$$I_{s',t'}(f)(x,y) = \left\| L_{s',t'}(f)(x,y,\cdot,\cdot)\right\|_{L^q(\mathcal{R}^2)}.$$

Therefore, it follows from the generalized Minkowski inequality that

$$\begin{aligned}(3.7)\quad \big\| \|F_k(f)\|_{L^q(\mathcal{R}^2)}\big\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} &\le k^2 \iint_{\mathcal{R}^2} \left\| I_{s',t'}(f)\right\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} ds' dt' \\ &= k^2 \iint_{\mathcal{R}^2} \big\| \|L_{s',t'}(f)\|_{L^q(\mathcal{R}^2)}\big\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} ds' dt'.\end{aligned}$$

In what follows, we estimate $\|\|L_{s',t'}(f)\|_{L^q(\mathcal{R}^2)}\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)}$ for different $1 < p, q < \infty$.

By (3.5) and (3.2), it is easy to see that

$$\begin{aligned}(3.8)\quad & \big\| \|L_{s',t'}(f)\|_{L^q(\mathcal{R}^2)}\big\|_{L^q(\mathcal{R}^m\times\mathcal{R}^n)} \\ &= \big\| \|L_{s',t'}(f)\|_{L^q(\mathcal{R}^m\times\mathcal{R}^n)}\big\|_{L^q(\mathcal{R}^2)} \\ &\le CA_k \Big\| \Big(\iint_{\mathcal{R}^2} \left|\Phi_{k(s+s')} \otimes \Psi_{k(t+t')} * f\right|^q ds dt\Big)^{1/q}\Big\|_{L^q(\mathcal{R}^m\times\mathcal{R}^n)} \\ &\le Ck^{-2/q} A_k \|f\|_{\dot F_q^{0,q}(\mathcal{R}^m\times\mathcal{R}^n)}.\end{aligned}$$

On the other hand, the Plancherel theorem gives

$$\begin{aligned}&\big\| \|L_{s',t'}(f)\|_{L^2(\mathcal{R}^2)}\big\|^2_{L^2(\mathcal{R}^m\times\mathcal{R}^n)} \\ &\quad = \iint_{\mathcal{R}^2} \iint_{\mathcal{R}^m\times\mathcal{R}^n} \left|\widehat{\Phi}_{k(s+s')}(\xi)\widehat{\Psi}_{k(t+t')}(\eta)\right|^2 |\widehat{\nu}_{s,t,k}(\xi,\eta)|^2 |\widehat{f}(\xi,\eta)|^2 d\xi d\eta ds dt \\ &\quad \le C \iint_{\mathcal{R}^2} \iint_{B^{s',t'}_{s,t,k}} |\widehat{\nu}_{s,t,k}(\xi,\eta)|^2 |\widehat{f}(\xi,\eta)|^2 d\xi d\eta ds dt,\end{aligned}$$

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where

$$B_{s,t,k}^{s',t'} = \big\{ (\xi,\, \eta) \in \mathcal{R}^m \times \mathcal{R}^n : 2^{-k(s+s')-1} < |\xi| \\ \le 2^{-k(s+s')+1}, 2^{-k(t+t')-1} < |\eta| \le 2^{-k(t+t')+1} \big\}.$$

And it follows from (3.4) that, for $(\xi,\eta) \in B_{s,t,k}^{s',t'}$,

$$\begin{aligned}\big|\widehat{\nu}_{s,t,k}(\xi,\eta)\big| &\le CA_k \min\big\{ |2^{ks}\xi|^{1/6k}|2^{kt}\eta|^{1/6k}, |2^{ks}\xi|^{1/6k}|2^{kt}\eta|^{-1/6k}, \\ &\qquad |2^{ks}\xi|^{-1/6k}|2^{kt}\eta|^{1/6k}, |2^{ks}\xi|^{-1/6k}|2^{kt}\eta|^{-1/6k}\big\} \\ &\le CA_k \min\big\{ 2^{-(s'+t')/6},\, 2^{-(s'-t')/6},\, 2^{(s'-t')/6},\, 2^{(s+t')/6}\big\}.\end{aligned}$$

Thus

(3.9)
$$\begin{aligned}\big\|\|L_{s',t'}(f)\|_{L^2(\mathcal{R}^2)}\big\|_{L^2(\mathcal{R}^m\times\mathcal{R}^n)} &\le CA_k k^{-1} 2^{-(s'+t')/6}\|f\|_{L^2(\mathcal{R}^m\times\mathcal{R}^n)} \\ &\le CA_k k^{-1} 2^{-(s'+t')/6}\|f\|_{\dot F_2^{0,2}(\mathcal{R}^m\times\mathcal{R}^n)}, \quad s'>0,\, t'>0;\end{aligned}$$

(3.10)
$$\big\|\|L_{s',t'}(f)\|_{L^2(\mathcal{R}^2)}\big\|_{L^2(\mathcal{R}^m\times\mathcal{R}^n)} \le CA_k k^{-1} 2^{-(s'-t')/6}\|f\|_{\dot F_2^{0,2}(\mathcal{R}^m\times\mathcal{R}^n)}, \quad s'>0,\, t'<0;$$

(3.11)
$$\big\|\|L_{s',t'}(f)\|_{L^2(\mathcal{R}^2)}\big\|_{L^2(\mathcal{R}^m\times\mathcal{R}^n)} \le CA_k k^{-1} 2^{(s'-t')/6}\|f\|_{\dot F_2^{0,2}(\mathcal{R}^m\times\mathcal{R}^n)}, \quad s'<0,\, t'>0;$$

(3.12)
$$\big\|\|L_{s',t'}(f)\|_{L^2(\mathcal{R}^2)}\big\|_{L^2(\mathcal{R}^m\times\mathcal{R}^n)} \le CA_k k^{-1} 2^{(s'+t')/6}\|f\|_{\dot F_2^{0,2}(\mathcal{R}^m\times\mathcal{R}^n)}, \quad s'<0,\, t'<0.$$

If $p > q$, let $r = (p/q)' = p/(p-q)$. By duality, we can take a nonnegative $h \in L^r(\mathcal{R}^m\times\mathcal{R}^n)$ with $\|h\|_{L^r(\mathcal{R}^m\times\mathcal{R}^n)} = 1$ such that

$$\begin{aligned}&\big\|\,\|L_{s',t'}(f)\|_{L^q(\mathcal{R}^2)}\big\|^q_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} \\ &= \iint_{\mathcal{R}^m\times\mathcal{R}^n}\iint_{\mathcal{R}^2}\big|L_{s',t'}(f)(x,y)\big|^q ds dt h(x,y) dx dy \\ &= \iint_{\mathcal{R}^2}\iint_{\mathcal{R}^m\times\mathcal{R}^n}\big|L_{s',t'}(f)(x,y)\big|^q h(x,y) dx dy ds dt \\ &\le \iint_{\mathcal{R}^2}\iint_{\mathcal{R}^m\times\mathcal{R}^n}\big|\nu_{s,t,k} * \big(\Phi_{k(s+s')}\otimes\Psi_{k(t+t')}\big) * f(x,y)\big|^q h(x,y) dx dy ds dt \\ &\le C\|\nu_{s,t,k}\|^{q/q'}\iint_{\mathcal{R}^2}\iint_{\mathcal{R}^m\times\mathcal{R}^n}\big|\nu_{s,t,k}\big| * \big|\big(\Phi_{k(s+s')}\otimes\Psi_{k(t+t')}\big) * f\big|^q\end{aligned}$$

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$$\times (x,y)h(x,y)dxdydsdt$$

$$\le CA_k^{q/q'} \iint_{\mathcal{R}^2}\iint_{\mathcal{R}^m\times\mathcal{R}^n} \left|\left(\Phi_{k(s+s')}\otimes\Psi_{k(t+t')}\right) * f(x,y)\right|^q \times (\nu^{(k)})^*(\widetilde{h})(-x,-y)dxdydsdt$$

$$= CA_k^{q/q'} \iint_{\mathcal{R}^m\times\mathcal{R}^n}\iint_{\mathcal{R}^2} \left|\left(\Phi_{k(s+s')}\otimes\Psi_{k(t+t')}\right) * f(x,y)\right|^q dsdt \times (\nu^{(k)})^*(\widetilde{h})(-x,-y)dxdy,$$

where $\widetilde{h}(x,y) = h(-x,-y)$. By (3.5), we know that

$$\left\|(\nu^{(k)})^*(\widetilde{h})\right\|_{L^r(\mathcal{R}^m\times\mathcal{R}^n)} \le CA_k\|\widetilde{h}\|_{L^r(\mathcal{R}^m\times\mathcal{R}^n} \le CA_k\|h\|_{L^r(\mathcal{R}^m\times\mathcal{R}^n)} = CA_k.$$

Thus by Hölder's inequality, we get

(3.13)
$$\left\|\|L_{s',t'}(f)\|_{L^q(\mathcal{R}^2)}\right\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} \le CA_k^{1/q'}\left\|(\nu^{(k)})^*(\widetilde{h})\right\|^{1/q}_{L^r(\mathcal{R}^m\times\mathcal{R}^n)} \times \left\|\left(\iint_{\mathcal{R}^2}\left|\Phi_{k(s+s')}\otimes\Psi_{(k(t+t')} * f\right|^q dsdt\right)^{1/q}\right\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} \le CA_k k^{-2/q}\|f\|_{\dot{F}^{0,q}_p(\mathcal{R}^m\times\mathcal{R}^n)}.$$

Note that $q > p$ implies $p' > q'$, by duality, for all $g(x,y,s,t)$ satisfying

$$\left\|\|g\|_{L^{q'}(\mathcal{R}^2)}\right\|_{L^{p'}(\mathcal{R}^m\times\mathcal{R}^n)} = 1,$$

we have

$$(3.14)\quad \left|\langle L_{s',t,}(f),g\rangle\right| \le \left\|\left(\iint_{\mathcal{R}^2}\left|\left(\Phi_{k(s+s')}\otimes\Psi_{k(t+t')}\right) * f\right|^q dsdt\right)^{1/q}\right\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} \times \left\|\|F_k^*(g)\|_{L^{q'}(\mathcal{R}^2)}\right\|_{L^{p'}(\mathcal{R}^m\times\mathcal{R}^n)} = \left\|\|F_k^*(g)\|_{L^{q'}(\mathcal{R}^2)}\right\|_{L^{p'}(\mathcal{R}^m\times\mathcal{R}^n)} k^{-2/q}\|f\|_{\dot{F}^{0,q}_p(\mathcal{R}^m\times\mathcal{R}^n)},$$

where

$$F_k^*(g)(x,y,s,t) = \iint_{\mathcal{R}^m\times\mathcal{R}^n} \nu_{s,t,k}(u,v)g(x+u,y+v,s,t)dudv.$$

Let $\gamma = p'/q' > 1$ and let γ' be the dual exponent of γ. Then there is a positive

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function $h \in L^{\gamma'}(\mathcal{R}^m \times \mathcal{R}^n)$ with $\|h\|_{L^{\gamma'}(\mathcal{R}^m\times\mathcal{R}^n)} = 1$, such that

$$\begin{aligned}
&\big\| \|F_k^*(g)\|_{L^{q'}(\mathcal{R}^2)}\big\|^{q'}_{L^{p'}(\mathcal{R}^m\times\mathcal{R}^n)} \\
&= \iint_{\mathcal{R}^m\times\mathcal{R}^n}\iint_{\mathcal{R}^2}\Big|\iint_{\mathcal{R}^m\times\mathcal{R}^n}\nu_{s,t,k}(u,v)g(x+u,y+v,s,t)dudv\Big|^{q'}dsdt\, h(x,y)dxdy \\
&\le C\|\nu_{s,t,k}\|^{q'/q}\iint_{\mathcal{R}^m\times\mathcal{R}^n}\Big(\sup_{s,t\in\mathcal{R}}\iint_{\mathcal{R}^m\times\mathcal{R}^n}|\nu_{s,t,k}(u-x,v-y)|h(x,y)dxdy\Big) \\
&\quad\times\iint_{\mathcal{R}^2}|g(u,v,s,t)|^{q'}dsdtdudv \\
&\le CA_k^{q'/q}\iint_{\mathcal{R}^m\times\mathcal{R}^n}(\nu^{(k)})^*(h)(u,v)\iint_{\mathcal{R}^2}|g(u,v,s,t)|^{q'}dsdtdudv \\
&\le CA_k^{q'/q}\big\|(\nu^{(k)})^*(h)\big\|_{L^{\gamma'}(\mathcal{R}^m\times\mathcal{R}^n)}\Big(\iint_{\mathcal{R}^m\times\mathcal{R}^n}\Big(\iint_{\mathcal{R}^2}|g(x,y,s,t)|^{q'}dsdt\Big)^{\gamma}dudv\Big)^{1/\gamma} \\
&\le CA_k^{q'/q}A_k\|h\|_{L^{\gamma'}(\mathcal{R}^m\times\mathcal{R}^n)}\big\|\|g\|_{L^{q'}(\mathcal{R}^2)}\big\|^{q'}_{L^{p'}(\mathcal{R}^m\times\mathcal{R}^n)} \\
&= CA_k^{1+q'/q}.
\end{aligned}$$

This together with (3.14) shows that, for all $q > p$,

$$\Big\|\|L_{s',t'}(f)\|_{L^q(\mathcal{R}^2)}\Big\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} \le CA_k k^{-2/q}\|f\|_{\dot F_p^{0,q}(\mathcal{R}^m\times\mathcal{R}^n)}. \tag{3.15}$$

Applying an interpolation theorem to (3.8)–(3.13) and (3.15), we obtain a $\delta > 0$ such that

$$\Big\|\|L_{s',t'}(f)\|_{L^q(\mathcal{R}^2)}\Big\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} \le C2^{-\delta(|s'|+|t'|)}k^{-2/q}A_k\|f\|_{\dot F_p^{0,q}(\mathcal{R}^m\times\mathcal{R}^n)}. \tag{3.16}$$

Thus by (3.7) and (3.16), for any $k \in D$,

$$\begin{aligned}
\big\|\|F_k(f)\|_{L^q(\mathcal{R}^2)}\big\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} &\le k^2\iint_{\mathcal{R}^2}\big\|\|L_{s',t'}(f)\|_{L^q(\mathcal{R}^2)}\big\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)}ds'dt' \\
&\le Ck^{2-2/q}A_k\iint_{\mathcal{R}^2}2^{-\delta(|s'|+|t'|)}ds'dt'\|f\|_{\dot F_p^{0,q}(\mathcal{R}^m\times\mathcal{R}^n)} \\
&\le Ck^{2-2/q}A_k\|f\|_{\dot F_p^{0,q}(\mathcal{R}^m\times\mathcal{R}^n)}.
\end{aligned}$$

On the other hand, notice that $\|\Omega_0\|_{L^2(S^{m-1}\times S^{n-1})} < \infty$, by the similar and simpler arguments, we can get

$$\big\|\|F_0(f)\|_{L^q(\mathcal{R}^2)}\big\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} \le C\|f\|_{\dot F_p^{0,q}(\mathcal{R}^m\times\mathcal{R}^n)}.$$

Therefore, by (3.6) and (2.2)

$$\begin{aligned}\|\mu_{\Omega,q}(f)\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} &\le C\Big(1+\sum_{k\in D}k^2A_k\Big)\|f\|_{\dot F_p^{0,q}(\mathcal{R}^m\times\mathcal{R}^n)}\\ &\le C\big(1+\|\Omega\|_{L(\log^+L)^2(S^{m-1}\times S^{n-1})}\big)\|f\|_{\dot F_p^{0,q}(\mathcal{R}^m\times\mathcal{R}^n)}\\ &\le C\|f\|_{\dot F_p^{0,q}(\mathcal{R}^m\times\mathcal{R}^n)}.\end{aligned}$$

This completes the proof of Theorem 1.3. ■

4 Proof of Theorem 1.2

Following the same proof as in Theorem 1.1, we have

$$\big\|\|\phi_{s,t,k}*f\|_{L^2(\mathcal{R}^2)}\big\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} \le CkA_k\|f\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} \approx CkA_k\|f\|_{\dot F_p^{0,2}(\mathcal{R}^m\times\mathcal{R}^n)}. \tag{4.1}$$

On the other hand, it follows from Theorem 1.3 that for $1<q<2$, $1<p<\infty$,

$$\big\|\|\phi_{s,t,k}*f\|_{L^q(\mathcal{R}^2)}\big\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} \le C\|\Omega_k\|_{L(\log^+L)^2(S^{m-1}\times S^{n-1})}\|f\|_{\dot F_p^{0,2}(\mathcal{R}^m\times\mathcal{R}^n)}.$$

By the definition of Ω_k, it is easy to see that

$$\|\Omega_k\|_{L(\log^+L)^2(S^{m-1}\times S^{n-1})} \le Ck^2A_k,$$

which implies

$$\big\|\|\phi_{s,t,k}*f\|_{L^q(\mathcal{R}^2)}\big\|_{L^p(\mathcal{R}^m\times\mathcal{R}^n)} \le Ck^2A_k\|f\|_{\dot F_p^{0,p}(\mathcal{R}^m\times\mathcal{R}^n)}. \tag{4.2}$$

Taking q in (4.2) sufficiently close to 1 and interpolating between (4.1) and (4.2), we obtain the desired result and complete the proof of Theorem 1.2. ■

5 Concluding Remarks

We remark that our method also works for the one-parameter case. Precisely, let Ω denote a homogeneous function of degree zero on $\mathcal{R}^n$, which is integrable on the unit sphere S^{n-1} and satisfies

$$\int_{S^{n-1}}\Omega(y')d\sigma(y')=0,$$

where $d\sigma$ represents the normalized measure on S^{n-1}. For $n\ge 2$, we define the generalized Marcinkiewicz integral operator on $\mathcal{R}^n$ as follows

$$\mu_{\Omega,q}(f)(x)=\|\phi_t*f(x)\|_{L^q(\mathcal{R},dt)},$$

where

$$\phi_t*f(x)=\frac{1}{2^t}\int_{|y|\le 2^t}\frac{\Omega(y)}{|y|^{n-1}}f(x-y)dy.$$

Obviously, $\mu_{\Omega,2}$ is the classical Marcinkiewicz integral operator, which was first introduced and studied by Stein [12] and subsequently received the most attention (see [2, 8, 10, 14, 18, 19] among others for examples). In particular, the following theorem can be found in [2].

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Theorem B *Let* $1 < p < \infty$. *If* $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$, *then* $\mu_{\Omega,2}$ *is bounded on* $L^p(\mathcal{R}^n)$.

By similar arguments as in the proofs of Theorem 1.1 and 1.2, we have the following generalization of Theorem B.

Theorem 5.1 *Let* $2 \le q \le \infty$, $1 < p < \infty$. *If* $\Omega \in L(\log^+ L)^{1/q}(S^{n-1})$, *then* $\mu_{\Omega,q}$ *is bounded on* $L^p(\mathcal{R}^n)$.

Theorem 5.2 *Let* $1 < q < 2$, $1 < p < \infty$. *If* $\Omega \in L(\log^+ L)^{1/q+\varepsilon}(S^{n-1})$ *for any* $\varepsilon > 0$, *then*

$$\|\mu_{\Omega,q}(f)\|_{L^p(\mathcal{R}^n)} \le C\|f\|_{\dot{F}_p^{0,q}(\mathcal{R}^n)},$$

where $\dot{F}_p^{0,q}(\mathcal{R}^n)$ *is the Triebel–Lizorkin spaces on* $\mathcal{R}^n$ *(see* [13] *for the definition).*

Remark 5.3 We remark that the result of Theorem 5.1 is also valid for $\Omega \in H^1(S^{n-1})$, which is the Hardy space on S^{n-1}, by similar arguments as in the proof of Theorem 1.1 and using the result in [8]. But we do not know whether the result of Theorem 5.2 is also available for $\Omega \in H^1(S^{n-1})$, and whether the ε in Theorem 5.2 can be removed.

On the other hand, Theorems 1.1 to 1.3 can also be extended to the multiple parameter cases. Let $k \in \mathcal{N}$, $n_1, \dots, n_k \ge 2$ and $\Omega(x_1', \dots, x_k')$ be an integrable function on $S^{n_1-1} \times \cdots \times S^{n_k-1}$. Suppose that Ω satisfies the following cancellation condition

$$\int_{S^{n_j-1}} \Omega(x_1', \dots, x_k') d\sigma(x_j') = 0, \qquad j = 1, 2, \dots, k.$$

The corresponding generalized Marcinkiewicz integral operator on $\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k}$ is defined by

$$\mu_{\Omega,q}(f)(x_1, \dots, x_k) = \left(\int_{\mathcal{R}} \cdots \int_{\mathcal{R}} |\phi_{t_1,\dots,t_k} * f(x_1, \dots, x_k)|^q \, dt_1 \cdots dt_k \right)^{1/q},$$

where

$$\phi_{t_1,\dots,t_k} * f(x_1, \dots, x_k) = \frac{1}{2^{t_1+\cdots+t_k}} \int_{|y_1| \le 2^{t_1}} \cdots \int_{|y_k| \le 2^{t_k}} \frac{\Omega(y_1, \dots, y_k)}{|y_1|^{n_1-1} \cdots |y_k|^{n_k-1}} \times f(x_1 - y_1, \dots, x_k - y_k) dy_1 \cdots dy_k.$$

Also, following the definition of (1.2) we can define the Triebel–Lizorkin spaces on $\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k}$. Let $U_j \in C^\infty(\mathcal{R}^{n_j})$ satisfy $\operatorname{supp}(U_j) \subseteq \{x_j \in \mathcal{R}^{n_j} : 1/2 < |x_j| \le 2\}$ and $U_j(x_j) > c > 0$ if $3/5 \le |x_j| \le 5/3$ for $j \in \{1, 2, \dots, k\}$. Let Φ_j be the Fourier transform of U_j, $j = 1, 2, \dots, k$. The Triebel–Lizorkin spaces $\dot{F}_p^{0,q}(\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k})$ is the set of all distributions f on $\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k}$ such that

$$\|f\|_{\dot{F}_p^{0,q}(\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k})} = \left\| \left(\int_{\mathcal{R}} \cdots \int_{\mathcal{R}} \left| \left(\Phi_{1,t_1} \otimes \cdots \otimes \Phi_{k,t_k} \right) * f \right|^q dt_1 \cdots dt_k \right)^{1/q} \right\|_{L^p(\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k})} < \infty,$$

where $\Phi_{j,t_j}(x_j) = 2^{-n_j t_j}\Phi(2^{-t_j}x_j)$ for $j \in \{1, 2, \dots, k\}$. Employing the ideas in [13], it is not difficult to show

$$\big(\dot F_p^{0,q}(\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k})\big)^* = \dot F_{p'}^{0,q'}(\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k}),\ 1/p + 1/p' = 1 = 1/q + 1/q',$$
$$\dot F_p^{0,2}(\mathcal{R}^{n_1} \times \cdots \mathcal{R}^{n_k}) = L^p(\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k}), \quad 1 < p < \infty.$$

In [1], Al-Qassem, Al-Salman, Cheng, and Pan showed the following result.

Theorem C *Let $1 < p < \infty$. If $\Omega \in L(\log^+ L)^{k/2}(S^{n_1-1} \times \cdots \times S^{n_k-1})$, then $\mu_{\Omega,2}$ is bounded on $L^p(\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k})$.*

Here, we have the following generalization of Theorem C.

Theorem 5.4 *Let $2 \le q \le \infty$, $1 < p < \infty$. If $\Omega \in L(\log^+ L)^{k/q}(S^{n_1-1} \times \cdots \times S^{n_k-1})$, then $\mu_{\Omega,q}$ is bounded on $L^p(\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k})$.*

Theorem 5.5 *Let $1 < q < 2$, $1 < p < \infty$. If $\Omega \in L(\log^+ L)^{k/q+\varepsilon}(S^{n_1-1} \times \cdots \times S^{n_k-1})$ for any $\varepsilon > 0$, then*

$$\big\|\mu_{\Omega,q}(f)\big\|_{L^p(\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k})} \le C\|f\|_{\dot F_p^{0,q}(\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k})}.$$

Theorem 5.6 *Let $1 < q \le \infty$, $1 < p < \infty$. If $\Omega \in L(\log^+ L)^{k}(S^{n_1-1} \times \cdots \times S^{n_k-1})$ for any $\varepsilon > 0$, then*

$$\big\|\mu_{\Omega,q}(f)\big\|_{L^p(\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k})} \le C\|f\|_{\dot F_p^{0,q}(\mathcal{R}^{n_1} \times \cdots \times \mathcal{R}^{n_k})}.$$

Obviously, Theorems 1.1–1.3 treat the special case $k = 2$. The proofs of Theorems 1.1–1.3 easily extend to the case $k > 2$. We omit the details.

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