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# THE RATIONALITY PROBLEM FOR NORM ONE TORI

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***To the memory of Professor Masayoshi Nagata***

**Abstract**. We consider the problem of whether the norm one torus defined by a finite separable field extension $K/k$ is stably (or retract) rational over $k$. This has already been solved for the case where $K/k$ is a Galois extension. In this paper, we solve the problem for the case where $K/k$ is a non-Galois extension such that the Galois group of the Galois closure of $K/k$ is nilpotent or metacyclic.

## Introduction

Let $K/k$ be a finite separable field extension, and denote by $R^{(1)}_{K/k}(\mathbb{G}_m)$ the norm one torus defined by $K/k$, as usual (see, e.g., [V]).

The purpose of this paper is to determine whether the torus $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably (or retract) rational over $k$. For the case where $K/k$ is Galois, this problem was solved completely in [EM2] and [S]. Hence, we have only to consider this for the case where $K/k$ is non-Galois.

Assume that $K/k$ is non-Galois, and let $L/k$ be the Galois closure of $K/k$. Let $G = \mathrm{Gal}(L/k)$, and let $H = \mathrm{Gal}(L/K)$. The main results in this paper are the following.

[I] Assume that $G$ is a nilpotent group. Then $T$ is not retract rational over $k$.

[II] Assume that $G$ is a metacyclic group. Then $T$ is always retract rational over $k$, and the following conditions are equivalent:
- (1) $T$ is stably rational over $k$;
- (2) $G$ is the dihedral group $D_n$ of order $2n$ with $n$ odd or the direct product of the cyclic group $C_m$ of order $m$ and the dihedral group $D_n$ of order $2n$, where $m, n$ are odd, $m, n \geqq 3$, $(m, n) = 1$, and $H \subseteq D_n$ is of order 2.

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[III] Assume that $G = A_n, n \geqq 3$, the alternating group on $n$ letters, and that $H = A_{n-1} \subseteq G$, where $H$ is the stabilizer of one of the letters in $G$. Then,
(1) $T$ is retract rational over $k$ if and only if $n$ is a prime;
(2) for some $t \geqq 1$, $T^{(t)}$, the product of $t$ copies of $T$, is stably rational over $k$ if and only if $n = 3, 5$.

For the case of $G$ metacyclic, it is an immediate consequence of [EM2, (1.5)] and [S, (3.14)] that $T$ is retract rational over $k$. It should be noted that partial results of [I] and [II] have already been given in [CS1].

[I] and [II] are final answers to the problem for the cases of nilpotent groups and metacyclic groups, respectively. [III] can be regarded as an additional remark on the result for symmetric groups in [CS2], [lB], [CK], [LL], and so forth. We will also give another proof of the result for symmetric groups.

## §1. Preliminaries

Let $G$ be a finite group. A $G$-module means a finitely generated left $G$-module, and a $G$-module with a $\mathbb{Z}$-basis is said to be a *$G$-lattice*. A $G$-lattice $M$ is said to be a *permutation* $G$-lattice if it has a $\mathbb{Z}$-basis permuted by $G$, that is, if $M \cong \bigoplus_{1 \leq i \leq m} \mathbb{Z}G/H_i$ for subgroups $H_1, H_2, \ldots, H_m$. $M$ is said to be *invertible* if it is a direct summand of a permutation $G$-lattice. $M$ is said to be a *quasi-permutation* if there exists an exact sequence of $G$-lattices

$$0 \to M \to U \to V \to 0,$$

where $U$ and $V$ are permutation lattices. $M$ is said to be *quasi-invertible* if it is a direct summand of a quasi-permutation $G$-lattice. The dual lattice $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ of a $G$-lattice $M$ is denoted by $M^\circ$.

For a subgroup $H$ of $G$, there exists an exact sequence of $G$-lattices

$$0 \to I_{G/H} \to \mathbb{Z}G/H \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\varepsilon$ is the augmentation map and $I_{G/H} = \mathrm{Ker}\,\varepsilon$. The dual lattice $J_{G/H} = (I_{G/H})^\circ$ of $I_{G/H}$ will play a central part in this paper.

When $I_{G/H}$ and $J_{G/H}$ are examined, $H$ can be assumed to contain no normal subgroup of $G$ except $\{1\}$. In fact, let $N \subseteq H$ be a maximal subgroup which is normal in $G$, set $\overline{G} = G/N$, and set $\overline{H} = H/N$. Then $I_{\overline{G}/\overline{H}} = I_{G/H}$ and $J_{\overline{G}/\overline{H}} = J_{G/H}$, and therefore we may use $\overline{G}$ and $\overline{H}$ instead of $G$ and $H$, where $\overline{H}$ contains no normal subgroup of $\overline{G}$ except $\{1\}$.

Throughout this paper, a finite group is said to be a *metacyclic* group if all its Sylow subgroups are cyclic.

Let $k$ be a field, let $L$ be a finite Galois extension of $k$, and let $G = \mathrm{Gal}(L/k)$. Let $M$ be a $G$-lattice with a $\mathbb{Z}$-basis $\{u_1, u_2, \ldots, u_n\}$. Define the action of $G$ on the rational function field $L(X_1, X_2, \ldots, X_n)$ with variables $X_1, X_2, \ldots, X_n$ over $L$, as an extension of the action of $G$ over $L$, as follows. For each $\sigma \in G$,

$$\sigma(X_i) = \prod_{j=1}^{n} X_j^{m_{ij}}, \quad 1 \leqq i \leqq n,$$

when $\sigma u_i = \sum_{j=1}^{n} m_{ij} u_j, m_{ij} \in \mathbb{Z}$, and denote $L(X_1, X_2, \ldots, X_n)$ with this action of $G$ by $L(M)$.

For a given $G$-lattice $M$, there exists an algebraic torus $T$ defined over $k$ and split over $L$ such that the character group of $T$ is isomorphic to $M$ as $G$-lattices, and the invariant subfield $L(M)^G$ of $L(M)$ can be identified with the function field of $T$.

An extension field $F$ of a basic field $k$ is said to be *rational* over $k$ if it is generated over $k$ by a finite number of elements of $F$ which are algebraically independent over $k$. $F$ is said to be *stably rational* over $k$ if there exists an extension field of $F$ which is rational over each of $k$ and $F$. Further, $F$ is said to be *retract rational* over $k$ if there exists an extension field $k(x_1, x_2, \ldots, x_n)$ of $F$ rational over $k$ where $x_1, x_2, \ldots, x_n$ are algebraically independent over $k$, and if $F$ is the quotient field of a $k$-subalgebra $A$ of $F$ such that, for some nonzero element $s$ of $k[x_1, x_2, \ldots, x_n]$, we have $A \subseteq k[x_1, x_2, \ldots, x_n][1/s]$ and a $k$-algebra homomorphism

$$\theta\colon k[x_1, x_2, \ldots, x_n][1/s] \to A$$

whose restriction to $A$ is the identity on $A$. More generally, $F$ is said to be *unirational* over $k$ if there exists an extension field of $F$ which is rational over $k$.

It is easy to see that

$$\text{rational} \Longrightarrow \text{stably rational} \Longrightarrow \text{retract rational} \Longrightarrow \text{unirational}.$$

It should be noted that every algebraic torus defined by a separable extension of a field $k$ is unirational over $k$.

We now have the following.

THEOREM 1.1. *Let $L/k$ be a finite Galois field extension with a group $G$, and let $M$ be a $G$-lattice. Then,*

(1) *$M$ is a quasi-permutation $G$-lattice if and only if $L(M)^G$ is stably rational over $k$ (see, e.g., [EM1, (1.6)]);*
(2) *$M$ is a quasi-invertible $G$-lattice if and only if $L(M)^G$ is retract rational over $k$ (see [S, (3.14)]).*

Let $k$ be a field, and let $K/k$ be a finite separable extension. Let $L/k$ be the Galois closure of $K/k$, let $G = \mathrm{Gal}(L/k)$, and let $H = \mathrm{Gal}(L/K) \subseteq G$. The norm one torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ defined by $K/k$ has the lattice $J_{G/H}$ as its character lattice and the field $L(J_{G/H})^G$ as its function field (see [V]). Note that $H$ contains no normal subgroup of $G$ except $\{1\}$, since $L/k$ is the Galois closure of $K/k$. For the case where $K/k$ is Galois (i.e., $H = \{1\}$), the $G$-lattices $I_{G/H}$ and $J_{G/H}$ are denoted by $I_G$ and $J_G$, respectively.

For the case where $K/k$ is Galois, we have the following.

THEOREM 1.2. *Let $K/k$ be a finite Galois field extension with a group $G$. Then,*

(1) *$R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract rational over $k$ if and only if $G$ is metacyclic (see [EM2, (1.5)], [S, (3.14)]);*
(2) *$R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably rational over $k$ if and only if $G$ is a cyclic group, or a direct product of a cyclic group of order $m$ and a group $\langle \sigma, \tau \mid \sigma^n = \tau^{2^d} = 1, \tau\sigma\tau^{-1} = \sigma^{-1}\rangle$, where $d, m \geqq 1$, $n \geqq 3$, $m, n$ odd, and $(m, n) = 1$ (see [EM2, (2.3)]).*

Therefore, in this paper, we will consider only the case where $K/k$ is non-Galois, that is, the case where $H \neq \{1\}$.

Let $G$ be a finite group. Let $H_1, H_2, \ldots, H_t, t \geqq 2$ be subgroups of $G$, and let $\varepsilon_i \colon \mathbb{Z}G/H_i \to \mathbb{Z}, 1 \leqq i \leqq t$, be the augmentation maps. Then the *multiaugmentation* map

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t) \colon \mathbb{Z}G/H_1 \oplus \mathbb{Z}G/H_2 \oplus \cdots \oplus \mathbb{Z}G/H_t \to \mathbb{Z}$$

is defined by sending $u = (u_i) \in \bigoplus_{i=1}^t \mathbb{Z}G/H_i$ to $\sum_{i=1}^t \varepsilon_i(u_i) \in \mathbb{Z}$.

The following proposition on multiaugmentation maps is simple but very useful.

PROPOSITION 1.3. *Let $G$ be a finite group, and let $H_1, H_2, \ldots, H_t$, $t \geqq 2$ be subgroups of $G$ such that $H_{t-1} \supseteq H_t$. Let $\varepsilon_i \colon \mathbb{Z}G/H_i \to \mathbb{Z}, 1 \leqq i \leqq t$, be the augmentation maps. Further, let*

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{t-1}, \varepsilon_t)\colon \mathbb{Z}G/H_1 \oplus \mathbb{Z}G/H_2 \oplus \cdots \oplus \mathbb{Z}G/H_{t-1} \oplus \mathbb{Z}G/H_t \to \mathbb{Z},$$
$$\varepsilon' = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{t-1})\colon \mathbb{Z}G/H_1 \oplus \mathbb{Z}G/H_2 \oplus \cdots \oplus \mathbb{Z}G/H_{t-1} \to \mathbb{Z},$$

*be the multiaugmentation maps, set* $I = \mathrm{Ker}\,\varepsilon$, $I' = \mathrm{Ker}\,\varepsilon'$, *and set* $J = I^\circ$, $J' = (I')^\circ$. *Then* $I \cong I' \oplus \mathbb{Z}G/H_t$ *and* $J \cong J' \oplus \mathbb{Z}G/H_t$.

*Proof.* Define $\delta_t\colon \mathbb{Z}G/H_t \to \mathbb{Z}G/H_{t-1}$ by $\rho H_t \to \rho H_{t-1}, \rho \in G$, and define

$$\begin{aligned}\delta = (1, 1, \ldots, 1, \delta_t)\colon &\mathbb{Z}G/H_1 \oplus \mathbb{Z}G/H_2 \oplus \cdots \oplus \mathbb{Z}G/H_{t-1} \oplus \mathbb{Z}G/H_t \\ &\to \mathbb{Z}G/H_1 \oplus \mathbb{Z}G/H_2 \oplus \cdots \oplus \mathbb{Z}G/H_{t-1}\end{aligned}$$

by sending $(u_1, u_2, \ldots, u_{t-1}, u_t)$ to $(u_1, u_2, \ldots, u_{t-1} + \delta_t(u_t))$. Then $\delta$ is a split surjection and $\mathrm{Ker}\,\delta$ $(\cong \mathbb{Z}G/H_t) \subseteq I$. Hence, we can form the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & \\
 & \mathbb{Z}G/H_t \arrow[rr, equal] \arrow[d] & & \mathbb{Z}G/H_t \arrow[d] & & \\
0 \arrow[r] & I \arrow[rr] \arrow[d] & & \mathbb{Z}G/H_1 \oplus \cdots \oplus \mathbb{Z}G/H_{t-1} \oplus \mathbb{Z}G/H_t \arrow[r, "\varepsilon"] \arrow[d, "\delta"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I' \arrow[rr] \arrow[d] & & \mathbb{Z}G/H_1 \oplus \cdots \oplus \mathbb{Z}G/H_{t-1} \arrow[r, "\varepsilon'"] \arrow[d] & \mathbb{Z} \arrow[r] & 0 \\
 & 0 & & 0 & &
\end{tikzcd}$$

Then the first column is also split, and so $I \cong I' \oplus \mathbb{Z}G/H_t$ and $J \cong J' \oplus \mathbb{Z}G/H_t$. ∎

Corollary 1.4. *Let* $G$ *be a finite group, and let*

$$\varepsilon^{(t)}\colon [\mathbb{Z}G]^{(t)} \to \mathbb{Z}, \quad t \geqq 2,$$

*be the multiaugmentation map of* $[\mathbb{Z}G]^{(t)}$, *the direct sum of* $t$ *copies of* $\mathbb{Z}G$, *on* $\mathbb{Z}$ *defined as in Proposition* 1.3 *by augmentation map* : $\mathbb{Z}G \to \mathbb{Z}$. *Let* $I = \mathrm{Ker}\,\varepsilon^{(t)}$, *and let* $J = I^\circ$. *Then* $I \cong I_G \oplus [\mathbb{Z}G]^{(t-1)}$, *and hence* $J \cong J_G \oplus [\mathbb{Z}G]^{(t-1)}$.

Note that special cases of Proposition 1.3 and Corollary 1.4 have been used in [E] and [CK].

A lattice $M$ over a finite group $G$ is said to be *coflasque* if $H^1(G', M) = 0$ for any subgroup $G'$ of $G$. Every invertible lattice is coflasque. For any $G$-lattice $M$, we can construct an exact sequence

$$0 \to N \to U \to M \to 0,$$

where $U$ is permutation and $N$ is coflasque (see [EM2, (1.1)]). This is said to be a *coflasque resolution* of $M$.

Proposition 1.5. *Let $G$ be a finite group, and let $0 \to N \to U \to M \to 0$ be an exact sequence of $G$-lattices with $U$ permutation. Then,*

(1) *$M^\circ$ is a quasi-permutation if and only if $N$ is a quasi-permutation;*

(2) *$M^\circ$ is quasi-invertible if and only if $N$ is quasi-invertible.*

*Suppose further that $N$ is coflasque. Then,*

(3) *$M^\circ$ is quasi-invertible if and only if $N$ is invertible.*

*Proof.* For example, see the proof of [EM2, (1.6)]. □

Corollary 1.6. *A lattice over a finite group $G$ is quasi-invertible if and only if it is quasi-invertible over every Sylow subgroup of $G$.*

*Proof.* It is well known (see, e.g., [EM2, (1.4)]) that a $G$-lattice is invertible if and only if it is invertible over every Sylow subgroup of $G$. Therefore, the assertion follows directly from Proposition 1.5. □

The following proposition is only a slight generalization of Theorem 1.2(1), but this is useful for our problem.

Proposition 1.7. *Let $G$ be a finite group, and let $H$ be a nonnormal Hall subgroup of $G$. Then $J_{G/H}$ is quasi-invertible over $G$ if and only if all Sylow $p$-subgroups of $G$ are cyclic for any prime $p \mid [G:H]$.*

*Proof.* Suppose that there exists a noncyclic Sylow $p$-subgroup $P$ of $G$ for some prime $p \mid [G:H]$. Then $\mathbb{Z}G/H$ is $\mathbb{Z}P$ free, and therefore $\mathbb{Z}G/H \cong [\mathbb{Z}P]^{(t)}, t \geqq 1$. Hence, by Corollary 1.4, $J_{G/H} \cong J_P \oplus [\mathbb{Z}P]^{(t-1)}$. However, since $P$ is noncyclic, $J_P$ is not quasi-invertible over $P$. Thus, $J_{G/H}$ is not quasi-invertible over $G$.

On the other hand, suppose that Sylow $p$-subgroups of $G$ are cyclic for any prime $p \mid [G:H]$. Let $p$ be a prime divisor of $|G|$, and let $P$ be a Sylow $p$-subgroup. Assume first that $p \mid [G:H]$. Then, as above, $\mathbb{Z}G/H \cong [\mathbb{Z}P]^{(t)}, t \geqq 1$, as $P$-lattices, and hence, by Corollary 1.4, $J_{G/H} \cong J_P \oplus [\mathbb{Z}P]^{(t-1)}$. Since $P$ is cyclic by assumption, $J_P$ is quasi-invertible. This shows that $J_{G/H}$ is

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quasi-invertible over $P$. Next, assume that $p \mid |H|$. As $H$ is a Hall subgroup, we have $p \nmid [G:H]$, and then the action of $P$ on $G/H$ has a fixed point. Therefore, $J_{G/H}$, as a $P$-lattice, is a direct summand of $\mathbb{Z}G/H$, which shows also that $J_{G/H}$ is invertible over $P$. Hence, in both cases, $J_{G/H}$ is quasi-invertible over $P$. Thus, it follows from Corollary 1.6 that $J_{G/H}$ is quasi-invertible over $G$. □

## §2. Nilpotent groups

In this section, we will prove the following.

THEOREM 2.1. *Let $K/k$ be a finite non-Galois, separable field extension, and let $L/k$ be the Galois closure of $K/k$. Assume that the Galois group of $L/k$ is nilpotent. Then the norm one torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ defined by $K/k$ is not retract rational over $k$.*

Let $G = \mathrm{Gal}(L/k)$, and let $H = \mathrm{Gal}(L/K) \subseteq G$. In order to prove Theorem 2.1, it suffices to show by Theorem 1.1(2) that the $G$-lattice $J = J_{G/H}$ is not quasi-invertible.

We can reduce Theorem 2.1 to the case where $G$ is a $p$-group for a prime $p$. In fact, given a nilpotent group $G$ and a nonnormal subgroup $H \subseteq G$, there exists a Sylow $p$-subgroup $P$ for some $p \mid |G|$ such that $P' = P \cap H$ is nonnormal in $P$, because nilpotent groups $G$ and $H$ are expressible uniquely as the direct products of their Sylow subgroups. Then we have $\mathbb{Z}G/H \cong [\mathbb{Z}P/P']^{(t)}$ for some $t \geqq 1$ as $P$-lattices, and so, by Proposition 1.3, $J_{G/H} \cong J_{P/P'} \oplus [\mathbb{Z}P/P']^{(t-1)}$ as $P$-lattices. Accordingly, it follows that $J_{G/H}$ is not quasi-invertible over $G$ when $J_{P/P'}$ is not quasi-invertible over $P$.

From now on, we assume that $G$ is a $p$-group and that $H \subseteq G$ contains no normal subgroup of $G$ except $\{1\}$.

We will prove step by step that $J_{G/H}$ is not quasi-invertible over $G$.

STEP 1. Case where the center of G is not cyclic.

*Proof.* Let $Z = Z(G)$ be the center of $G$. Since $H$ contains no normal subgroup of $G$ except $\{1\}$, we have $H \cap Z = \{1\}$, and so $\mathbb{Z}G/H \cong [\mathbb{Z}Z]^{(t)}$ for some $t \geqq 1$ as $Z$-lattices. Then, from Corollary 1.4, it follows that $J_{G/H} \cong J_Z \oplus [\mathbb{Z}Z]^{(t-1)}$. Since $Z$ is not cyclic by the assumption, $J_Z$ is not quasi-invertible over $Z$ by Theorem 1.2(1), and so $J_{G/H}$ is not quasi-invertible over $G$. □

According to Step 1, we may assume from now that the center $Z(G)$ of $G$ is cyclic.

STEP 2. Case where $p$ is odd.

*Proof.* By [Be, (1.4)], there exists a normal subgroup of $G$ as follows:

$$N = \langle \sigma, \tau \mid \sigma^p = \tau^p = 1, \sigma\tau = \tau\sigma \rangle.$$

Then, by the above assumption, we may suppose that $N \cap Z(G) = \langle \sigma \rangle$.

Suppose first that $H \cap N = \{1\}$. Then we have $\mathbb{Z}G/H \cong [\mathbb{Z}N]^{(t)}$ for some $t \geqq 1$, as $N$-lattices, and therefore by Corollary 1.4, $J_{G/H} \cong J_N \oplus [\mathbb{Z}N]^{(t-1)}$, as $N$-lattices. Since $N$ is not cyclic, $J_N$ is not quasi-invertible over $N$ by Theorem 1.2(1), and hence $J_{G/H}$ is also not quasi-invertible over $G$.

Next, suppose that $H \cap N \neq \{1\}$. Then we may assume that $H \cap N = \langle \tau \rangle$. As is easily seen, the subgroups $\langle \tau \rangle, \langle \tau\sigma \rangle, \langle \tau\sigma^2 \rangle, \ldots, \langle \tau\sigma^{p-1} \rangle$ are conjugate under $G$, because $\langle \tau \rangle$ is not normal in $G$ and $\sigma \in Z(G)$, and so, as $N$-lattices,

$$\mathbb{Z}G/H \cong [\mathbb{Z}N/\langle \tau \rangle \oplus \mathbb{Z}N/\langle \tau\sigma \rangle \oplus \cdots \oplus \mathbb{Z}N/\langle \tau\sigma^{p-1} \rangle]^{(t)}$$

for some $t \geqq 1$. Let

$$\varepsilon \colon \mathbb{Z}N/\langle \tau \rangle \oplus \mathbb{Z}N/\langle \tau\sigma \rangle \oplus \cdots \oplus \mathbb{Z}N/\langle \tau\sigma^{p-1} \rangle \to \mathbb{Z}$$

be the multiaugmentation map, and set $J = [\operatorname{Ker} \varepsilon]^\circ$. Then it follows from Proposition 1.3 that

$$J_{G/H} \cong J \oplus [\mathbb{Z}N/\langle \tau \rangle \oplus \mathbb{Z}N/\langle \tau\sigma \rangle \oplus \cdots \oplus \mathbb{Z}N/\langle \tau\sigma^{p-1} \rangle]^{(t-1)}.$$

Since $J$ is not quasi-invertible over $N$ by [E, Theorem 2(2)], this implies that $J_{G/H}$ is not quasi-invertible over $G$. $\square$

The following 2-groups are said to be of maximal class (see [Be, p. 26, Definition 2 and (1.7)]):

- the dihedral group
  $D_{2^n} = \langle \sigma, \tau \mid \sigma^{2^n} = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle, n \geqq 2,$
- the generalized quaternion group
  $Q_{2^n} = \langle \sigma, \tau \mid \sigma^{2^n} = 1, \sigma^{2^{n-1}} = \tau^2, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle, n \geqq 2,$
- the semidihedral group
  $SD_{2^n} = \langle \sigma, \tau \mid \sigma^{2^n} = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1+2^{n-1}} \rangle, n \geqq 3.$

Any subgroup $\neq \{1\}$ of the group $Q_{2^n}$ contains the center $Z(Q_{2^n}) = \langle \sigma^{2^{n-1}} \rangle$, and so $Q_{2^n}$ can be omitted from the object of our consideration.

STEP 3. Case where $p = 2$ and $G$ is of maximal class.

*Proof.* Assume that $G = D_{2^n}$. Then $H$ is one of the subgroups $\langle\tau\rangle, \langle\tau\sigma\rangle, \dots, \langle\tau\sigma^{2^{n-1}}\rangle$, and therefore we may assume that $H = \langle\tau\rangle$. Define $N = \langle\sigma^2, \tau\sigma\rangle$. Then $N$ is normal in $G$ and $N \cong D_{2^{n-1}}$ $(n \geqq 3)$ or the elementary abelian group of order 4. Further, we have $\mathbb{Z}G/H \cong \mathbb{Z}N$ as $N$-lattices, and hence $J_{G/H} = J_N$ is not quasi-invertible over $N$, again by Theorem 1.2(1). Thus, we conclude that $J_{G/H}$ is not quasi-invertible over $G$.

Next, assume that $G = SD_{2^n}$. Then $H$ is one of the subgroups $\langle\tau\rangle, \langle\tau\sigma^2\rangle, \dots, \langle\tau\sigma^{2(2^{n-1}-1)}\rangle$, and therefore we may assume that $H = \langle\tau\rangle$. Note that the subgroups $\langle\tau\sigma\rangle, \langle\tau\sigma^3\rangle, \dots, \langle\tau\sigma^{2^n-1}\rangle$ of $G$ contain the center $Z(G) = \langle\sigma^{2^{n-2}}\rangle$. Set $N = \langle\sigma^2, \tau\sigma\rangle$. Then $N$ is normal in $G$ and $N \cong Q_{2^{n-1}}$ $(n \geqq 3)$. Further, we have $\mathbb{Z}G/H \cong \mathbb{Z}N$ as $N$-lattices, and hence, along the same lines as in the dihedral case, we can show that $J_{G/H}$ is not quasi-invertible over $G$. ∎

STEP 4. Case where a 2-group $G$ is not of maximal class and does not have the elementary abelian group of order 8 as its normal subgroup.

*Proof.* Since $G$ is not of maximal class, there exists an elementary abelian normal subgroup $E$ of order 4 in $G$ by [Be, (1.4)]. The centralizer $C_G(E)$ of $E$ in $G$ is normal in $G$, and by [Be, (1.8)], we have $E \subsetneq C_G(E)$. Then there is $\rho \in C_G(E) - E$ such that the class $\bar{\rho}$ of $\rho$ in $G/E$ is contained in the center of $G/E$ and is of order 2. Then $N = \langle\rho, E\rangle$ is an abelian, noncyclic normal subgroup of order 8 in $G$. However, by the assumption, $N$ is not elementary abelian, and therefore it can be expressed as follows:

$$N = \langle\sigma, \tau \mid \sigma^4 = \tau^2 = 1, \sigma\tau = \tau\sigma\rangle.$$

Since the conjugacy class of $\sigma$ in $G$ is contained in $\{\sigma, \sigma^3, \sigma\tau, \sigma^3\tau\}$, the conjugacy class of $\sigma^2$ in $G$ is $\{\sigma^2\}$, and so we have $\sigma^2 \in Z(G)$. However, by assumption, $Z(G)$ is cyclic. Accordingly, the elements $\tau$ and $\sigma^2\tau$ of order 2 in $N$ must be conjugate under $G$.

Assume first that $H \cap N = \{1\}$. Then $\mathbb{Z}G/H \cong [\mathbb{Z}N]^{(t)}, t \geqq 1$, as $N$-lattices, and therefore, by Corollary 1.4, $J_{G/H} \cong J_N \oplus [\mathbb{Z}N]^{(t-1)}$, as $N$-lattices. Since $N$ is not cyclic, we can conclude that $J_{G/H}$ is not quasi-invertible.

Next, assume that $H \cap N \neq \{1\}$. Because $H \cap Z(G) = \{1\}$, $\sigma^2 \notin H$, and so $H \cap N = \{\tau\}$ or $\{\tau\sigma^2\}$. As noted above, $\tau$ and $\tau\sigma^2$ are conjugate under $G$. Hence, we have $\mathbb{Z}G/H \cong [\mathbb{Z}N/\langle\tau\rangle \oplus \mathbb{Z}N/\langle\tau\sigma^2\rangle]^{(s)}, s \geqq 1$, as $N$-lattices. Let

$$\varepsilon = (\varepsilon_1, \varepsilon_2) \colon U = \mathbb{Z}N/\langle\tau\rangle \oplus \mathbb{Z}N/\langle\tau\sigma^2\rangle \to \mathbb{Z}$$

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be the multiaugmentation map, and set $J = [\operatorname{Ker}\varepsilon]^\circ$. Then, by Proposition 1.3, $J_{G/H} \cong J \oplus [\mathbb{Z}N/\langle\tau\rangle \oplus \mathbb{Z}N/\langle\tau\sigma^2\rangle]^{(s-1)}$ as $N$-lattices. According to Lemma 2.2 given at the end of this section, $J$ is not quasi-invertible over $N$. Thus, $J_{G/H}$ is not quasi-invertible over $G$. □

STEP 5. Case where the 2-group $G$ has the elementary abelian group $E$ of order 8 as a normal subgroup.

*Proof.* Let $E = \langle\rho,\sigma,\tau \mid \rho^2 = \sigma^2 = \tau^2 = 1, \rho\sigma = \sigma\rho, \sigma\tau = \tau\sigma, \tau\rho = \rho\tau\rangle$. Since $Z(G)$ is cyclic and $E \cap Z(G) \neq \{1\}$, we may assume that $E \cap Z(G) = \langle\rho\rangle$. Since $H$ contains no normal subgroup of $G$ except $\{1\}$, we have $H \cap Z(G) = \{1\}$, and hence $\rho$ is not contained in any subgroup conjugate to $H$.

First, suppose that $|H \cap E| = 1$, that is, that $H \cap E = \{1\}$. Then we have $\mathbb{Z}G/H \cong [\mathbb{Z}E]^{(t)}$ for some $t \geqq 1$ as $E$-lattices, and so the proof is similar to the previous one.

Second, suppose that $|H \cap E| = 2$. Then we may assume that $H \cap E = \langle\sigma\rangle$. Let $E_0 = \langle\rho,\sigma\rangle$. If $E_0$ is normal in $G$, then $\{\sigma,\sigma\rho\}$ is a conjugacy class of $G$. Then the subgroup $E_1 = \langle\rho,\tau\rangle$ does not contain any of $\sigma$ and $\sigma\rho$. Hence, we have $\mathbb{Z}G/H \cong [\mathbb{Z}E_1]^{(t)}$ for some $t \geqq 1$ as $E_1$-lattices. On the other hand, if $E_0$ is not normal in $G$, then one of the subgroups $\langle\rho,\tau\rangle$ and $\langle\rho,\sigma\tau\rangle$ is normal in $G$, and we denote it by $E_1$. Then $E_1 \cap H = \{1\}$, and therefore we have $\mathbb{Z}G/H \cong [\mathbb{Z}E_1]^{(t)}$ for some $t \geqq 1$ as $E_1$-lattices. Thus, the proof is done in the same way as in the first case.

Finally, suppose that $|H \cap E| = 4$. In this case, we may assume that $H \cap E = \langle\sigma,\tau\rangle$. Now, all the subgroups of order 4 in $E$ are expressible as follows:

$$\langle\sigma,\tau\rangle, \qquad \langle\rho\sigma,\tau\rangle, \qquad \langle\sigma,\rho\tau\rangle, \qquad \langle\rho\sigma,\sigma\tau\rangle,$$
$$\langle\rho,\sigma\rangle, \qquad \langle\rho,\tau\rangle, \qquad \langle\rho,\sigma\tau\rangle.$$

The groups in the second row are not conjugate to those in the first row under $G$, because $E \cap Z(G) = \langle\rho\rangle$.

We will now show that the groups in the first row of the above list are conjugate under $G$. Let $E_1 = \langle\rho,\sigma\tau\rangle$, let $E_2 = \langle\rho,\sigma\rangle$, and let $E_3 = \langle\rho,\tau\rangle$. It is easy to see that at least one of $E_1$, $E_2$, and $E_3$ is normal in $G$, and so we may assume that $E_1$ is normal in $G$. Then the centralizer $C_G(\sigma\tau)$ of $\sigma\tau$ in $G$ is a maximal subgroup of $G$; that is, $[G : C_G(\sigma\tau)] = 2$. Note that $E_2$ and $E_3$ are either both normal or both nonnormal in $G$.

We first consider the case where both $E_2$ and $E_3$ are normal in $G$. Then both $C_G(\sigma)$ and $C_G(\tau)$, the centralizers of $\sigma$ and $\tau$ in $G$, are maximal in

$G$. These three maximal subgroups of $G$ are distinct. In fact, if $C_G(\sigma) = C_G(\tau)$, for example, then $C_G(\sigma) = C_G(\tau) = C_G(\sigma\tau)$, since $C_G(\sigma) \cap C_G(\tau) \subseteq C_G(\sigma\tau)$. Setting $C = C_G(\sigma) = C_G(\tau) = C_G(\sigma\tau)$ and taking $\mu \in G - C$, we have $\mu\sigma\mu^{-1} = \rho\sigma$, $\mu\tau\mu^{-1} = \rho\tau$, and $\mu\sigma\tau\mu^{-1} = \rho\sigma\tau$, because $E_1$, $E_2$, and $E_3$ are normal in $G$. From the equalities $\mu\sigma\mu^{-1} = \rho\sigma$ and $\mu\tau\mu^{-1} = \rho\tau$, it follows that $(\mu\sigma\mu^{-1})(\mu\tau\mu^{-1}) = (\rho\sigma)(\rho\tau) = \sigma\tau$. This contradicts obviously the third equality $\mu\sigma\tau\mu^{-1} = \rho\sigma\tau$. Now, let $\mu \in C_G(\sigma) - C_G(\tau)$ and $\nu \in C_G(\tau) - C_G(\sigma)$. Then we have

$$\begin{aligned} \mu\sigma\mu^{-1} &= \sigma, & \mu\tau\mu^{-1} &= \rho\tau, \\ \nu\sigma\nu^{-1} &= \rho\sigma, & \nu\tau\nu^{-1} &= \tau, \\ (\nu\mu)\sigma(\nu\mu)^{-1} &= \rho\sigma, & (\nu\mu)\tau(\nu\mu)^{-1} &= \rho\tau. \end{aligned}$$

This implies that the groups given in the first row are conjugate under $G$.

Second, we consider the case where both $E_2$ and $E_3$ are nonnormal in $G$. In this case, the set $\{\sigma, \rho\sigma, \tau, \rho\tau\}$ is a conjugacy class of $G$, because $[G : C_G(\sigma)] = [G : C_G(\tau)] = 4$. If $C_G(\sigma) = C_G(\tau)$, then $C = C_G(\sigma) = C_G(\tau) \subsetneq C_G(\sigma\tau) \subsetneq G$. Let $\mu \in G - C_G(\sigma\tau)$. Since $\mu \notin C$, we have

$$\mu\sigma\mu^{-1} = \tau, \qquad \mu\tau\mu^{-1} = \rho\sigma$$

or

$$\mu\sigma\mu^{-1} = \rho\tau, \qquad \mu\tau\mu^{-1} = \sigma.$$

Then we have further

$$\mu^2\sigma\mu^{-2} = \rho\sigma, \qquad \mu^2\tau\mu^{-2} = \rho\tau$$

and

$$\mu^3\sigma\mu^{-3} = \rho\tau, \qquad \mu^3\tau\mu^{-3} = \sigma$$

or

$$\mu^3\sigma\mu^{-3} = \tau, \qquad \mu^3\tau\mu^{-3} = \rho\sigma.$$

Therefore, the subgroups given in the first row are conjugate under $G$. On the other hand, if $C_G(\sigma) \neq C_G(\tau)$, then there exist $\mu \in C_G(\sigma) - C_G(\tau)$ and $\nu \in C_G(\tau) - C_G(\sigma)$. Using these $\mu, \nu$, we can show in the same way as in the first case that the four subgroups are conjugate under $G$. Thus, in both cases, we conclude that the four subgroups are conjugate under $G$.

Since $H \cap E = \langle\sigma, \tau\rangle$, we have

$$\mathbb{Z}G/H \cong [\mathbb{Z}E/\langle\sigma,\tau\rangle \oplus \mathbb{Z}E/\langle\rho\sigma,\tau\rangle \oplus \mathbb{Z}E/\langle\sigma,\rho\tau\rangle \oplus \mathbb{Z}E/\langle\rho\sigma,\sigma\tau\rangle]^{(t)}$$

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for some $t \geqq 1$ as $E$-lattices. Let

$$\varepsilon \colon \mathbb{Z}E/\langle\sigma,\tau\rangle \oplus \mathbb{Z}E/\langle\rho\sigma,\tau\rangle \oplus \mathbb{Z}E/\langle\sigma,\rho\tau\rangle \oplus \mathbb{Z}E/\langle\rho\sigma,\sigma\tau\rangle \to \mathbb{Z}$$

be the multiaugmentation map, and set $J = [\operatorname{Ker}\varepsilon]^\circ$. Then it follows from Proposition 1.3 that

$$J_{G/H} \cong J \oplus [\mathbb{Z}E/\langle\sigma,\tau\rangle \oplus \mathbb{Z}E/\langle\rho\sigma,\tau\rangle \oplus \mathbb{Z}E/\langle\sigma,\rho\tau\rangle \oplus \mathbb{Z}E/\langle\rho\sigma,\sigma\tau\rangle]^{(t-1)}.$$

Since $J$ is not quasi-invertible over $E$ by [E, Theorem 2(1)], this implies that $J_{G/H}$ is not quasi-invertible over $G$. This completes the proof of this step, and so the proof of Theorem 2.1. □

Finally, we show the following lemma, which has been used in Step 4.

Lemma 2.2. *Let $G = \langle\sigma,\tau \mid \sigma^4 = \tau^2 = 1, \sigma\tau = \tau\sigma\rangle$ be the direct product of the cyclic groups $\langle\sigma\rangle$ and $\langle\tau\rangle$. Let*

$$\varepsilon = (\varepsilon_1,\varepsilon_2) \colon U = \mathbb{Z}G/\langle\tau\rangle \oplus \mathbb{Z}G/\langle\tau\sigma^2\rangle \to \mathbb{Z}$$

*be the multiaugmentation map, set $I = \operatorname{Ker}\varepsilon$, and set $J = I^\circ$. Then $J$ is not quasi-invertible.*

*Proof.* We construct a concrete coflasque resolution of $I$. The subgroups of $G$ are as follows:

| | |
|---|---|
| order 1 | $\{1\}$ |
| order 2 | $H_0 = \langle\sigma^2\rangle, H_1 = \langle\tau\rangle, H_2 = \langle\tau\sigma^2\rangle$ |
| order 4 | $N_0 = \langle\sigma^2,\tau\rangle, N_1 = \langle\sigma\rangle, N_2 = \langle\sigma\tau\rangle$ |
| order 8 | $G$ |

Under this notation, we have $U = \mathbb{Z}G/H_1 \oplus \mathbb{Z}G/H_2$. Both $\mathbb{Z}G/H_1$ and $\mathbb{Z}G/H_2$ have $\{1,\sigma,\sigma^2,\sigma^3\}$ as representatives of the cosets, so $I$ can be expressed as follows:

$$I = \mathbb{Z}\langle\sigma\rangle(1,-1) + \mathbb{Z}\langle\sigma\rangle(0,\sigma-1) = \mathbb{Z}\langle\sigma\rangle(1,-1) + \mathbb{Z}\langle\sigma\rangle(\sigma-1,0).$$

Here, note that $\tau(1,-1) = (1,-1) - (0,\sigma^2-1)$. Then we have

$$\begin{aligned}
I^{H_1} &= \mathbb{Z}\langle\sigma\rangle(1+\sigma^2)(1,-1) + \mathbb{Z}\langle\sigma\rangle(\sigma-1,0),\\
I^{H_2} &= \mathbb{Z}\langle\sigma\rangle(1+\sigma^2)(1,-1) + \mathbb{Z}\langle\sigma\rangle(0,\sigma-1),\\
I^{H_0} = I^{N_0} &= \mathbb{Z}\langle\sigma\rangle(1+\sigma^2)(1,-1) + \mathbb{Z}(1+\sigma^2)(\sigma-1,0)\\
&= \mathbb{Z}\langle\sigma\rangle(1+\sigma^2)(1,-1) + \mathbb{Z}(1+\sigma^2)(0,\sigma-1),
\end{aligned}$$

$$I^G = I^{N_1} = I^{N_2} = \mathbb{Z}(1+\sigma+\sigma^2+\sigma^3)(1,-1) \cong \mathbb{Z}.$$

Define now the $G$-homomorphisms

$$\begin{aligned}
\delta_0 &: \mathbb{Z}G \to I && \text{by } \mapsto (1,-1),\\
\delta_1 &: \mathbb{Z}G/H_1 \to I && \text{by } 1 \mapsto (\sigma-1,0),\\
\delta_2 &: \mathbb{Z}G/H_2 \to I && \text{by } 1 \mapsto (0,\sigma-1),\\
\delta_3 &: \mathbb{Z}G/N_0 \to I && \text{by } 1 \mapsto (1+\sigma^2)(1,-1).
\end{aligned}$$

Set $V = \mathbb{Z}G \oplus \mathbb{Z}G/H_1 \oplus \mathbb{Z}G/H_2 \oplus \mathbb{Z}G/N_0$, let

$$\delta = (\delta_0, \delta_1, \delta_2, \delta_3) : V \to I,$$

and let $W = \operatorname{Ker}\delta$. Then it is easy to see that $H^1(H, W) = 0$ for every subgroup $H$ of $G$. This shows that $W$ is coflasque.

We denote by $X^*$ the completion of a lattice $X$ at 2. Then, $\mathbb{Z}^*G$ is a local ring, and, for any subgroup $H$ of $G$, $\mathbb{Z}^*G/H$ is indecomposable.

Suppose now that $J$ is quasi-invertible. Then $W$ is invertible by Proposition 1.5(3). Since the Krull-Schmidt theorem holds for permutation lattices over $\mathbb{Z}^*G$, the completion $W^*$ of $W$ at 2 must be a permutation.

From the exact sequences

$$0 \to I \to U \to \mathbb{Z} \to 0,$$

$$0 \to W \to V \to I \to 0,$$

we get the list of the $\mathbb{Z}$-rank for these $G$-lattices as follows:

| $H$ | $\operatorname{rank}_{\mathbb{Z}} U^H$ | $\operatorname{rank}_{\mathbb{Z}} I^H$ | $\operatorname{rank}_{\mathbb{Z}} V^H$ | $\operatorname{rank}_{\mathbb{Z}} W^H$ |
|---|---|---|---|---|
| $\{1\}$ | 8 | 7 | 18 | 11 |
| $H_0$ | 4 | 3 | 10 | 7 |
| $H_1$ | 6 | 5 | 12 | 7 |
| $H_2$ | 6 | 5 | 12 | 7 |
| $N_0$ | 4 | 3 | 8 | 5 |
| $N_1$ | 2 | 1 | 5 | 4 |
| $N_2$ | 2 | 1 | 5 | 4 |
| $G$ | 2 | 1 | 4 | 3 |

From this list, we can deduce that $W^* \cong \mathbb{Z}^*G \oplus \mathbb{Z}^*G/N_0 \oplus \mathbb{Z}^*$. Now, we have the exact sequence

$$0 \to W^* \to V^* \to I^* \to 0,$$

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with $V^* \cong \mathbb{Z}^*G \oplus \mathbb{Z}^*G/H_1 \oplus \mathbb{Z}^*G/H_2 \oplus \mathbb{Z}^*G/N_0$. Setting $V' = \mathbb{Z}^*G/H_1 \oplus \mathbb{Z}^*G/H_2 \oplus \mathbb{Z}^*G/N_0$ and $W' = \mathbb{Z}^*G/N_0 \oplus \mathbb{Z}^*$, and forming the pushout of

$$\begin{array}{ccc} W^* & \longrightarrow & V^* \\ \downarrow & & \\ W' & & \end{array}$$

we obtain the exact sequence

$$0 \to W' \to V' \to I^* \to 0,$$

which is obviously a contradiction, because the image of $V'$ in $I^*$ cannot contain the element $(1, -1)$. This concludes that $J$ is not quasi-invertible. $\square$

Remark 2.3. In [CS1, (d3)], it was shown that the torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not rational over $k$ in the case where $\mathrm{Gal}(L/k)$ is the dihedral group $D_4$ of order 8 and $\mathrm{Gal}(L/K)$ is the subgroup of order 2.

## §3. Metacyclic groups

The main result of this section is the following.

Theorem 3.1. *Let $K/k$ be a finite non-Galois, separable field extension, and let $L/k$ be the Galois closure of $K/k$. Let $G = \mathrm{Gal}(L/k)$, and let $H = \mathrm{Gal}(L/K) \subseteq G$. Assume that $G$ is metacyclic. Then the following conditions are equivalent.*

(1) *The norm one torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ defined by $K/k$ is stably rational over $k$.*
(2) *$G$ is the dihedral group $D_n$ of order $2n$ with $n$ odd ($n \geqq 3$) or the direct product of the cyclic group $C_m$ of order $m$ and the dihedral group $D_n$ of order $2n$, where $m, n$ are odd, $m, n \geqq 3$, $(m, n) = 1$, and $H \subseteq D_n$ is of order 2.*

Note that Theorem 3.1(2) is equivalent to the following.

(2′) $H = C_2$ is the cyclic group of order 2, and $G$ is isomorphic to a semidirect product $C_r \rtimes H$, $r \geqq 3$ odd, where $H$ acts nontrivially on the cyclic group $C_r$ of order $r$.

Let $G$ be a nonabelian metacyclic group. Then $G$ is expressible as the semidirect product of the cyclic normal subgroup $N_0 = C_l$ of order $l$ by the cyclic subgroup $H_0 = C_f$ of order $f$, all Sylow subgroups of which are nonnormal in $G$, where $l \geqq 3$ odd, $f \geqq 2$, and $(f,l)=1$. We define

$$i(G) = |\mathrm{Im}(H_0 \to \mathrm{Aut}\, N_0)|.$$

Theorem 3.1 is only a restatement of the following (see [EM2, (2.3) and p. 92, (1′)]).

Theorem 3.2. *Let $G$ be a nonabelian metacyclic group, and let $H$ be a nonnormal subgroup of $G$ which contains no normal subgroup of $G$ except $\{1\}$. Then the following conditions are equivalent:*

(1) *$i(G)=2$;*

(2) *$J_{G/H}$ is a quasi-permutation $G$-lattice;*

(3) *$[J_{G/H}]^{(t)}$ is a quasi-permutation $G$-lattice for some $t \geqq 1$.*

Remark 3.3. The partial results of Theorems 3.1 and 3.2 were obtained in [CS1, (R4) and (d1)] and [F, (2.3)]. It is given without proof in [CS1, (d1)] that, for the case of $\mathrm{Gal}(L/k) = D_n$ with $n$ odd, the torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ is rational over $k$.

Now we will prove Theorem 3.2. In Theorem 3.2 the implication (2) $\Rightarrow$ (3) is obvious, and so it suffices to prove the implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

*The proof of* (3) $\Rightarrow$ (1). Assume that $i(G) \geqq 3$.

Case 1. Suppose that $|H| \geqq 3$. Then there exist a subgroup $H'$ of $H$ with $|H'| = 4$ or $q$ an odd prime and a subgroup $N'$ of $N_0$ with $|N'| = p$ an odd prime such that $H'$ acts faithfully on $N'$ by conjugation. Set $G' = N'H'$, and regard $\mathbb{Z}G/H'$ as a $G'$-lattice. Then we have $\mathbb{Z}G/H \cong \mathbb{Z}G'/H' \oplus S'$ as $G'$-lattices, where $S' = 0$ or $S' = \bigoplus_{i=1}^{d} \mathbb{Z}G'/H'_i, d \geqq 1$, for subgroups $H'_i \subseteq H'$, and so, by Proposition 1.3, $J_{G/H} \cong J_{G'/H'} \oplus S'$. Therefore, it suffices to show that $[J_{G'/H'}]^{(t)}$ is not a quasi-permutation over $G'$ for any $t \geqq 1$.

Suppose that $[J_{G'/H'}]^{(t)}$ is a quasi-permutation for some $t \geqq 1$. We have an exact sequence

$$0 \to I_{G'/H'} \to \mathbb{Z}G'/H' \to \mathbb{Z} \to 0.$$

Let $\sigma'$ be a generator of $N'$. Then $I_{G'/H'}$ is generated by $\sigma' - 1$. Therefore, defining the map $\phi : \mathbb{Z}G' \to I_{G'/H'}$ by $\phi(1) = \sigma' - 1$ and setting $B' = \mathrm{Ker}\,\phi$, we have an exact sequence

$$0 \to B' \to \mathbb{Z}G' \to I_{G'/H'} \to 0.$$

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It is easy to see that $B'$ is coflasque, and so, by Proposition 1.5, $B'$ is invertible. By assumption, we have $[B']^{(t)} \oplus U' \cong V'$ for some permutation $G'$-lattices $U'$ and $V'$, and so $[B']^{(t)} \oplus U' \cong [[B']^\circ]^{(t)} \oplus U'$. From this it follows that $H^i(G', B') \cong H^i(G', [B']^\circ)$ for any $i$. Computing the two-dimensional cohomology groups, we obtain

$$H^2(G', B') \cong H^1(G', I'_{G'/H'}) \cong \mathbb{Z}/p\mathbb{Z},$$

$$H^2(G', [B']^\circ) \cong H^3(G', J_{G'/H'}) \cong H^4(G', \mathbb{Z})_p \cong H^4(N', \mathbb{Z})^{H'} \cong [\mathbb{Z}/p\mathbb{Z}]^{H'},$$

where the $p$-part of a finite abelian group $A$ is denoted by $A_p$. This implies that $H'$ acts trivially on $H^4(N', \mathbb{Z})$, but, according to [Br, p. 159, Example 6], this is not the case because of $|H'| = 4$ or $q$.

Case 2. Suppose that $|H| = 2$.

If there exists an odd prime $q \mid i(G)$, then there exist a subgroup $H'$ of $H_0$ with $|H'| = q$ and a subgroup $N'$ of $N_0$ with $|N'| = p$ an odd prime such that $H'$ acts nontrivially on $N'$ by conjugation. Setting $G' = N'H'$, and regarding $\mathbb{Z}G/H$ as a $G'$-lattice, we have $\mathbb{Z}G/H \cong [\mathbb{Z}G']^{(s)}$ for some $s \geqq 1$, as $G'$-lattices, and therefore, by Corollary 1.4, $J_{G/H} \cong J_{G'} \oplus [\mathbb{Z}G']^{(s-1)}$. According to [EM2, (2.3)], $[J_{G'}]^{(t)}$ is not a quasi-permutation over $G'$ for any $t \geqq 1$. Thus, we conclude that $[J_{G/H}]^{(t)}$ is not a quasi-permutation over $G$ for any $t \geqq 1$.

If $i(G) (\geqq 3)$ is a power of 2, then there exist a subgroup $H'$ of $H_0$ with $|H'| = 4$ containing $H$ and a subgroup $N'$ of $N_0$ with $|N'| = p$ an odd prime such that $H'$ acts faithfully on $N'$ by conjugation. Setting $G' = N'H'$, and regarding $\mathbb{Z}G/H$ as a $G'$-lattice, we have $\mathbb{Z}G/H \cong \mathbb{Z}G'/H \oplus S'$ as $G'$-lattices, where $S' = 0$ or $S' = \bigoplus_{i=1}^d \mathbb{Z}G'/H_i$, $d \geqq 1$, for subgroups $H_i \subseteq H$, and so, by Proposition 1.3, $J_{G/H} \cong J_{G'/H} \oplus S'$. Therefore, it suffices to show that $[J_{G'/H}]^{(t)}$ is not a quasi-permutation over $G'$. Suppose that $[J_{G'/H}]^{(t)}$ is a quasi-permutation for some $t \geqq 1$. We have an exact sequence

$$0 \to I_{G'/H} \to \mathbb{Z}G'/H \to \mathbb{Z} \to 0.$$

Let $N' = \langle \sigma \rangle$, and let $H' = \langle \tau \rangle$. Then we have $I_{G'/H} = (\sigma - 1, \tau - 1)$. Noticing that $[I_{G'/H}]^{N'} = \mathbb{Z}(\sum_{i=0}^{p-1} \sigma^i)(\tau - 1)$, we can construct the following coflasque resolution of $I_{G'/H}$:

$$0 \to B' \to \mathbb{Z}G' \oplus \bigoplus_{i=1}^{s} \mathbb{Z}G'/H'_i \to I_{G'/H} \to 0,$$

where each $H_i'$ is a subgroup of $H'$. Then $B'$ is invertible by [EM2, (1.5)]. Since $[J_{G'/H}]^{(t)}$ is a quasi-permutation, $[B']^{(t)}$ is a quasi-permutation by Proposition 1.5, and so we have $[B']^{(t)} \oplus U' \cong V'$ for some permutation $G'$-lattices $U'$ and $V'$. From this it follows that $H^i(G', B') \cong H^i(G', [B']^\circ)$ for any $i$. Computing the cohomology groups $H^2(G', B')$ and $H^2(G', [B']^\circ)$ along the same lines as in Case 1, and considering only the $p$-parts of the cohomology groups, we finally see that

$$\mathbb{Z}/p\mathbb{Z} \cong H^4(N', \mathbb{Z})^{H'} \cong [\mathbb{Z}/p\mathbb{Z}]^{H'},$$

which is a contradiction. This completes the proof (3) $\Rightarrow$ (1). $\square$

*The proof of* (1) $\Rightarrow$ (2). Assume that $i(G) = 2$.

Under this assumption, the subgroups of $H$ of $G$ as in the theorem are of order 2. Therefore, $G$ and $H$ are expressible as follows:

$$G = \langle \mu, \nu, \tau \mid \mu^m = \nu^n = \tau^2 = 1, \mu\nu = \nu\mu, \mu\tau = \tau\mu, \tau\nu\tau^{-1} = \nu^{-1} \rangle$$

and $H = \langle \tau \rangle$, where $m, n$ are odd, $m \geqq 1$, $n \geqq 3$, and $(m, n) = 1$, that is, that $G = \langle \mu \rangle \times \langle \nu, \tau \rangle$, the direct product of the cyclic group $C_m$ of order $m$ and the dihedral group $D_n$ of order $2n$.

Now we will prove that $J_{G/H}$ is a quasi-permutation, by induction on the number of prime divisors of $n$. We denote by $\Phi_a(X)$ the $a$th cyclotomic polynomial and by $\zeta_a$ the primitive $a$th root of unity.

Set $\sigma = \mu\nu$, and set $l = mn$. Let $p$ be a prime divisor of $n$. Let $n = p^c n'$, $p \nmid n'$, and let $l' = l/p^c$. Further, let $\Psi(X) = \prod_{r \mid l'} \Phi_{p^c r}(X)$ and $\Psi_0(X) = \Psi(X)/\Phi_{p^c}(X)$, and let $\Gamma = \mathbb{Z}G/(\Psi(\sigma))$, $\Gamma_0 = \mathbb{Z}G/(\Psi_0(\sigma))$, and $\Gamma_1 = \mathbb{Z}G/(\Phi_{p^c}(\sigma))$. Then there is an exact sequence of $G$-lattices

$$0 \to \Gamma_1 \to \Gamma \to \Gamma_0 \to 0.$$

From now on, the tensor products $\otimes$ mean those over $\mathbb{Z}G$ for brevity. As is easily seen, $\Gamma_1 \otimes I_{G/H}$, $\Gamma \otimes I_{G/H}$, and $\Gamma_0 \otimes I_{G/H}$ are torsion free, and hence the following sequence is exact:

$$0 \to \Gamma_1 \otimes I_{G/H} \to \Gamma \otimes I_{G/H} \to \Gamma_0 \otimes I_{G/H} \to 0.$$

From the fact that $\Psi_0(1) = \pm 1$, it follows that $\Gamma_0 \otimes \mathbb{Z} = 0$, and so, tensoring $\Gamma_0$ with the exact sequence $0 \to I_{G/H} \to \mathbb{Z}G/H \to \mathbb{Z} \to 0$, we have

$$\Gamma_0 \otimes I_{G/H} \cong \Gamma_0 \otimes \mathbb{Z}G/H.$$

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Let $\eta(X) = (X^l - 1)/\Psi_0(X) = (X^{l/p} - 1)\Phi_{p^c}(X)$, and let $\Gamma' = \mathbb{Z}G/(\eta(\sigma))$. Then we have the following exact sequence:

$$0 \to \Gamma' \to \mathbb{Z}G \to \Gamma_0 \to 0.$$

Tensoring $\mathbb{Z}G/H$ with this exact sequence, we obtain the following exact sequence:

$$0 \to \Gamma' \otimes \mathbb{Z}G/H \to \mathbb{Z}G/H \to \Gamma_0 \otimes \mathbb{Z}G/H \to 0.$$

Using these facts, we can form the following pullback diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Gamma' \otimes \mathbb{Z}G/H \arrow[r, equal] \arrow[d] & \Gamma' \otimes \mathbb{Z}G/H \arrow[d] & \\
0 \arrow[r] & \Gamma_1 \otimes I_{G/H} \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & \mathbb{Z}G/H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma_1 \otimes I_{G/H} \arrow[r] & \Gamma \otimes I_{G/H} \arrow[r] \arrow[d] & \Gamma_0 \otimes I_{G/H} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Now, $\Gamma_1 \cong \mathbb{Z}[\zeta_{p^c}, \tau]$ is the twisted group ring of $H$ over $\mathbb{Z}[\zeta_{p^c}]$, and $\Gamma_1 \otimes I_{G/H} \cong (\zeta_{p^c} - 1) \subseteq \mathbb{Z}[\zeta_{p^c}]$ is an ambiguous ideal of $\mathbb{Z}[\zeta_{p^c}]$. As is easily seen, $\mathbb{Q}(\zeta_{p^c})$ is tamely ramified over $\mathbb{Q}(\zeta_{p^c})^H = \mathbb{Q}(\zeta_{p^c} + \zeta_{p^c}^{-1})$, and then $\Gamma_1$ is a nonmaximal, hereditary order in the full matrix algebra $M_2(\mathbb{Q}(\zeta_{p^c} + \zeta_{p^c}^{-1}))$ of degree 2 over $\mathbb{Q}(\zeta_{p^c} + \zeta_{p^c}^{-1})$. Setting $S = \mathbb{Z}[\zeta_{p^c}]$ and $P = (\zeta_{p^c} - 1)$, we have $\Gamma_1 \cong S \oplus P$ as $\Gamma_1$-lattices and $(\Gamma_1)^\circ \cong \Gamma_1$, $S^\circ \cong S$. Hence, all of $\Gamma_1$, $(\Gamma_1)^\circ$, $S^\circ$, $S$, $P$, $P^\circ$ are $\Gamma_1$-projective (see [R], [CR, Section 28]). Since $\Gamma_1$ is $\mathbb{Z}H$ free, so is $\Gamma_1 \otimes I_{G/H}$. Therefore, we have

$$\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}G/H, \Gamma_1 \otimes I_{G/H}) \cong H^1(H, \Gamma_1 \otimes I_{G/H}) = 0.$$

Accordingly, the second row of the above diagram is split, and so we obtain the exact sequence

$$0 \to \Gamma' \otimes \mathbb{Z}G/H \to [\Gamma_1 \otimes I_{G/H}] \oplus \mathbb{Z}G/H \to \Gamma \otimes I_{G/H} \to 0.$$

We further see that $[\Gamma_1 \otimes I_{G/H}]^\circ (\cong P^\circ)$ is a quasi-permutation $G/\langle \sigma^{p^c} \rangle$-lattice, because both $\Gamma_1$ and $S$ are quasi-permutations. On the other hand, there is an exact sequence

$$0 \to \mathbb{Z}G/\langle \sigma^{l/p} \rangle \to \Gamma' \to \Gamma_1 \to 0.$$

Tensoring $\mathbb{Z}G/H$ with this, we obtain an exact sequence

$$0 \to \mathbb{Z}G/\langle \sigma^{l/p} \rangle \otimes \mathbb{Z}G/H \to \Gamma' \otimes \mathbb{Z}G/H \to \Gamma_1 \otimes \mathbb{Z}G/H \to 0.$$

Since $\mathbb{Z}G/\langle \sigma^{l/p} \rangle \otimes \mathbb{Z}G/H$ is a permutation and $\Gamma_1 \otimes \mathbb{Z}G/H \cong \mathbb{Z}[\zeta_{p^c}]$, $[\Gamma' \otimes \mathbb{Z}G/H]^\circ$ is also a quasi-permutation. Hence, setting $U = [\Gamma_1 \otimes I_{G/H}]^\circ \oplus \mathbb{Z}G/H$ and $V = [\Gamma' \otimes \mathbb{Z}G/H]^\circ$, we have an exact sequence

(i) $$0 \to [\Gamma \otimes I_{G/H}]^\circ \to U \to V \to 0,$$

where both $U$ and $V$ are quasi-permutations.

Let $\overline{G} = G/\langle \sigma^{mn'p^{c-1}} \rangle$. Note that $\mathbb{Z}\overline{G} = \mathbb{Z}G/\langle \sigma^{mn'p^{c-1}} \rangle \cong \Psi(\sigma)\mathbb{Z}G$ and that $\Psi(1) = p$. Then we can form the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & I_{G/H} \arrow[r] \arrow[d] & \Gamma \otimes I_{G/H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}\overline{G}/H \arrow[r] \arrow[d] & \mathbb{Z}G/H \arrow[r] \arrow[d] & \Gamma \otimes \mathbb{Z}G/H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/p\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

It is easy to see that $Y \cong I_{\overline{G}/H}$, and so we have an exact sequence

(ii) $$0 \to [\Gamma \otimes I_{G/H}]^\circ \to J_{G/H} \to J_{\overline{G}/H} \to 0.$$

Finally, we show that, for any subgroup $G'$ of $G$,

(iii) $$H^0(G', [\Gamma \otimes I_{G/H}]^\circ) = H^0(G', \Gamma \otimes I_{G/H}) = 0.$$

In order to show (iii), we first prove that

(iii′) $$H^0(G', \Gamma \otimes \mathbb{Z}G/H) = 0 \quad \text{for any } G' \subseteq G.$$

By definition, $\Psi(X) = \Phi_p(X^{mn'p^{c-1}})$, and so $\Gamma \otimes \mathbb{Z}G/H = \mathbb{Z}[\sigma]/(\Psi(\sigma)) = \mathbb{Z}[\sigma]/(\Phi_p(\sigma^{mn'p^{c-1}})) \cong \mathbb{Z}[\zeta_p] + \mathbb{Z}[\zeta_p]\sigma + \mathbb{Z}[\zeta_p]\sigma^2 + \cdots + \mathbb{Z}[\zeta_p]\sigma^{mn'p^{c-1}(p-1)-1}$, where $\sigma^{mn'p^{c-1}} = \zeta_p$. From this it follows that $[\Gamma \otimes \mathbb{Z}G/H]^{N_0} = 0$, where $N_0 = \langle \sigma^{mn'p^{c-1}} \rangle$.

Assume first that $N_0 \subseteq G'$. Then $[\Gamma \otimes \mathbb{Z}G/H]^{G'} = 0$, so that $H^0(G', \Gamma \otimes \mathbb{Z}G/H) = 0$. Next assume that $G' \subseteq N$. From the exact sequence

$$0 \to \mathbb{Z}G/\langle \sigma^{mn'p^{c-1}}, \tau \rangle \to \mathbb{Z}G/H \to \Gamma \otimes \mathbb{Z}G/H \to 0,$$

we obtain the following exact sequence:

$$\to H^0(G', \mathbb{Z}G/H) \to H^0(G', \Gamma \otimes \mathbb{Z}G/H) \to H^1(G', \mathbb{Z}G/\langle \sigma^{mn'p^{c-1}}, \tau \rangle) \to$$

Since $\mathbb{Z}G/H \cong \mathbb{Z}N$ as $N$-lattices, $H^0(G', \mathbb{Z}G/H) = 0$, and since $\mathbb{Z}G/\langle \sigma^{mn'p^{c-1}}, \tau \rangle$ is a permutation, $H^1(G', \mathbb{Z}G/\langle \sigma^{mn'p^{c-1}}, \tau \rangle) = 0$. Thus, we have $H^0(G', \Gamma \otimes \mathbb{Z}G/H) = 0$. Further, assume that $G' = H = \langle \tau \rangle$ (or one of its conjugates). Then, we have $N_{G'}(-(\zeta_p + \zeta_p^2 + \cdots + \zeta_p^{(p-1)/2})u) = u$ for any $u \in [\Gamma \otimes \mathbb{Z}G/H]^{G'}$, which implies that $H^0(G', \Gamma \otimes \mathbb{Z}G/H) = 0$. In the other cases, we may assume that $G' = \langle \sigma^{m'n''p^c}, \tau \rangle$, where $m' \mid m$, $n'' \mid n'$ and $m' < m$ or $n'' < n$. Set $N' = \langle \sigma^{m'n''p^c} \rangle$. Then we have $[\Gamma \otimes \mathbb{Z}G/H]^{N'} = (1 + \mu^{m'} + \cdots + (\mu^{m'})^{m/m'-1})(1 + \nu^{n''} + \cdots + (\nu^{n''})^{n'/n''-1})\Gamma \otimes \mathbb{Z}G/H$, and therefore $[\Gamma \otimes \mathbb{Z}G/H]^{G'} = (1 + \mu^{m'} + \cdots + (\mu^{m'})^{m/m'-1})(1 + \nu^{n''} + \cdots + (\nu^{n''})^{n'/n''-1})[\Gamma \otimes \mathbb{Z}G/H]^H = N_{G'}(\Gamma \otimes \mathbb{Z}G/H)$, which implies that $H^0(G', \Gamma \otimes \mathbb{Z}G/H) = 0$. This concludes the proof of (iii′).

From (iii′) and the exact sequence

$$0 \to \Gamma \otimes I_{G/H} \to \Gamma \otimes \mathbb{Z}G/H \to \mathbb{Z}/p\mathbb{Z} \to 0,$$

we obtain an exact sequence

$$H^{-1}(G', \Gamma \otimes \mathbb{Z}G/H) \xrightarrow{\theta} H^{-1}(G', \mathbb{Z}/p\mathbb{Z}) \to H^0(G', \Gamma \otimes I_{G/H}) \to 0$$

for any $G' \subseteq G$. The above map $\theta : H^{-1}(G', \Gamma \otimes \mathbb{Z}G/H) \to H^{-1}(G', \mathbb{Z}/p\mathbb{Z})$ is surjective. In fact, we have $H^{-1}(G', \mathbb{Z}/p\mathbb{Z}) = 0, \mathbb{Z}/p\mathbb{Z}$ when $p \nmid |G'|$, $p \mid |G'|$, respectively. For the case where $p \mid |G'|$, $N_0 = \langle \sigma^{mn'p^{c-1}} \rangle \subseteq G'$ and $N_{N_0}(\Gamma \otimes \mathbb{Z}G/H) = (1 + \zeta_p + \zeta_p^2 + \cdots + \zeta_p^{p-1})\Gamma \otimes \mathbb{Z}G/H = 0$, and therefore $\operatorname{Ker} N_{G'} = \operatorname{Ker} N_{N_0} = \Gamma \otimes \mathbb{Z}G/H$. Thus, $\theta$ is surjective; that is, $H^0(G', \Gamma \otimes I_{G/H}) = 0$, which completes the proof of (iii).

By (i), (iii), and [EM2, (2.2)], $[\Gamma \otimes I_{G/H}]^\circ$ is a quasi-permutation. Further, by (ii) and [EM2, (2.2)], $J_{G/H}$ is a quasi-permutation if and only if $J_{\overline{G}/H}$ is so. Note that, for the case where $n = p$, $\overline{G}$ is cyclic of order $2m$, and therefore $J_{\overline{G}/H}$ is a quasi-permutation. Hence, by induction, we can show that $J_{G/H}$ is a quasi-permutation. This completes the proof of the implication (1) $\Rightarrow$ (2). □

Remark 3.4. The above proof of (3) $\Rightarrow$ (1) was done in the same way as in [CS1, (R4)]. The proof of (1) $\Rightarrow$ (2) was done by making some modifications on that in [EM2, (2.3)].

## §4. Symmetric groups and alternating groups

In this section, we consider the problem for $S_n$ (resp., $A_n$), the symmetric (resp., alternating) group on $n$ letters. We also assume that the subgroup $S_{n-1}$ (resp., $A_{n-1}$) of $S_n$ (resp., $A_n$) is the stabilizer of one of the letters in $S_n$ (resp., $A_n$).

Let $K/k$ be a non-Galois separable field extension of degree $n$, and let $L/k$ be the Galois closure of $K/k$. Let $T_n = R^{(1)}_{K/k}(\mathbb{G}_m)$ be the norm one torus defined by $K/k$.

We give first the following.

Theorem 4.1. *Assume that* $\operatorname{Gal}(L/k) = S_n, n \geqq 2$, *and that* $\operatorname{Gal}(L/K) = S_{n-1}$. *Then,*

(1) *$T_n$ is retract rational over $k$ if and only if $n$ is a prime;*
(2) *$T_n$ is (stably) rational over $k$ if and only if $n = 2, 3$.*

Remark 4.2. The "if" part of Theorem 4.1(1) was first proved in [CS2]. It is well known that, for $n = 2, 3$, $T_n$ is rational over $k$. The "only if" part of Theorem 4.1(2) was proved in [lB] for the case where $n$ is a prime, and in [CK] for the general case. Note that the "only if" parts of Theorem 4.1(1), (2) were proved implicitly in [LL].

Theorem 4.1 can be restated as follows.

THEOREM 4.3. *Let $S_n, n \geqq 2$ be the symmetric group on $n$ letters. Then we have that*

(1) *$J_{S_n/S_{n-1}}$ is quasi-invertible over $S_n$ if and only if $n$ is a prime;*
(2) *$J_{S_n/S_{n-1}}$ is a quasi-permutation over $S_n$ if and only if $n = 2, 3$.*

*Proof.* The "if" part of (1) is only a corollary to Proposition 1.7 because $S_{n-1}$ is a Hall subgroup of $S_n$ if $n$ is a prime. Suppose now that $n$ is not a prime.

First assume that there is an odd prime $p \mid n$, and set $m = n/p \geqq 2$. Let $P$ be the elementary abelian $p$-subgroup of $S_n$ generated by $\rho_1 = (1\ 2\ \cdots\ p)$, $\rho_2 = (p+1\ \ p+2\ \ \cdots\ \ 2p), \ldots,\ \rho_m = ((m-1)p+1\ \ (m-1)p+2\ \ \cdots\ \ mp)$, and set further $P_1 = \langle \rho_2, \rho_3, \ldots, \rho_m \rangle$, $P_2 = \langle \rho_1, \rho_3, \ldots, \rho_m \rangle, \ldots,\ P_m = \langle \rho_1, \rho_2, \ldots, \rho_{m-1} \rangle$. Regarding $\mathbb{Z}S_n/S_{n-1}$ as $P$-lattices, we have

$$\mathbb{Z}S_n/S_{n-1} \cong \mathbb{Z}P/P_1 \oplus \mathbb{Z}P/P_2 \oplus \cdots \oplus \mathbb{Z}P/P_m,$$

and therefore, by [E, Theorem 2(2)], $J_{S_n/S_{n-1}}$ is not quasi-invertible over $P$. This implies that $J_{S_n/S_{n-1}}$ is not quasi-invertible over $S_n$.

Assume next that $n = 2^h, h \geqq 2$. Let $P$ be the subgroup of $S_n$ generated by $(1\ 2)(3\ 4) \cdot (5\ 6)(7\ 8) \cdots (2^h-3\ \ 2^h-2)(2^h-1\ \ 2^h)$ and $(1\ 3)(2\ 4) \cdot (5\ 7)(6\ 8) \cdots (2^h-3\ \ 2^h-1)(2^h-2\ \ 2^h)$. Then $P$ is an elementary abelian group of order 4, and, as is easily seen, $\mathbb{Z}S_n/S_{n-1} \cong [\mathbb{Z}P]^{(2^{h-2})}$ as $P$-lattices. Since $J_P$ is not quasi-invertible by Theorem 1.2(1), it follows from Corollary 1.4 that $J_{S_n/S_{n-1}}$ is not quasi-invertible over $S_n$.

For assertion (2), the "if" part is well known, and so it suffices to prove the "only if" part. However, for $n$ a nonprime, this follows directly from assertion (1). Hence it remains to prove this for $n = p \geqq 5$ a prime. Let $\sigma = (1\ 2\ \cdots\ p)$, and let $\tau$ be a $(p-1)$ cycle on the letters $2, 3, \ldots, p$ acting faithfully on $\langle \sigma \rangle$ by conjugation. Set $G' = \langle \sigma, \tau \rangle$, and set $H' = \langle \tau \rangle$. Then we have $\mathbb{Z}S_p/S_{p-1} \cong \mathbb{Z}G'/H'$, and so $J_{S_p/S_{p-1}} \cong J_{G'/H'}$ as $G'$-lattices. Since $p-1 \geqq 4$, it follows from Theorem 3.2 that $J_{G'/H'}$ is not a quasi-permutation, and so $J_{S_p/S_{p-1}}$ is not a quasi-permutation over $S_p$. Thus, the proof is complete. □

Note that Theorem 4.3(2) can be replaced by the following:

(2′) $[J_{S_n/S_{n-1}}]^{(t)}$ is a quasi-permutation for some $t \geqq 1$ if and only if $n = 2, 3$.

Next, we give the following.

THEOREM 4.4. *Assume that* $\mathrm{Gal}(L/k) = A_n, n \geqq 3$ *and that* $\mathrm{Gal}(L/K) = A_{n-1}$. *Then,*

(1) *$T_n$ is retract rational over $k$ if and only if $n$ is a prime;*
(2) *$[T_n]^{(t)}$ is stably rational over $k$ for some $t \geqq 1$ if and only if $n = 3, 5$.*

This can also be reduced to the following.

THEOREM 4.5. *Let $A_n, n \geqq 3$ be the alternating group on $n$ letters. Then,*
(1) *$J_{A_n/A_{n-1}}$ is quasi-invertible if and only if $n$ is a prime;*
(2) *$[J_{A_n/A_{n-1}}]^{(t)}$ is a quasi-permutation for some $t \geqq 1$ if and only if $n = 3, 5$.*

*Proof.* The "if" part of (1) is only a corollary to Proposition 1.7 because $A_{n-1}$ is a Hall subgroup of $A_n$ if $n$ is a prime. In the case where $n$ is not a prime, the assertions can be proved by the same way as in Theorem 4.3. Thus, the proof of (1) is complete. In order to show (2), we may assume that $n = p \geqq 3$ is a prime. The $(p-1)$ cycle $\tau$ in the proof of Theorem 4.3(2) is not contained in $A_p$. Therefore, we must use $G'' = \langle \sigma, \tau^2 \rangle$ and $H'' = \langle \tau^2 \rangle$ instead of $G'$ and $H'$, respectively, in the proof of Theorem 4.3. Then, by Theorem 3.2, we see that $[J_{G''/H''}]^{(t)}$ is a quasi-permutation for some $t \geqq 1$ if and only if $p = 3, 5$. Since $\mathbb{Z}A_p/A_{p-1} \cong \mathbb{Z}G''/H''$ and $J_{A_p/A_{p-1}} \cong J_{G''/H''}$ as $G''$-lattices, this also shows that $[J_{A_p/A_{p-1}}]^{(t)}$ is not a quasi-permutation over $A_p$ for any $t \geqq 1$ when $p \geqq 7$. On the other hand, $J_{A_3/A_2}$ is a quasi-permutation because $A_3$ is cyclic of order 3. Further, according to [D, (3.3)], $[J_{A_5/A_4}]^{(t)}$ is a quasi-permutation for some $t \geqq 1$. This completes the proof of (2). $\square$

REMARK 4.6. It is an open problem whether $J_{A_5/A_4}$ is a quasi-permutation. This is an interesting problem because we do not know any example of the norm one torus defined by non-Galois separable extension $K/k$ which is stably rational over $k$ except those in Theorem 3.1.

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