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ON WEAK SPECTRAL SYNTHESIS

K. PARTHASARATHY AND SUJATHA VARMA

Weak spectral synthesis fails in the group algebra and the generalised group algebra of any non compact locally compact abelian group and also in the Fourier algebra of any infinite compact Lie group.

1. INTRODUCTION

The main purpose of this note is to observe that the celebrated theorem of Malliavin on the failure of spectral synthesis for the group algebra $L^1(G)$ of a non compact locally compact abelian group G can be strengthened to get the failure of weak spectral synthesis (in the sense of Warner [7]) for $L^1(G)$ and also for the "generalised group algebra" $L^1(G, A)$.

Let A be a commutative, semisimple, regular Banach algebra with maximal ideal space Δ. For a closed subset E of Δ let

$$I(E) = \{x \in A : \widehat{x} = 0 \text{ on } E\},$$
$$j(E) = \{x \in A : \widehat{x} \text{ has compact support disjoint from } E\}$$

and $J(E) = \overline{j(E)}$. E is called a weak spectral set for A if for each x in $I(E)$ there is a positive integer $n = n(x)$ such that x^n belongs to $J(E)$. Warner [6] proved the pleasant result that if E is a weak spectral set then n can be chosen to be independent of x in $I(E)$. When $n = 1$, E is a spectral set or a set of spectral synthesis.

We point out here that E is a weak spectral set if and only if there is a positive integer n such that the product $x_1 \dots . x_n$ is in $J(E)$ for every choice of $x_1, \dots, x_n$ in $I(E)$. This is because any n-fold product is a linear combination of nth powers.

In Section 2 we prove a result on certain weak spectral sets for $L^1(G, A)$, the Banach algebra of A-valued integrable functions on a locally compact abelian group G. In the last section we point out that weak spectral synthesis fails in $L^1(G)$ and in $L^1(G, A)$ when G is non-compact.

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2. Weak Spectral sets in $L^1(G, A)$

$L^1(G, A)$ is a commutative semisimple regular Banach algebra with maximal ideal space $\Gamma \times \Delta$ where Γ is the dual group of G. (An extensive discussion of spaces of this type can be found in [**3**].)

Theorem 2.1. *Let E and M be closed sets in Γ and Δ respectively. If $E \times M$ is a weak spectral set for $L^1(G, A)$ then E and M are weak spectral set for $L^1(G)$ and A respectively.*

Proof: Suppose $E \times M$ is weak spectral for $L^1(G, A)$. To prove that E is weak spectral for $L^1(G)$, let $f \in I(E)$. Choose $\phi \in M$ and $a \in A$ with $\phi(a) = 1$. Then $af \in I(E \times M)$ where $(af)(x) = f(x)a$, $x \in G$. Since $E \times M$ is a weak spectral set for $L^1(G, A)$, some convolution power $(af)^n$ can be approximated by functions $F \in j(E \times M)$. Then it is easy to see that $\phi \circ F \in j(E)$. Further, $\phi \circ (af)^n = f^n$ and

$$\begin{aligned} \|f^n - \phi \circ F\|_{L^1(G)} &= \|\phi \circ ((af)^n - F)\|_{L^1(G)} \\ &\leqslant \|(af)^n - F\|_{L^1(G, A)}. \end{aligned}$$

Thus the convolution power f^n can be approximated by functions $\phi \circ F \in j(E)$. It follows that E is weak spectral for $L^1(G)$.

The proof that M is weak spectral for A is similar. Let $a \in I(M)$ and choose $\gamma \in E$, $g \in L^1(G)$ such that $\widehat{g}(\gamma) = 1$. Then $ag \in I(E \times M)$. If $F \in j(E \times M)$ approximates $(ag)^n$, it is easy to see that the A-valued Fourier transform $b = \mathcal{F}F(\gamma)$ belongs to $j(M)$ and approximates a^n and the theorem is proved. ∎

We shall use this result in the next section to get the failure of weak synthesis for $L^1(G, A)$.

3. Failure of weak synthesis

We consider the Fourier algebra $A(G) = L^1(\Gamma)^\wedge$ (with the norm carried over from $L^1(\Gamma)$) in place of the group algebra $L^1(\Gamma)$. We then have the following improvement of Malliavin's theorem.

Theorem 3.1. *Weak spectral synthesis fails in $A(G)$ for every non discrete locally compact abelian group G. In fact, every nonempty open set in G contains a closed set which is not a weak spectral set for $A(G)$.*

Proof: A detailed exposition of the Varopoulos proof of Malliavin's theorem is given in Section 42 of Hewitt and Ross [**2**]. Each of the following steps can be proved by a suitable modification of the proof of the corresponding result for synthesis given there.

(1) Weak synthesis fails for $A(D)$, D being the product of countably infinite copies of the two element group.

(2) If weak synthesis fails for $A(G)$, G compact abelian, then it fails for the Varopoulos algebra $V(G) = C(G)\widehat{\otimes}C(G)$.
(3) Let E be a closed set in G. If weak synthesis fails for the restriction algebra $A(E)$, then it fails for $A(G)$.
(4) Weak synthesis fails for $A(G)$ for any compact, infinite abelian group G.
(5) Proof of the statements of the theorem.

As a sample, we sketch a proof of (3) to indicate the kind of modifications involved. Suppose S is a closed subset of E which is not weak spectral for $A(E)$. Then, using obvious notation, for each positive integer n there exist a function f_n in $I_E(S)$ and a positive real number α_n such that

$$d(f_n^n, j_E(S)) \geqslant \alpha_n.$$

Let $g_n \in A(G)$ be such that $g_n = f_n$ on E; then $g_n \in I(S)$. To see that S is not weak spectral for $A(G)$, suppose, to get a contradiction, that $g_n^n \in J(S)$ for some n. Choose $k_n \in j(S)$ such that

$$\|g_n^n - k_n\|_{A(G)} < \alpha_n.$$

If h_n denotes the restriction of k_n to E, then $h_n \in j_E(S)$ and

$$\alpha_n \leqslant \|f_n^n - h_n\|_{A(E)} \leqslant \|g_n^n - k_n\|_{A(G)} < \alpha_n.$$

This gives the required contradiction and completes the proof. $\square$

Combining Theorems 2.1 and 3.1 we get the failure of weak synthesis in $L^1(G, A)$.

THEOREM 3.2. *Weak spectral synthesis fails for $L^1(G, A)$ when G is non compact.*

PROOF: There is a closed set E of Γ which is not weak spectral for $L^1(G)$ (Theorem 3.1). Then $E \times \Delta$ is not weak spectral for $L^1(G, A)$ (Theorem 2.1). $\square$

REMARK 3.3. (i) Theorem 3.1 implies that weak synthesis fails for any Segal algebra $S(G)$ for non compact G. This is because a Segal algebra has "the same ideal theory" as the group algebra. (The relevant definitions and results can be found in Reiter [4].)

(ii) Step (3) of the proof of Theorem 3.1 is valid for compact nonabelian G. It then follows that weak synthesis fails for the Fourier algebra $A(G)$ of any compact non-abelian group G containing an infinite abelian subgroup. In particular, weak synthesis fails for $A(G)$ when G is any infinite compact Lie group (compare [2], 42.27).

(iii) Bloom [1] defined and studied p-spectral sets in G, $1 \leqslant p < \infty$. The 1-spectral sets are just the spectral sets and he showed that every closed set is p-spectral for $p \geqslant 2$. Following the proof of Malliavin's theorem as expounded in Rudin [5, Chapter 7], he proved that every non discrete locally compact abelian group contains closed sets which are not p-spectral for any p, $1 \leqslant p < 2$. Weak p-spectral sets can be defined in an obvious way and his arguments can be modified to give the existence of closed sets which are not weak p-spectral for any p, $1 \leqslant p < 2$, in every non discrete locally compact abelian group. We omit the details which any interested reader would be able to work out.

We conclude with an open problem. It is a well known result of Schwartz that the unit sphere S^n in $\mathbb{R}^{n+1}$ is not a spectral set for $L^1(\mathbb{R}^{n+1})$, $n > 1$. But Varopoulos [6] has shown that S^n is a weak spectral set for $L^1(\mathbb{R}^{n+1})$.

PROBLEM: Does every non discrete locally compact abelian group Γ contain a weak spectral set which is not a spectral set for $L^1(G)$?

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Ramanujan Institute for
Advanced Study in Mathematics
University of Madras
Madras – 600 005
India

Indira Gandhi National
Open University
New Delhi
India

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