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ON $\mathcal{T}$-NONCOSINGULAR MODULES

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Abstract

In this paper we introduce $\mathcal{T}$-noncosingular modules. Rings for which all right modules are $\mathcal{T}$-noncosingular are shown to be precisely those for which every simple right module is injective. Moreover, for any ring R we show that the right R-module R is $\mathcal{T}$-noncosingular precisely when R has zero Jacobson radical. We also study the $\mathcal{T}$-noncosingular condition in association with (strongly) FI-lifting modules.

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1. Introduction

Throughout this paper S denotes the endomorphism ring of any module M. In [8], the authors investigate $\mathcal{K}$-nonsingular modules. Motivated by this work, we introduce the notion of $\mathcal{T}$-noncosingular modules as the dual notion to the notion of $\mathcal{K}$-nonsingular modules. A module M is called *$\mathcal{T}$-noncosingular* if, for every nonzero endomorphism φ of M, Im φ is not small in M. Following [10], the module M is called *noncosingular* if for every nonzero module N and every nonzero homomorphism $f : M \to N$, Im f is not a small submodule of N. It is clear that every noncosingular module is $\mathcal{T}$-noncosingular.

The aim of this paper is to study $\mathcal{T}$-noncosingular modules. It turns out that some results about $\mathcal{K}$-nonsingular modules have corresponding duals for $\mathcal{T}$-noncosingular modules.

Section 2 introduces the concept of $\mathcal{T}$-noncosingular modules. The structure of finitely generated $\mathcal{T}$-noncosingular $\mathbb{Z}$-modules is described. We show that in general the direct sum of $\mathcal{T}$-noncosingular modules is not a $\mathcal{T}$-noncosingular module. Then we provide a necessary and sufficient condition for a direct sum of $\mathcal{T}$-noncosingular modules to be $\mathcal{T}$-noncosingular. We also prove that $\mathcal{T}$-noncosingularity is inherited by direct summands.

Section 3 is concerned with the concept of FI-lifting modules. We prove some results concerning these types of modules using the notion of $\mathcal{T}$-noncosingularity. In particular, any $\oplus$-supplemented module is FI-lifting.

2. $\mathcal{T}$-noncosingular modules

Let M and N be two modules. We say that M is *$\mathcal{T}$-noncosingular relative to N* if, for every nonzero homomorphism $\varphi : M \longrightarrow N$, $\operatorname{Im}\varphi$ is not small in N. If M is $\mathcal{T}$-noncosingular relative to M, we say that M is *$\mathcal{T}$-noncosingular*. The ring R is said to be *right $\mathcal{T}$-noncosingular* if the right R-module R_R is $\mathcal{T}$-noncosingular. Left $\mathcal{T}$-noncosingular rings are defined similarly.

Recall (see, for example, [11, 23.1]) that a module M is called *cosemisimple* if each factor module of M has zero (Jacobson) radical and, for any ring R, the right R-module R_R is cosemisimple precisely when every simple right R-module is injective, that is, R is a right V-ring. Note, from the above definition, that every module with zero radical is $\mathcal{T}$-noncosingular. Consequently every cosemisimple module is $\mathcal{T}$-noncosingular.

It is clear that a module M is noncosingular if and only if it is a $\mathcal{T}$-noncosingular module relative to N for every module N. However, it is easy to check that the $\mathbb{Z}$-module $M = \mathbb{Z}/p\mathbb{Z}$, where p is a prime integer, is $\mathcal{T}$-noncosingular but not noncosingular.

For every module M, let

$$\overline{Z}(M) = \bigcap \{\operatorname{Ker} g \mid g : M \to T,\ \text{ where } T \text{ is small in its injective hull}\}$$

and let $\nabla(M) = \{\varphi \in S \mid \operatorname{Im}\varphi \ll M\}$. It is easy to see that $\nabla(M)$ is an ideal of S. By the *$\mathcal{T}$-noncosingular submodule* of M we mean $\overline{Z}_{\mathcal{T}}(M) = \bigcap_{\varphi\in\nabla(M)} \operatorname{Ker}\varphi$.

A module M is called a *lifting* module if for every submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll M_2$ or, equivalently, for every submodule N of M there is a direct summand K of M such that $N/K \ll M/K$. The module M is called *discrete* if it is lifting and satisfies the condition that, if N is a submodule of M for which M/N is isomorphic to a direct summand of M, then N is a direct summand of M.

EXAMPLE 2.1. Every injective module over a right hereditary ring R is $\mathcal{T}$-noncosingular. In fact, let f be an endomorphism of M such that $\operatorname{Im} f \ll M$. Since R is a right hereditary ring and $\operatorname{Im} f \cong M/\operatorname{Ker} f$, $\operatorname{Im} f$ is injective. Thus, $\operatorname{Im} f$ is a direct summand of M. Therefore, $f = 0$.

PROPOSITION 2.2. *Let M be a module. We have:*

(i) *M is $\mathcal{T}$-noncosingular if and only if $\overline{Z}_{\mathcal{T}}(M) = M$;*
(ii) *$\overline{Z}_{\mathcal{T}}(M)$ is a fully invariant submodule of M; moreover, $\overline{Z}(M) \subseteq \overline{Z}_{\mathcal{T}}(M)$;*
(iii) *if $M = \bigoplus_{i\in I} M_i$, then $\overline{Z}_{\mathcal{T}}(M) \subseteq \bigoplus_{i\in I} \overline{Z}_{\mathcal{T}}(M_i)$.*

PROOF. (i) This is clear.

(ii) Let $\varphi \in S$ such that $\operatorname{Im}\varphi \ll M$ and let $f \in S$. We have $\operatorname{Im}\varphi f \subseteq \operatorname{Im}\varphi$, and hence $\operatorname{Im}\varphi f \ll M$. Therefore, $\overline{Z}_{\mathcal{T}}(M)$ is fully invariant.

The inclusion $\overline{Z}(M) \subseteq \overline{Z}_{\mathcal{T}}(M)$ is clear from the definitions.

(iii) Since $\overline{Z}_{\mathcal{T}}(M)$ is fully invariant in M, we have $\overline{Z}_{\mathcal{T}}(M) = \bigoplus_{i \in I}(\overline{Z}_{\mathcal{T}}(M) \cap M_i)$. It is sufficient to show that $\overline{Z}_{\mathcal{T}}(M) \cap M_i \subseteq \overline{Z}_{\mathcal{T}}(M_i)$ for all $i \in I$.

Let $x_i \in \overline{Z}_{\mathcal{T}}(M) \cap M_i$ for a fixed $i \in I$. Let $\varphi_i \in \operatorname{End}(M_i)$ such that $\operatorname{Im}\varphi_i \ll M_i$. Extending φ_i to $\overline{\varphi_i} : M \longrightarrow M$ by $\overline{\varphi_i} \mid M_j = 0$ for $i \neq j$, we have $\operatorname{Im}\overline{\varphi_i} \ll M$. Thus, $\overline{\varphi_i}(x_i) = \varphi_i(x_i) = 0$. Therefore, $x_i \in \overline{Z}_{\mathcal{T}}(M_i)$. □

PROPOSITION 2.3. *Let M be a $\mathcal{T}$-noncosingular module and let N be a direct summand of M. Then N is $\mathcal{T}$-noncosingular.*

PROOF. Let $M = N \oplus N'$. Let $\varphi : N \to N$ with $\operatorname{Im}\varphi \ll N$. Consider the homomorphism $\varphi \oplus 0_{N'} : N \oplus N' \to N \oplus N'$ defined by $\varphi \oplus 0_{N'}(n + n') = \varphi(n)$. Now $\varphi \oplus 0_{N'}(N \oplus N') = \varphi(N) \ll M$. Since M is $\mathcal{T}$-noncosingular, $\varphi \oplus 0_{N'} = 0$, and hence $\varphi = 0$. □

Note that the $\mathbb{Z}$-module $\mathbb{Z}$ is $\mathcal{T}$-noncosingular, but $S = \operatorname{End}(\mathbb{Z})$ is not von Neumann regular. However, the following two results show that there is some connection between the $\mathcal{T}$-noncosingular condition and regular endomorphism rings.

PROPOSITION 2.4. *If M is a $\mathcal{T}$-noncosingular discrete module, then S is von Neumann regular.*

PROOF. By [7, Theorem 5.4], $\nabla(M) = J(S)$ the Jacobson radical of S and $S/J(S)$ is von Neumann regular. However, since M is $\mathcal{T}$-noncosingular, $\nabla(M) = 0$. □

PROPOSITION 2.5. *If M is a module such that S is von Neumann regular, then M is $\mathcal{T}$-noncosingular.*

PROOF. Let $f \in S$ such that $\operatorname{Im} f \ll M$. Since S is von Neumann regular, there exists $g \in S$ such that $fgf = f$. This gives that fg is an idempotent. Hence $\operatorname{Im} fg$ is a direct summand of M. But $\operatorname{Im} fg \leq \operatorname{Im} f$. Thus $\operatorname{Im} fg \ll M$. So $fg = 0$, and hence $f = fgf = 0$. □

PROPOSITION 2.6. *Let $M = xR$ be a cyclic module such that $Ann(x)$, the right annihilator of x, is an ideal of R. Then M is a $\mathcal{T}$-noncosingular module if and only if* $\operatorname{Rad}(M) = 0$.

PROOF. Suppose that M is a $\mathcal{T}$-noncosingular module and $\operatorname{Rad}(M) \neq 0$. Therefore there exists $a \in R$ such that $xa \neq 0$ and $xa \in \operatorname{Rad}(M)$. Consider the endomorphism f of M defined by $f(x\alpha) = xa\alpha$ for every $\alpha \in R$. The map f is well defined since $Ann(x)$ is an ideal of R. Thus, $\operatorname{Im} f \leq \operatorname{Rad}(M)$ and $f \neq 0$. However, $\operatorname{Rad}(M) \ll M$. Then M is not $\mathcal{T}$-noncosingular, a contradiction. The converse is clear. □

The following two corollaries are now immediate.

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COROLLARY 2.7. *A ring R is right (left) $\mathcal{T}$-noncosingular if and only if* $\mathrm{Rad}(R) = 0$.

COROLLARY 2.8. *Let M be a local module over a commutative ring R. Then M is a $\mathcal{T}$-noncosingular module if and only if M is a simple module.*

COROLLARY 2.9. *Let M be a finitely generated module over a commutative principal ideal domain R. Then M is a $\mathcal{T}$-noncosingular module if and only if* $\mathrm{Rad}(M) = 0$.

PROOF. This follows from Propositions 2.3, 2.6 and [9, Corollary, p. 179]. □

PROPOSITION 2.10. *A finitely generated $\mathbb{Z}$-module M is a $\mathcal{T}$-noncosingular module if and only if $M = \mathbb{Z}^{(n)} \oplus K$ for some $n \in \mathbb{N}$ and semisimple module K.*

PROOF. It is well known that every finitely generated $\mathbb{Z}$-module is a finite direct sum of cyclic modules. Since every direct summand of a $\mathcal{T}$-noncosingular module is a $\mathcal{T}$-noncosingular module, the Chinese remainder theorem implies that every cyclic torsion $\mathbb{Z}$-module is a $\mathcal{T}$-noncosingular module if and only if it is semisimple by Corollary 2.8. The result follows. On the other hand, it is clear that if K is semisimple, then $\mathbb{Z}^{(n)} \oplus K$ is $\mathcal{T}$-noncosingular because $\mathrm{Rad}(\mathbb{Z}^{(n)} \oplus K) = 0$. □

PROPOSITION 2.11. *Let $(M_i)_{i\in I}$ be a family of modules. Then $M = \bigoplus_{i\in I} M_i$ is a $\mathcal{T}$-noncosingular module if and only if M_i is a $\mathcal{T}$-noncosingular module relative to M_j for all $i, j \in I$.*

PROOF. ($\Rightarrow$) Let (i, j) be any pair in $I \times I$. Let $\varphi \in \mathrm{Hom}(M_i, M_j)$ such that $\mathrm{Im}\,\varphi \ll M_j$. Consider the homomorphism $f : M_i \oplus M_j \to M_i \oplus M_j$ defined by $f(x_i + x_j) = \varphi(x_i)$ with $x_i \in M_i$ and $x_j \in M_j$. Then $\mathrm{Im}\, f = \varphi(M_i) \ll M_i \oplus M_j$. However, $M_i \oplus M_j$ is a $\mathcal{T}$-noncosingular module by Proposition 2.3. Thus, $f = 0$ and hence $\varphi = 0$. This completes the proof.

($\Leftarrow$) Let f be an endomorphism of M such that $\mathrm{Im}\, f \ll M$. Consider the homomorphisms $\pi_i : M \to M_i$ (the projections) and $\phi_i : M_i \to M$ (the inclusion maps). Let (i, j) be any pair in $I \times I$. Since $\mathrm{Im}(f\phi_i) \ll M$, we have $\mathrm{Im}(\pi_j f \phi_i) \ll M_j$. By hypothesis, $\pi_j f \phi_i = 0$. Now, for all $x \in M$, we have $f(x) = \sum_{i\in I} \sum_{j\in I} \pi_j[f(\phi_i(\pi_i(x)))]$ (The sum is finite.) Thus, $f = 0$. Consequently, M is a $\mathcal{T}$-noncosingular module. □

In general, a direct sum of $\mathcal{T}$-noncosingular modules is not a $\mathcal{T}$-noncosingular module, as the following example shows.

If R is a Dedekind domain, then R is said to be *proper* if R is not a field.

If R is a proper Dedekind domain, then for each nonzero prime ideal P of R, $R(P^\infty)$ will denote the P-primary component of the torsion R-module K/R, where K is the quotient field of R.

EXAMPLE 2.12. Let R be a proper Dedekind domain. Let P be any nonzero prime ideal of R. Consider the module $M = R(P^\infty) \oplus R/P$ and the endomorphism $f : M \longrightarrow M$ defined by $f(x + \overline{y}) = cy$ with $x \in R(P^\infty)$, $y \in R$ and c is a nonzero element of $R(P^\infty)$ such that $cP = 0$. It is clear that $\mathrm{Im}\, f = cR$ which is nonzero

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and small in M. So M is not a $\mathcal{T}$-noncosingular module. In particular, for any prime integer p, the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}/p\mathbb{Z}$ is not a $\mathcal{T}$-noncosingular $\mathbb{Z}$-module.

PROPOSITION 2.13. *The following are equivalent for a ring R.*

(i) *Every right R-module is $\mathcal{T}$-noncosingular.*
(ii) *Every right R-module is noncosingular.*
(iii) *R is a right V-ring, that is, every simple right R-module is injective.*

PROOF. (i) $\Rightarrow$ (ii) Let M and N be two modules. Since $M \oplus N$ is $\mathcal{T}$-noncosingular, M is $\mathcal{T}$-noncosingular relative to N by Proposition 2.11. Therefore, M is noncosingular. The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) follow from [10, Proposition 2.5]. □

PROPOSITION 2.14. *Let M be a $\mathcal{T}$-noncosingular module. If $N \leq X$, $X/N \ll M/N$ and N is a direct summand of M, then N is unique.*

PROOF. Let M be $\mathcal{T}$-noncosingular. Assume that $X/N_i \ll M/N_i$ with $M = N_i \oplus P_i$, $i = 1, 2$ and assume that $N_1 \neq N_2$. Without loss of generality, suppose that $N_1 \nsubseteq N_2$. Consider the projections $\pi_{N_1} : M \to N_1$ and $\pi_{P_2} : M \to P_2$. Then we have the nonzero homomorphism $\varphi = \pi_{P_2}\pi_{N_1}$. On the other hand, $\operatorname{Im} \varphi = (N_1 + N_2) \cap P_2 \subseteq X \cap P_2 \ll P_2$ implies that $\varphi = 0$, a contradiction. Therefore, $N_1 = N_2$. □

Let M be a module and $N \leq M$. The submodule N is called *coclosed* if $N/K \ll M/K$ implies $N = K$ for every submodule K of M contained in N. Let $K \leq N \leq M$. If K is coclosed in M and $N/K \ll M/K$, then K is called a *coclosure* of N in M. The module M is called a *UCC* module if every submodule of M has a unique coclosure in M (see [3]).

COROLLARY 2.15. *Every lifting $\mathcal{T}$-noncosingular module is UCC.*

PROPOSITION 2.16. *Let M be a $\mathcal{T}$-noncosingular module and X fully invariant in M. Let $N \leq X$ such that $X/N \ll M/N$ and N a direct summand of M. Then N is (unique) fully invariant in M.*

PROOF. Let P be a submodule of M such that $M = N \oplus P$. Assume that N is not fully invariant in M. Then there exist an endomorphism φ of M and $x \in N$ such that $\varphi(x) \notin N$. Let $\psi = \pi_P \varphi \pi_N : M \to P$, where $\pi_N : M \to N$ and $\pi_P : M \to P$ are the projections. Note that $\psi \neq 0$ ($\varphi(x) \notin N$) and $\operatorname{Im} \psi \subseteq X \cap P \ll M$. This contradicts the fact that M is $\mathcal{T}$-noncosingular. Thus, N is fully invariant in M. □

COROLLARY 2.17. *We have the following results.*

(i) *Let M be a noncosingular module and $X \leq M$. Let $N \leq X$ such that $X/N \ll M/N$ and N is a direct summand of M. Then N is unique.*
(ii) *Let M be a noncosingular module and X a fully invariant submodule of M. Let $N \leq X$ such that $X/N \ll M/N$ and N is a direct summand of M. Then N is unique and fully invariant in M.*

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PROOF. Part (i) follows from Proposition 2.14 while part (ii) follows from Proposition 2.16. □

3. FI-lifting and strongly FI-lifting modules

A module M is called FI-*lifting* if for every fully invariant submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll M_2$ or, equivalently, for every fully invariant submodule N of M there is a direct summand K of M such that $N/K \ll M/K$. The module M is called *strongly* FI-*lifting* if, for every fully invariant submodule N of M, there is a fully invariant direct summand K of M such that $N/K \ll M/K$. It is easy to prove that any direct summand of a strongly FI-lifting module is strongly FI-lifting.

Let M be a module. If $N \leq M$, then N is called a *supplement* submodule of M if there exists a submodule K of M such that $M = N + K$ and $N \cap K \ll N$ (in this case we say that N is a *supplement of* K *in* M). If every submodule of M has a (direct summand) supplement in M, then M is called ($\oplus$-)*supplemented*. If for every submodule N of M there exists a submodule K of M with $M = N + K$ and $N \cap K \ll M$, then M is called *weakly supplemented*.

By [6, Theorem 3.4], any finite direct sum of FI-lifting modules is again FI-lifting. The following two examples show that this property is not true in general for infinite direct sums of FI-lifting modules. Let R be a discrete valuation ring with maximal ideal m. Let $M = \bigoplus_{i=1}^{\infty} R/m^i$ or $M = R^{\mathbb{N}}$. By [12, Corollary 2, p. 48], $\mathrm{Rad}(M)$ does not have a supplement in M. Since $\mathrm{Rad}(M)$ is a fully invariant submodule of M, M is not FI-lifting. On the other hand, it is clear that R/m^i $(i \geq 1)$ and R are lifting modules.

PROPOSITION 3.1. *Let M be a $\mathcal{T}$-noncosingular module. Then M is FI-lifting if and only if M is strongly FI-lifting.*

PROOF. Let M be FI-lifting and X a fully invariant submodule of M. Then there exists a direct summand N of M such that $X/N \ll M/N$. By Proposition 2.16, N is fully invariant in M. Thus, M is strongly FI-lifting. The converse is clear. □

COROLLARY 3.2. *Let M be a noncosingular module. Then M is FI-lifting if and only if M is strongly FI-lifting.*

The following proof uses the concept of a *left semicentral idempotent* of a ring S: this is an idempotent e of S for which $exe = xe$ for all $x \in S$.

LEMMA 3.3. *If K is a fully invariant submodule of M having a coclosure L which is a fully invariant direct summand of M, then L is the unique direct summand coclosure of K.*

PROOF. By [1, Lemma 1.9] and our hypothesis, there is a left semicentral idempotent $e \in S$ such that $L = e(M)$ and $K/e(M) \ll M/e(M)$. Let $c \in S$ be an idempotent such that $K/c(M) \ll M/c(M)$. Then $(1-c)(M) \cap K \ll (1-c)(M)$. Let us show that

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$L = c(M)$. Since K is fully invariant in M, we have $(1-c)(K) = (1-c)(M) \cap K$. Thus, $(1-c)(K) \ll M$. Therefore, $e(1-c)(K) \ll e(M)$ and hence $e(1-c)(K) \ll K$ since $e(M) \subseteq K$. So $e(1-c)e(K) \subseteq e(1-c)(K) \ll K$. Then, since e is left semicentral, $(1-c)e(K) = e(1-c)e(K) \ll K$ and $(1-c)e$ is an idempotent of S. Therefore, $(1-c)e(K) = 0$. Since $e(M) = e(K)$, we have $(1-c)e(M) = 0$, and hence $e = ce$. It follows that $e(M) \subseteq c(M)$. Since $c(M)/e(M) \subseteq K/e(M) \ll M/e(M)$, we obtain $c(M) = e(M)$. This completes the proof. □

PROPOSITION 3.4. *If M is a strongly FI-lifting module and K is a fully invariant submodule of M, then there exists a unique (fully invariant) direct summand L of M such that $K/L \ll M/L$.*

PROOF. This follows from Lemma 3.3. □

PROPOSITION 3.5. *Let M be an FI-lifting module and X a fully invariant submodule of M. If one of the following conditions is satisfied, then M/X is strongly FI-lifting:*

(i) *M/X is indecomposable;*
(ii) *M/X is $\mathcal{T}$-noncosingular.*

PROOF. By [6, Proposition 3.3], M/X is FI-lifting.

(i) Clearly, indecomposable FI-lifting modules are strongly FI-lifting.

(ii) This follows from Proposition 3.1. □

PROPOSITION 3.6. *Let M be a lifting (respectively noncosingular weakly supplemented FI-lifting) module such that every small submodule is fully invariant. Then every factor module of M is lifting (respectively strongly FI-lifting).*

PROOF. Let X, Y be submodules of M such that $M = X + Y$ and $X \cap Y \ll M$. Note that $M/(X \cap Y) = X/(X \cap Y) \oplus Y/(X \cap Y)$. By hypothesis, $X \cap Y$ is fully invariant in M. If M is lifting, then $M/(X \cap Y)$ is lifting by [2, 22.2]. Since the lifting property is inherited by direct summands, M/X is lifting. Now assume that M is a noncosingular weakly supplemented FI-lifting module. Then the result follows from [6, Proposition 3.3], Corollary 3.2 and the fact that any direct summand of a strongly FI-lifting module is strongly FI-lifting. □

PROPOSITION 3.7. *Let M be a module. The following are equivalent:*

(i) *M is FI-lifting;*
(ii) *every fully invariant submodule of M has a direct summand supplement;*
(iii) *for each fully invariant submodule X of M, there is a coclosed submodule K of M and a direct summand supplement L of K such that $K \leq X$, $X/K \ll M/K$ and every homomorphism $f : M \to M/(L \cap K)$ can be lifted to an endomorphism $g : M \to M$, that is, such that $g(m) + (L \cap K) = f(m)$ for all $m \in M$.*

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PROOF. (i) ⇔ (ii) Let X be a fully invariant submodule of M. First assume that M is FI-lifting. Then there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq X$ and $M_2 \cap X \ll M_2$. Then $M = X + M_2$ and M_2 is a direct summand supplement of X. Conversely, let K be a direct summand supplement of X in M. Then $M = K + X = K \oplus K'$ and $K \cap X \ll K$ for some submodule K' of M. Consider the natural projection map $\phi : M \to K'$. Since X is fully invariant,

$$\phi(X) = (X + K) \cap K' = M \cap K' = K' \leq X.$$

Thus, M is FI-lifting.

(i) ⇒ (iii) Let X be a fully invariant submodule of M. Since M is FI-lifting, there exists a decomposition $M = L \oplus K$ such that $K \leq X$ and $X/K \ll M/K$. Since $L \cap K = 0$, clearly any homomorphism $f : M \to M/(L \cap K)$ lifts to a $g : M \to M$.

(iii) ⇒ (i) Let X be a fully invariant submodule of M. By (iii), there is a coclosed submodule K of M and a direct summand supplement L of K such that $K \leq X$ and $X/K \ll M/K$. Since K is a supplement in M by [4, Proposition 3], it follows from [5, Lemma 2.2] that K is a direct summand of M. Thus, M is FI-lifting. □

PROPOSITION 3.8. *Let M be a module. The following are equivalent:*

(i) *M is strongly FI-lifting;*
(ii) *every fully invariant submodule of M has a supplement K which is a direct summand of M with $M = K \oplus N$ for some fully invariant submodule N of M.*

PROOF. We completely follow the proof of Proposition 3.7((i) ⇔ (ii)). □

PROPOSITION 3.9. *Let M be an FI-lifting module and let U be a fully invariant submodule of M. Then M/U is FI-lifting. If, moreover, U is coclosed in M, then U is also FI-lifting.*

PROOF. By [6, Proposition 3.3], M/U is FI-lifting. Assume that U is coclosed in M. Let V be a fully invariant submodule of U. Then V is fully invariant in M. So, there exist submodules K and K' of M such that $M = K \oplus K'$, $K' \leq V$ and $K \cap V \ll K$. Thus, $U = V + (U \cap K)$. Since U is fully invariant in M, $U = (U \cap K) \oplus (U \cap K')$. Hence, $U \cap K$ is a direct summand of U. Moreover, $V \cap (U \cap K) = V \cap K \ll K$. This implies that $V \cap (U \cap K) \ll U \cap K$ since $U \cap K$ is coclosed in M by [2, 3.7]. Therefore, $U \cap K$ is a direct summand supplement of V in U. By Proposition 3.7, U is FI-lifting. □

A module M is called a *duo* module provided that every submodule of M is fully invariant.

PROPOSITION 3.10. *Let M be a module. Consider the following statements:*

(i) *M is lifting;*
(ii) *M is $\oplus$-supplemented;*
(iii) *M is FI-lifting.*

Then (i) ⇒ (ii) ⇒ (iii). *If M is a duo module, then* (iii) ⇒ (i).

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PROOF. (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (iii) This is clear by Proposition 3.7.

The rest is clear from the definitions. □

REMARK. (1) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$. It is well known that M is not lifting, but it is FI-lifting by [6, Theorem 3.4].

(2) Consider $\mathbb{Q}$ the additive group of rational numbers. Let f be any nonzero $\mathbb{Z}$-endomorphism of $\mathbb{Q}$. Let r be a nonzero element of $\mathbb{Q}$ such that $f(1) = r$. Let a and b be two nonzero integers. Then $f(1) = f((1/b) \times b) = f(1/b)b = r$. So $f(1/b) = r/b$. Thus, $f(a/b) = f(1/b)a = (r/b)a = (a/b)r$. Now let N be a nonzero fully invariant submodule of $\mathbb{Q}$. Let s be a nonzero element of N. Let g be the endomorphism of $\mathbb{Q}$ defined by $g(x) = (1/s)x$ for every $x \in \mathbb{Q}$. Since N is fully invariant, $g(s) \in N$. Thus, $1 \in N$. Hence, $\mathbb{Q} \leq N$ since $h(1) \in N$ for every $h \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Q})$. Consequently, the only fully invariant submodules of $\mathbb{Q}$ are 0 and $\mathbb{Q}$. Therefore, $\mathbb{Q}$ is strongly FI-lifting. On the other hand, $\mathbb{Q}$ is not $\oplus$-supplemented since $\mathbb{Q}$ is an indecomposable $\mathbb{Z}$-module which is not hollow.

THEOREM 3.11. *Let M be a $\mathcal{T}$-noncosingular module and X a fully invariant submodule of M. Then M is (strongly) FI-lifting if and only if $M = M_1 \oplus M_2$ such that M_1 and M_2 are (strongly) FI-lifting and M_1 is the unique fully invariant direct summand of M with $M_1 \subseteq X$ and $X/M_1 \ll M/M_1$.*

PROOF. ($\Rightarrow$) Since X is fully invariant in M and M is FI-lifting, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq X$ and $X/M_1 \ll M/M_1$. By Proposition 2.16, M_1 is unique and fully invariant in M. Then by Proposition 3.9, M_1 and M_2 are FI-lifting. The remainder of the proof is a consequence of Propositions 2.3 and 3.1.

($\Leftarrow$) This follows from [6, Theorem 3.4] and Proposition 3.1. □

PROPOSITION 3.12. *Let $M = M_1 \oplus M_2$. Then M_2 is FI-lifting if and only if for every fully invariant submodule N/M_1 of M/M_1, there exists a direct summand K of M such that $K \leq M_2$, $M = K + N$ and $N \cap K \ll M$.*

PROOF. Suppose that M_2 is FI-lifting. Let N/M_1 be any fully invariant submodule of M/M_1. It is easy to see that $N \cap M_2$ is fully invariant in M_2. Since M_2 is FI-lifting, there exists a decomposition $M_2 = K \oplus K'$ such that $M_2 = (N \cap M_2) + K$ and $N \cap K \ll K$. Clearly, $M = N + K$.

Conversely, suppose that M/M_1 has the stated property. Let H be a fully invariant submodule of M_2. It is easy to see that $(H \oplus M_1)/M_1$ is fully invariant in M/M_1. By hypothesis, there exists a direct summand L of M such that $L \leq M_2$, $M = L + H + M_1$ and $L \cap (H + M_1) \ll M$. By modularity, $M_2 = L + H$. It follows easily that L is a supplement of H in M_2. Therefore, M_2 is FI-lifting by Proposition 3.7. □

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