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REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY A LÉVY PROCESS

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Abstract

In this paper, we deal with a class of reflected backward stochastic differential equations (RBSDEs) corresponding to the subdifferential operator of a lower semi-continuous convex function, driven by Teugels martingales associated with a Lévy process. We show the existence and uniqueness of the solution for RBSDEs by means of the penalization method. As an application, we give a probabilistic interpretation for the solutions of a class of partial differential-integral inclusions.

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1. Introduction

The theory of backward stochastic differential equations (BSDEs) was developed by Pardoux and Peng [21]. Given data (ξ, f) consisting of a progressively measurable process f and a square integrable random variable ξ, they proved the existence and uniqueness of an adapted process (Y, Z) solution to the following BSDEs:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s, \quad 0 \le t \le T.$$

These equations have attracted great interest due to their connections with mathematical finance [7], stochastic control and stochastic games [10–12]. Furthermore, it was shown in various papers that BSDEs give the probabilistic representation for solution (at least in the viscosity sense) of a large class of systems of semi-linear parabolic partial differential equations (PDEs) [20, 23]. Further,

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other settings of BSDEs have been introduced. Gegout-Petit [8] proposed a class of reflected backward stochastic differential equations (RBSDEs) associated with a multivalued maximal monotone operator defined by the subdifferential of a convex function. Further, Pardoux and Rǎşcanu [22] proved the existence and uniqueness of the solution for RBSDEs, on a random (possibly infinite) time interval, involving a subdifferential operator in order to give the probabilistic interpretation for the viscosity solution of some parabolic and elliptic variational inequalities. Then Ouknine [19], N'Zi and Ouknine [18] and Bahlali *et al.* [1, 2] discussed a type of RBSDEs driven by a Brownian motion or the combination of a Brownian motion and a Poisson random measure under Lipschitz conditions, locally Lipschitz conditions, or monotone conditions on the coefficients. El Karoui *et al.* [6] introduced another type of RBSDEs, where one of the components of the solution is forced to stay above a given barrier, which provided a probabilistic formula for the viscosity solution of an obstacle problem for a parabolic PDE. Since then, there have been many papers on this topic; see, for example, Matoussi [15], Hamadène [9, 13], Lepeltier and Xu [14] and Ren *et al.* [24, 25].

The main tool in the theory of BSDEs is the martingale representation theorem for a martingale which is adapted to the filtration of a Brownian motion or a Poisson point process [21, 28]. Recently, Nualart and Schoutens [16] gave a martingale representation theorem associated with a Lévy process. This class of Lévy processes includes Brownian motion and the Poisson, gamma, negative binomial and Meixner processes as special cases. Based on [16], they showed the existence and uniqueness of the solution for BSDEs driven by Teugels martingales associated with a Lévy process in [17]. These results were important from a pure mathematical point of view as well as from an applied point of view (in the world of finance). Specifically, they could be used for the purpose of option pricing in a Lévy market and gave the probabilistic interpretation for the solutions of a related partial differential equation which provided an analogue of the famous Black–Scholes formula.

Motivated by the above work, the purpose of the present paper is to consider RBSDEs related to the subdifferential operator of a lower semi-continuous convex function driven by Teugels martingales, associated with a Lévy process, which are considered in Nualart and Schoutens [16, 17]. We obtain the existence and uniqueness of the solutions for such RBSDEs. As an application, we give a probabilistic interpretation for the solutions of a class of partial differential–integral inclusions (PDIIs).

The paper is organized as follows. In Section 2 we introduce some preliminaries. Section 3 is devoted to the proof of the existence and uniqueness of the solution for RBSDEs driven by a Lévy process by means of the penalization method. A probabilistic interpretation for the solutions of a class of PDIIs by our RBSDEs is given in the final section.

2. Preliminaries

Let $T > 0$ be a fixed terminal time and $(\Omega, \mathcal{F}, \mathbb{P})$ be a completed probability space on which an $\mathbb{R}$-valued Lévy process $(L_t)_{t\in[0,T]}$ with càdlàg paths is defined. Let

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$\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be the right-continuous filtration generated by L, that is, $\mathcal{F}_t = \sigma\{L_s; 0 \leq s \leq t\} \vee \mathcal{N}$, and assume that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets $\mathcal{N}$ of $\mathcal{F}$. The process L is characterized by its so-called local characteristics in the Lévy–Khintchine formula so that

$$\mathbb{E}e^{iuL_t} = e^{-t\Psi(u)}$$

with

$$\Psi(u) = -iau + \frac{\sigma^2}{2}u^2 - \int_{\mathbb{R}} \left(e^{iux} - 1 - iux1_{\{|x|\leq 1\}}\right) \nu(dx).$$

Thus L is characterized by its Lévy triplet (a, σ, ν) where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure defined in $\mathbb{R}\backslash\{0\}$ which satisfies:

(i) $\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < +\infty$,
(ii) $\int_{(-\varepsilon,\varepsilon)^c} e^{\lambda|x|}\nu(dx) < +\infty$, for every $\varepsilon > 0$ and for some $\lambda > 0$.

This implies that the random variables L_t have moments of all orders, that is,

$$\int_{-\infty}^{+\infty} |x|^i \nu(dx) < \infty, \quad \forall i \geq 2.$$

For background on Lévy processes, we refer the reader to [3, 27].

We use $L_{t-} = \lim_{s\nearrow t} L_s$ and $\Delta L_t = L_t - L_{t-}$ and define the power jumps of the Lévy process L by

$$L_t^{(1)} = L_t \quad \text{and} \quad L_t^{(i)} = \sum_{0<s\leq t} (\Delta L_s)^i, \quad i \geq 2.$$

Let $m_1 = \mathbb{E}[L_1] = a + \int_{|x|\geq 1} x\nu(dx)$ and $m_i = \int_{-\infty}^{+\infty} x^i \nu(dx)$ for $i \geq 2$. For $i \geq 1$ let us define $Y_t^{(i)} = L_t^{(i)} - m_i t$, the so-called Teugels martingales. We associate with the Lévy process $(L_t)_{0\leq t\leq T}$ the family of processes $(H^{(i)})_{i\geq 1}$ defined by $H_t^{(i)} = \sum_{j=1}^{i} a_{ij} Y_t^{(j)}$. The coefficients a_{ij} correspond to the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $\pi(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$. Specifically, the polynomials q_n defined by $q_n(x) = \sum_{k=1}^{n} a_{nk}x^{k-1}$ are orthonormal with respect to the measure π:

$$\int_{\mathbb{R}} q_n(x)q_m(x)\pi(dx) = 0 \text{ if } n \neq m \quad \text{and} \quad \int_{\mathbb{R}} q_n(x)^2\pi(dx) = 1.$$

We set

$$p_n(x) = xq_{n-1}(x) = a_{n,n}x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n,1}x.$$

The martingales $H^{(i)}$, called the orthonormalized ith-power-jump processes, are strongly orthogonal and their predictable quadratic variation processes are

$$\langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij}t.$$

REMARK 1. If $\nu = 0$, we have the classic Brownian case and all nonzero-degree polynomials $q_i(x)$ will vanish, giving $H_t^{(i)}$, $i = 2, 3, \ldots$. If μ only has a mass at 1, we have the Poisson case; here also $H_t^{(i)} = 0$, $i = 2, 3, \ldots$. Both cases are degenerate in this Lévy framework.

Let us introduce the following appropriate spaces.

- $\ell^2 = \Big\{x = (x_n)_{n\geq 1};\ \|x\|_{\ell^2} = \big(\sum_{n=1}^{\infty} |x_n|^2\big)^{1/2} < \infty\Big\}$.
- $\mathcal{S}^2(\mathbb{R})$, the subspace of the $\mathcal{F}_t$-adapted, rcll (right continuous having left-hand limits), $\mathbb{R}$-valued processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|^2_{\mathcal{S}^2(\mathbb{R})} = \mathbb{E}\bigg(\sup_{0\leq t\leq T} |Y_t|^2\bigg) < +\infty.$$

- $\mathcal{P}^2(\ell^2)$, the space of predictable processes $(Z)_{t\in[0,T]}$ taking values in ℓ^2 such that

$$\|Z\|^2_{\mathcal{P}^2(\ell^2)} = \mathbb{E}\int_0^T \|Z_s\|^2_{\ell^2}\, ds = \sum_{i=1}^{\infty} \mathbb{E}\int_0^T |Z_s^{(i)}|^2\, ds < \infty.$$

Throughout the paper, we work under the following standing assumptions.

(H1) The terminal value $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

(H2) The coefficient $f : [0, T] \times \Omega \times \mathbb{R} \times \ell^2 \to \mathbb{R}$ is $\mathcal{F}_t$-progressively measurable and satisfies

$$\mathbb{E}\int_0^T |f(s, 0, 0)|^2\, ds < \infty.$$

(H3) There exists a constant $C > 0$ such that for every $(\omega, t) \in \Omega \times [0, T]$, $(y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \ell^2$,

$$|f(t, y_1, z_1) - f(t, y_2, z_2)|^2 \leq C\big(|y_1 - y_2|^2 + \|z_1 - z_2\|^2_{\ell^2}\big).$$

(H4) Let $\Phi : \mathbb{R} \to (-\infty, +\infty]$ be a proper lower semi-continuous convex function. Define:

$$\mathrm{Dom}(\Phi) := \{u \in \mathbb{R} \mid \Phi(u) < +\infty\}, \quad \mathrm{Dom}(\partial\Phi) := \{u \in \mathbb{R} \mid \partial\Phi \neq \emptyset\},$$
$$\partial\Phi(u) := \big\{u^* \in \mathbb{R} \mid \langle u^*, v - u\rangle + \Phi(u) \leq \Phi(v),\ \text{for all } v \in \mathbb{R}\big\},$$
$$\mathrm{Gr}(\partial\Phi)) := \Big\{(u, u^*) \in \mathbb{R}^2 \ \Big|\ u \in \mathrm{Dom}(\partial\Phi),\ u^* \in \partial\Phi(u)\Big\}.$$

(H5) Further, we assume that $\xi \in \overline{\mathrm{Dom}(\Phi)}$ and $\mathbb{E}\Phi(\xi) < \infty$.

Now, we introduce a multivalued maximal monotone operator on $\mathbb{R}$ defined by the subdifferential of the above function Φ. The details appeared in Brezis [4]. For all $x \in \mathbb{R}$, define

$$\Phi_n(x) = \min_y \left(\frac{n}{2}|x - y|^2 + \Phi(y)\right).$$

Let $J_n(x)$ be the unique solution of the differential inclusion $x \in J_n(x) + \partial\Phi(J_n(x))/n$, which is called the resolvent of the monotone operator $A = \partial\Phi$. Then, we have the following proposition.

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PROPOSITION 2.1.

(1) $\Phi_n : \mathbb{R} \to \mathbb{R}$ *is Lipschitz continuous.*

(2) *The Yosida approximation of* $\partial\Phi$ *defined by*

$$A_n(x) := \nabla\Phi_n(x) = n(x - J_n(x)), \quad x \in \mathbb{R},$$

is monotonic and Lipschitz continuous and there exists $a \in \operatorname{interior}(\operatorname{Dom}(\Phi))$ *and positive numbers* R, C *which satisfy*

$$\left\langle \nabla\Phi_n(z)^*, z - a \right\rangle \geq R|A_n(z)| - C|z| - C \quad \text{for all } z \in \mathbb{R} \text{ and } n \in \mathbb{N}. \tag{2.1}$$

(3) *For all* $x \in \mathbb{R}$, $A_n(x) \in A(J_n(x))$.

This paper is mainly concerned with the following RBSDE.

DEFINITION 2. The solution associated with the data (ξ, f, Φ) is a triple $(Y_t, Z_t, K_t)_{0\leq t\leq T}$ of progressively measurable processes such that, for all $0 \leq t \leq T$,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + K_T - K_t - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)}\, dH_s^{(i)} \tag{2.2}$$

satisfying:

(1) $(Y_t, Z_t)_{0\leq t\leq T} \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{P}^2(\ell^2)$;

(2) $\{Y_t, 0 \leq t \leq T\}$ is rcll and takes values in $\overline{\operatorname{Dom}(\Phi)}$;

(3) $\{K_t, 0 \leq t \leq T\}$ is absolutely continuous, $K_0 = 0$, and for all progressively measurable and right continuous processes $\{(\alpha_t, \beta_t), 0 \leq t \leq T\}$ valued in $Gr(\partial\Phi)$,

$$\int_0^T (Y_t - \alpha_t)(dK_t + \beta_t\, dt) \leq 0.$$

In order to obtain the existence and uniqueness of the solution for the RBSDE, for all $0 \leq t \leq T$, we consider the following BSDEs:

$$Y_t^n = \xi + \int_t^T \left[f(s, Y_s^n, Z_s^n) - A_n(Y_s^n) \right] ds - \sum_{i=1}^{\infty} \int_t^T Z_s^{n,(i)}\, dH_s^{(i)}, \tag{2.3}$$

where ξ and f satisfy the assumptions stated above and A_n is the Yosida approximation of the operator $A = \partial\Phi$. Since A_n is Lipschitz continuous, it follows from the result of [17] that (2.3) has a unique solution $(Y_t^n, Z_t^n)_{0\leq t\leq T} \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{P}^2(\ell^2)$.

Set $K_t^n = -\int_0^t A_n(Y_s^n)\, ds$, $0 \leq t \leq T$. Our aim is to prove that the sequence (Y^n, Z^n, K^n) converges to (Y, Z, K) which is the solution of the RBSDE.

3. Existence and uniqueness of the solutions

The main result of this section is the following theorem.

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THEOREM 3.1. *Suppose that assumptions* (H1)–(H5) *hold. Then the RBSDE* (2.2) *has a unique solution* $(Y_t, Z_t, K_t)_{0\le t\le T}$. *Moreover,*

$$\lim_{n\to\infty} \mathbb{E}\Big(\sup_{0\le t\le T} |Y_t^n - Y_t|^2\Big) = 0, \quad \lim_{n\to\infty} \mathbb{E}\int_0^T \|Z_t^n - Z_t\|_{\ell^2}^2 \, dt = 0,$$
$$\lim_{n\to\infty} \mathbb{E}\Big(\sup_{0\le t\le T} |K_t^n - K_t|^2\Big) = 0,$$

where (Y^n, Z^n) *is the solution of BSDEs* (2.3).

In the sequel, $C > 0$ denotes a constant whose value can vary from line to line. The proof of Theorem 3.1 is divided into the following lemmas.

LEMMA 3.2. *Under the assumptions of Theorem* 3.1, *there exists a constant* $C_1 > 0$ *such that for all* $n \ge 1$,

$$\mathbb{E}\left(\sup_{0\le t\le T} |Y_t^n|^2 + \int_0^T \|Z_s^n\|_{\ell^2}^2 \, ds + \int_0^T |A_n(Y_s^n)| \, ds\right) \le C_1. \tag{3.1}$$

PROOF. Applying Itô's formula to $|Y_t^n - a|^2$ yields that

$$\begin{aligned} |Y_t^n - a|^2 = |\xi - a|^2 &+ 2\int_t^T (Y_s^n - a) f(s, Y_s^n, Z_s^n) \, ds - 2\int_t^T (Y_s^n - a) A_n(Y_s^n) \, ds \\ &- \int_t^T \|Z_s^n\|_{\ell^2}^2 \, ds - 2\sum_{i=1}^{\infty} \int_t^T (Y_s^n - a) Z_s^{n,(i)} \, dH_s^{(i)}. \end{aligned} \tag{3.2}$$

Taking expectation on the both sides of (3.2) and considering (2.1), we obtain

$$\begin{aligned} \mathbb{E}|Y_t^n - a|^2 + \mathbb{E}\int_t^T \|Z_s^n\|_{\ell^2}^2 \, ds &\le \mathbb{E}|\xi - a|^2 + 2\mathbb{E}\int_t^T (Y_s^n - a) f(s, Y_s^n, Z_s^n) \, ds \\ &- 2R\mathbb{E}\int_t^T |A_n(Y_s^n)| \, ds + 2C\mathbb{E}\int_t^T |Y_s^n| \, ds + 2C. \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}|Y_t^n - a|^2 &+ \mathbb{E}\int_t^T \|Z_s^n\|_{\ell^2}^2 \, ds + 2R\mathbb{E}\int_t^T |A_n(Y_s^n)| \, ds \\ &\le \mathbb{E}|\xi - a|^2 + 2\mathbb{E}\int_t^T (Y_s^n - a) f(s, Y_s^n, Z_s^n) \, ds + 2C\mathbb{E}\int_t^T |Y_s^n| \, ds + 2C \\ &\le \mathbb{E}|\xi - a|^2 + C\mathbb{E}\int_t^T |Y_s^n - a|^2 \, ds + \frac{1}{2}\mathbb{E}\int_t^T \|Z_s^n\|_{\ell^2}^2 \, ds + C, \end{aligned} \tag{3.3}$$

where we have used the elementary inequality $2ab \leq \beta a^2 + b^2/\beta$, for all $\beta > 0$. From (3.3), we obtain

$$\mathbb{E}|Y_t^n - a|^2 + \frac{1}{2}\mathbb{E}\int_t^T \|Z_s^n\|_{\ell^2}^2 \, ds \leq C\left(1 + \mathbb{E}\int_t^T |Y_s^n - a|^2 \, ds\right). \tag{3.4}$$

Gronwall's inequality shows that $\mathbb{E}|Y_t^n - a|^2 \leq C \ \forall n$. Hence $\mathbb{E}|Y_t^n|^2 \leq C \ \forall n$. From (3.3) and (3.4), it is easy to show that

$$\mathbb{E}\int_0^T \left(\|Z_s^n\|_{\ell^2}^2 + |A_n(Y_s^n)|\right) \, ds \leq C \quad \forall n.$$

Finally, by the Bulkholder–Davis–Gundy inequality [5], the desired result follows. □

LEMMA 3.3. *Under the assumptions of Theorem 3.1, there exists a constant $C_2 > 0$ such that for all $n \geq 1$,*

$$\mathbb{E}\int_0^T |A_n(Y_s^n)|^2 \, ds \leq C_2.$$

PROOF. Without loss of generality, we assume that $\Phi \geq 0$, $\Phi(0) = 0$. Let $\psi_n \triangleq \Phi_n/n$. By the convexity of ψ_n, Itô's formula yields that

$$\begin{aligned}\psi_n(Y_t^n) \leq \psi_n(\xi) &+ \int_t^T \nabla\psi_n(Y_s^n)\left[f(s, Y_s^n, Z_s^n) - A_n(Y_s^n)\right] \, ds \\ &- \sum_{i=1}^{\infty}\int_t^T \nabla\psi_n(Y_s^n) Z_s^{n,(i)} \, dH_s^{(i)}.\end{aligned} \tag{3.5}$$

Taking expectation on both sides of (3.5), we obtain

$$\begin{aligned}\mathbb{E}\psi_n(Y_t^n) &\leq \mathbb{E}\psi_n(\xi) + \mathbb{E}\int_t^T \nabla\psi_n(Y_s^n) f(s, Y_s^n, Z_s^n) \, ds \\ &\quad - \mathbb{E}\int_t^T \nabla\psi_n(Y_s^n) A_n(Y_s^n) \, ds \\ &= \mathbb{E}\psi_n(\xi) + \mathbb{E}\int_t^T \nabla\psi_n(Y_s^n) f(s, Y_s^n, Z_s^n) \, ds \\ &\quad - \frac{1}{n}\mathbb{E}\int_t^T |A_n(Y_s^n)|^2 \, ds.\end{aligned} \tag{3.6}$$

Using the elementary inequality $2ab \leq \beta a^2 + b^2/\beta$, for all $\beta > 0$, we obtain

$$\begin{aligned}&\mathbb{E}\psi_n(Y_t^n) + \frac{1}{n}\mathbb{E}\int_t^T |A_n(Y_s^n)|^2 \, ds \\ &\quad \leq \mathbb{E}\psi_n(\xi) + \frac{1}{2n}\mathbb{E}\int_t^T |A_n(Y_s^n)|^2 \, ds + \frac{1}{2n}\mathbb{E}\int_t^T |f(s, Y_s^n, Z_s^n)|^2 \, ds.\end{aligned} \tag{3.7}$$

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Further, we have

$$
\begin{aligned}
&\mathbb{E}\psi_n(Y_t^n) + \frac{1}{n}\mathbb{E}\int_t^T |A_n(Y_s^n)|^2\,ds \\
&\quad \le \mathbb{E}\psi_n(\xi) + \frac{1}{2n}\mathbb{E}\int_t^T |A_n(Y_s^n)|^2\,ds + \frac{1}{2n}\mathbb{E}\int_t^T |f(s, Y_s^n, Z_s^n)|^2\,ds \\
&\quad \le \mathbb{E}\psi_n(\xi) + \frac{1}{2n}\mathbb{E}\int_t^T |A_n(Y_s^n)|^2\,ds + \frac{C}{n}\mathbb{E}\int_t^T |Y_s^n|^2\,ds \\
&\qquad + \frac{C}{n}\mathbb{E}\int_t^T \|Z_s^n\|_{\ell^2}^2\,ds + \frac{C}{n}\mathbb{E}\int_t^T |f(s, 0, 0)|^2\,ds. \qquad (3.8)
\end{aligned}
$$

By Lemma 3.2,

$$
\mathbb{E}\psi_n(Y_t^n) + \frac{1}{n}\mathbb{E}\int_t^T |A_n(Y_s^n)|^2\,ds \le \frac{C}{n},
$$

which implies the desired result. □

LEMMA 3.4. *Under the assumptions of Theorem 3.1, (Y^n, Z^n) is a Cauchy sequence in $\mathcal{S}^2(\mathbb{R}) \times \mathcal{P}^2(\ell^2)$.*

PROOF. Applying Itô's formula to $|Y_t^n - Y_t^m|^2$ yields that

$$
\begin{aligned}
&|Y_t^n - Y_t^m|^2 + \int_t^T \|Z_s^n - Z_s^m\|_{\ell^2}^2\,ds \\
&\quad = 2\int_t^T (Y_s^n - Y_s^m)[f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m)]\,ds \\
&\qquad - 2\int_t^T (Y_s^n - Y_s^m)(A_n(Y_s^n) - A_m(Y_s^m))\,ds \\
&\qquad - 2\sum_{i=1}^{\infty}\int_t^T (Y_s^n - Y_s^m)\big(Z_s^{n,(i)} - Z_s^{m,(i)}\big)\,dH_s^{(i)}. \qquad (3.9)
\end{aligned}
$$

From the relation

$$
I = J_n + \frac{1}{n}A_n = J_m + \frac{1}{m}A_m, \quad A_m(Y_s^m) \in \partial\Phi(J_m(Y_s^m)),\ A_n(Y_s^n) \in \partial\Phi(J_n(Y_s^n)),
$$

we have

$$
\begin{aligned}
&-(Y_s^n - Y_s^m)(A_n(Y_s^n) - A_m(Y_s^m)) \\
&\quad = -\big(A_n(Y_s^n) - A_m(Y_s^m), J_n(Y_s^n) - J_m(Y_s^m)\big) \\
&\qquad - \left(A_n(Y_s^n) - A_m(Y_s^m), \frac{1}{n}A_n(Y_s^n) - \frac{1}{m}A_m(Y_s^m)\right) \\
&\quad \le -\left(A_n(Y_s^n) - A_m(Y_s^m), \frac{1}{n}A_n(Y_s^n) - \frac{1}{m}A_m(Y_s^m)\right).
\end{aligned}
$$

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By using the elementary inequality $xy \le x^2/4 + y^2$, for all $x \ge 0$, $y \ge 0$, we obtain

$$\begin{aligned}
&-\left(A_n(Y_s^n) - A_m(Y_s^m),\ \frac{1}{n}A_n(Y_s^n) - \frac{1}{m}A_m(Y_s^m)\right)\\
&\quad= \left(\frac{1}{n} + \frac{1}{m}\right)(A_n(Y_s^n),\ A_m(Y_s^m)) - \frac{1}{n}|A_n(Y_s^n)|^2 - \frac{1}{m}|A_m(Y_s^m)|^2\\
&\quad\le \frac{1}{4m}|A_n(Y_s^n)|^2 + \frac{1}{4n}|A_m(Y_s^m)|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}|Y_t^n - Y_t^m|^2 + \mathbb{E}\int_t^T \|Z_s^n - Z_s^m\|_{\ell^2}^2\, ds\\
&\quad\le 2C\mathbb{E}\int_t^T \left(|Y_s^n - Y_s^m|^2 + |Y_s^n - Y_s^m|\|Z_s^n - Z_s^m\|_{\ell^2}\right)\, ds\\
&\qquad+ \mathbb{E}\int_t^T \left(\frac{1}{4m}|A_n(Y_s^n)|^2 + \frac{1}{4n}|A_m(Y_s^m)|^2\right)\, ds\\
&\quad\le 2C\mathbb{E}\int_t^T \left[(1+\beta)|Y_s^n - Y_s^m|^2 + \frac{1}{\beta}\|Z_s^n - Z_s^m\|_{\ell^2}^2\right]\, ds\\
&\qquad+ \mathbb{E}\int_t^T \left(\frac{1}{4m}|A_n(Y_s^n)|^2 + \frac{1}{4n}|A_m(Y_s^m)|^2\right)\, ds.
\end{aligned}$$

Choosing β such that $2C/\beta < 1/2$, we obtain

$$\sup_{0\le t\le T} \mathbb{E}|Y_t^n - Y_t^m|^2 + \frac{1}{2}\mathbb{E}\int_0^T \|Z_s^n - Z_s^m\|_{\ell^2}^2\, ds \le C\left(\frac{1}{n} + \frac{1}{m}\right).$$

Further, by the Bulkholder–Davis–Gundy inequality, it follows that

$$\mathbb{E}\left(\sup_{0\le t\le T} |Y_t^n - Y_t^m|^2\right) + \frac{1}{2}\mathbb{E}\int_0^T \|Z_s^n - Z_s^m\|_{\ell^2}^2\, ds \le C\left(\frac{1}{n} + \frac{1}{m}\right),$$

which gives the desired result. □

In order to obtain the existence of solutions to the RBSDE, we give the following lemma which appeared in Saisho [26].

LEMMA 3.5. *Let $\{K^{(n)}, n \in \mathbb{N}\}$ be a family of continuous functions of finite variation on $\mathbb{R}^+$. Assume that:*

(1) $\sup_n |K^{(n)}|_t \le C_t < \infty,\ \ 0 \le t < \infty$;
(2) $\lim_{n\to\infty} K^{(n)} = K \in C([0, +\infty); \mathbb{R})$.

Then K is of finite variation. Moreover, if $\{f^{(n)}, n \in \mathbb{N}\}$ is a family of continuous functions such that $\lim_{n\to\infty} f^{(n)} = f \in C([0, +\infty); \mathbb{R})$, then

$$\lim_{n\to\infty} \int_s^t \left\langle f_u^{(n)},\ dK_u^{(n)}\right\rangle = \int_s^t \langle f_u,\ dK_u\rangle \quad \textit{for all } 0 \le s \le t < \infty.$$

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Now, we give the proof of Theorem 3.1.

PROOF. Beginning with existence, Lemma 3.4 shows that (Y^n, Z^n) is a Cauchy sequence in space $\mathcal{S}^2(\mathbb{R}) \times \mathcal{P}^2(\ell^2)$. We denote its limit as (Y, Z). Now let us put

$$K_T - K_t = Y_t - \xi - \int_t^T f(s, Y_s, Z_s)\, ds + \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)}\, dH_s^{(i)}.$$

Simple computation implies that

$$\lim_{n\to\infty} \mathbb{E}\bigg(\sup_{0\le t\le T} |K_t^n - K_t|^2\bigg) = 0.$$

Denote by $H^2(0, T; \mathbb{R})$ the Sobolev space consisting of all absolutely continuous functions with derivatives in $L^2(0, T)$. By Lemma 3.3, it follows that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\|K^n\|^2_{H^2(0,T;\mathbb{R})} < \infty,$$

which implies that the sequence K^n is bounded in $L^2(\Omega; H^2(0, T; \mathbb{R}))$. So there exists an absolutely continuous function $K \in L^2(\Omega; H^2(0, T; \mathbb{R}))$ which is the weak limit of K^n. Further, $dK_t/dt = V_t$, where $-V_t \in \partial\Phi(Y_t)$.

Below we verify that (Y, Z, K) is the unique solution to the RBSDE. Taking a subsequence, if necessary, we can suppose that

$$\sup_{0\le t\le T} |Y_t^n - Y_t| \underset{n\to\infty}{\longrightarrow} 0, \qquad \sup_{0\le t\le T} |K_t^n - K_t| \underset{n\to\infty}{\longrightarrow} 0.$$

It follows that Y_t is rcll and K_t is continuous. Further, if (α, β) is a rcll process with values in $Gr(\partial\Phi)$, then

$$\big\langle J_n(Y_t^n) - \alpha_t, dK_t^n + \beta_t\, dt\big\rangle \le 0.$$

Since J_n is a contraction and Y^n uniformly converges to Y on $[0, T]$, it follows that $J_n(Y_t^n)$ converges to $\mathrm{pr}(Y)$ uniformly, where pr denotes the projection on $\overline{\mathrm{Dom}(\Phi)}$. By Lemma 3.5,

$$\langle \mathrm{pr}(Y_t) - \alpha_t, dK_t + \beta_t\, dt\rangle \le 0.$$

In order to complete the proof of the existence, we need to verify that

$$\mathbb{P}\left(Y_t \in \overline{\mathrm{Dom}(\Phi)}, 0 \le t \le T\right) = 1.$$

From the right continuity of Y, it suffices to prove that

$$\mathbb{P}\left(Y_t \in \overline{\mathrm{Dom}(\Phi)}\right) = 1, \quad 0 \le t \le T.$$

Assume that there exist $0 < t_0 < T$ and $B_0 \in \mathcal{F}$ such that $\mathbb{P}(B_0) > 0$ and $Y_{t_0}(\omega) \notin \overline{\mathrm{Dom}(\Phi)}$, $\forall \omega \in B_0$. By the right continuity, there exist $\delta > 0$, $B_1 \in \mathcal{F}$ such that $\mathbb{P}(B_1) > 0$, $Y_t(\omega) \notin \overline{\mathrm{Dom}(\Phi)}$ for all $(\omega, t) \in B_1 \times [t_0, t_0 + \delta]$.

Using the fact that $\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^T |A_n(Y_s^n)|\,ds < \infty$, which is a consequence of (3.1) and Fatou's lemma, we get

$$\int_{B_1}\int_{t_0}^{t_0+\delta} \liminf_{n\to\infty} |A_n(Y_s^n)|\,ds\,d\mathbb{P} < \infty.$$

This contradicts the fact that $\liminf_{n\to\infty} |A_n(Y_t^n)| = \infty$ on the set $B_1 \times [t_0, t_0+\delta]$, which holds because $A_n(x)$ is Lipschitz continuous for all $n \in \mathbb{N}$, Y_t^n uniformly converges to Y_t on $[0, T]$ and $Y_t(\omega) \notin \overline{\mathrm{Dom}(\Phi)}$ for all $(\omega, t) \in B_1 \times [t_0, t_0+\delta]$. This completes the existence proof.

Turning now to uniqueness, let $(Y_t, Z_t, K_t)_{0\le t\le T}$ and $(Y_t', Z_t', K_t')_{0\le t\le T}$ be two solutions for the RBSDE. Define

$$(\triangle Y_t, \triangle Z_t, \triangle K_t)_{0\le t\le T} = (Y_t - Y_t', Z_t - Z_t', K_t - K_t')_{0\le t\le T}.$$

Applying Itô's formula to $|\triangle Y_t|^2$ yields that

$$\begin{aligned}\mathbb{E}|\triangle Y_t|^2 &+ \mathbb{E}\int_t^T \|\triangle Z_s\|_{\ell^2}^2\,ds \\ &= 2\mathbb{E}\int_t^T \triangle Y_s[f(s, Y_s, Z_s) - f(s, Y_s', Z_s')]\,ds + 2\mathbb{E}\int_t^T \triangle Y_s\,d\triangle K_s.\end{aligned}$$

Since $\partial\Phi$ is monotone and $-dK_t/dt \in \partial\Phi(Y_t)$, $-dK_t'/dt \in \partial\Phi(Y_t')$, we obtain

$$\mathbb{E}\int_t^T \triangle Y_s\,d\triangle K_s \le 0.$$

Therefore, we get

$$\mathbb{E}|\triangle Y_t|^2 + \mathbb{E}\int_t^T \|\triangle Z_s\|_{\ell^2}^2\,ds \le C\mathbb{E}\int_t^T |\triangle Y_s|^2\,ds + \frac{1}{2}\mathbb{E}\int_t^T \|\triangle Z_s\|_{\ell^2}^2\,ds.$$

Hence, by Gronwall's inequality, we obtain the uniqueness of the solutions. □

4. Application

In this section, we study the link between the solution of the RBSDE driven by a Lévy process and the solution of a class of PDIIs. Suppose that our Lévy process has the form of $L_t = bt + X_t$, where X_t is a pure jump Lévy process with Lévy measure $\nu(dx)$.

In order to obtain our main result, we use the following lemma which appeared in [17].

LEMMA 4.1. *Let $c : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that*

$$|c(s, y)| \le a_s(y^2 \wedge |y|) \quad a.s.$$

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where $\{a_s,\ s \in [0, T]\}$ *is a non-negative predictable process such that* $\mathbb{E}\int_0^T a_s^2\, ds < \infty$*. Then, for each* $0 \le t \le T$*, we have*

$$\sum_{t<s\le T} c(s, \Delta L_s) = \sum_{i=1}^{\infty} \int_t^T \langle c(s, \cdot), p_i\rangle_{L^2(\nu)}\, dH_s^{(i)} + \int_t^T \int_{\mathbb{R}} c(s, y)\nu(dy)\, ds.$$

For all $0 \le t \le T$, consider the following coupled RBSDEs:

$$Y_t = h(L_T) + \int_t^T f(s, L_s, Y_s, Z_s)\, ds + K_T - K_t - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)}\, dH_s^{(i)}, \tag{4.1}$$

where $\mathbb{E}|h(L_T)|^2 < \infty$.

Define

$$u^1(t, x, y) = u(t, x + y) - u(t, x) - \frac{\partial u}{\partial x}(t, x)y,$$

where u is the solution for the following PDIIs:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) + a' \dfrac{\partial u}{\partial x}(t, x) + f\left(t, u(t, x), \{u^{(i)}(t, x)\}_{i=1}^{\infty}\right) \\ \qquad + \displaystyle\int_{\mathbb{R}} u^1(t, x, y)\nu(dy) \in \partial\Phi(u(t, x)), \\ u(T, x) = h(x) \in \overline{\mathrm{Dom}(\Phi)}, \end{cases}$$

where $a' = a + \int_{\{|y|\ge 1\}} y\nu(dy)$ and

$$u^{(1)}(t, x) = \int_{\mathbb{R}} u^1(t, x, y)p_1(y)\nu(dy) + \frac{\partial u}{\partial x}(t, x)\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2},$$

and for $i \ge 2$,

$$u^{(i)}(t, x) = \int_{\mathbb{R}} u^1(t, x, y)p_i(y)\nu(dy).$$

Suppose that u is a $\mathcal{C}^{1,2}$ function such that $\partial u/\partial t$ and $\partial^2 u/\partial x^2$ are bounded by polynomial function of x, uniformly in t. Then we have the following theorem.

THEOREM 4.2. *The unique adapted solution of* (4.1) *is given by*

$$Y_t = u(t, L_t), \quad K_t = \int_0^t V_s\, ds, \ -V_s \in \partial\Phi(u(s, L_s)),$$

$$Z_t^{(i)} = \int_{\mathbb{R}} u^1(t, L_{t-}, y)p_i(y)\nu(dy), \quad i \ge 2,$$

$$Z_t^{(1)} = \int_{\mathbb{R}} u^1(t, L_{t-}, y)p_1(y)\nu(dy) + \frac{\partial u}{\partial x}(t, L_{t-})\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2}.$$

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PROOF. Applying Itô's formula to $u(s, L_s)$ yields that

$$\begin{aligned}
&u(T, L_T) - u(t, L_t) \\
&\quad = \int_t^T \frac{\partial u}{\partial s}(s, L_{s-})\, ds + \int_t^T \frac{\partial u}{\partial x}(s, L_{s-})\, dL_s \\
&\quad\quad + \sum_{t<s\le T}\left[u(s, L_s) - u(s, L_{s-}) - \frac{\partial u}{\partial x}(s, L_{s-})\triangle L_s\right].
\end{aligned} \tag{4.2}$$

By Lemma 4.1,

$$\begin{aligned}
&\sum_{t<s\le T}\left[u(s, L_s) - u(s, L_{s-}) - \frac{\partial u}{\partial x}(s, L_{s-})\triangle L_s\right] \\
&\quad = \sum_{i=1}^{\infty}\int_t^T\left(\int_{\mathbb{R}} u^1(s, L_{s-}, y)p_i(y)\nu(dy)\right) dH_s^{(i)} \\
&\quad\quad + \int_t^T\int_{\mathbb{R}} u^1(s, L_{s-}, y)\nu(dy)\, ds.
\end{aligned} \tag{4.3}$$

Note that

$$L_t = Y_t^{(1)} + t\mathbb{E}L_1 = \left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2} H_t^{(1)} + t\mathbb{E}L_1, \tag{4.4}$$

where $\mathbb{E}L_1 = a + \int_{\{|y|\ge 1\}} y\nu(dy)$.

Hence, substituting (4.3) and (4.4) into (4.2) yields that

$$\begin{aligned}
&h(L_T) - u(t, L_t) \\
&\quad = \int_t^T\left[\frac{\partial u}{\partial s}(s, L_{s-}) + a\frac{\partial u}{\partial x}(s, L_{s-}) + \int_{\{|y|\ge 1\}} y\nu(dy)\right. \\
&\quad\quad \left. + \int_{\mathbb{R}} u^1(s, L_{s-}, y)\nu(dy)\right] ds \\
&\quad\quad + \int_t^T\left[u^1(s, L_{s-}, y)p_1(y)\nu(dy) + \frac{\partial u}{\partial x}(s, L_{s-})\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2}\right] dH_s^{(1)} \\
&\quad\quad + \sum_{i=2}^{\infty}\int_t^T\left(\int_{\mathbb{R}} u^1(s, L_{s-}, y)p_i(y)\nu(dy)\right) dH_s^{(i)},
\end{aligned}$$

which shows the desired result. □

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