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HECKE C^*-ALGEBRAS AND SEMI-DIRECT PRODUCTS

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Abstract We analyse Hecke pairs (G, H) and the associated Hecke algebra $\mathcal{H}$ when G is a semi-direct product $N \rtimes Q$ and $H = M \rtimes R$ for subgroups $M \subset N$ and $R \subset Q$ with M normal in N. Our main result shows that, when (G, H) coincides with its Schlichting completion and R is normal in Q, the closure of $\mathcal{H}$ in $C^*(G)$ is Morita–Rieffel equivalent to a crossed product $I \rtimes_\beta Q/R$, where I is a certain ideal in the fixed-point algebra $C^*(N)^R$. Several concrete examples are given illustrating and applying our techniques, including some involving subgroups of $\mathrm{GL}(2, K)$ acting on K^2, where $K = \mathbb{Q}$ or $K = \mathbb{Z}[p^{-1}]$. In particular we look at the $ax + b$ group of a quadratic extension of K.

Keywords: Hecke algebra; group C^*-algebra; Morita equivalence; semi-direct product

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1. Introduction

A *Hecke pair* (G, H) comprises a group G and a subgroup H for which every double coset is a finite union of left cosets, and the associated *Hecke algebra*, generated by the characteristic functions of double cosets, reduces to the group $*$-algebra of G/H when H is normal.

In [**5**] we introduced the *Schlichting completion* $(\bar{G}, \bar{H})$ of the Hecke pair (G, H) as a tool for analysing Hecke algebras, based in part on the work of Tzanev [**14**]. (A slight variation on this construction appears in [**3**].) The idea is that $\bar{H}$ is a compact open subgroup of $\bar{G}$ such that the Hecke algebra of $(\bar{G}, \bar{H})$ is naturally identified with the Hecke algebra $\mathcal{H}$ of (G, H). The characteristic function p of $\bar{H}$ is a projection in the group C^*-algebra $A := C^*(\bar{G})$, and $\mathcal{H}$ can be identified with $pC_c(\bar{G})p \subset A$; thus, the closure of $\mathcal{H}$ in A coincides with the corner pAp, which is Morita–Rieffel equivalent to the ideal $\overline{ApA}$.

In [**5**] we were mainly interested in studying the questions of when pAp is the enveloping C^*-algebra of the Hecke algebra $\mathcal{H}$, and when the projection p is full in A, making the C^*-completion pAp of $\mathcal{H}$ Morita–Rieffel equivalent to the group C^*-algebra A. We had the most success when $G = N \rtimes Q$ was a semi-direct product with the Hecke subgroup H contained in the normal subgroup N.

In this paper we again consider $G = N \rtimes Q$, but now we allow $H = M \rtimes R$, where M is a normal subgroup of N and R is a subgroup of Q which normalizes M. Briefly,

$$\begin{array}{ccccc} G & = & N & \rtimes & Q \\ \vee & & \triangledown & & \vee \\ H & = & M & \rtimes & R. \end{array} \tag{1.1}$$

This leads to a refinement of the Morita–Rieffel equivalence $\overline{ApA} \sim pAp$ (see Theorem 5.1).

We begin in § 2 by recalling our conventions from [**5**] regarding Hecke algebras. In § 3 we describe the main properties of our group-theoretic set-up (1.1). In particular, we characterize the reduced Hecke pairs in terms of N, Q, M and R.

In order to effectively analyse how our semi-direct-product decomposition affects the Hecke topology, we need to go into somewhat more detail than might be expected. In particular, we must exercise some care to obtain the semi-direct-product decomposition

$$\bar{G} = \bar{N} \rtimes \bar{Q}, \qquad \bar{H} = \bar{M} \rtimes \bar{R}$$

for the Schlichting completion (see Corollary 3.8), and to describe various bits of this completion as inverse limits of groups (see Theorem 3.14).

Section 4 is preparatory for § 5, but the results may be of independent interest. In Proposition 4.1 we show that if (B, Q, α) is an action, R is a compact normal subgroup of Q and $(B^R, Q/R, \beta)$ is the associated action, then the projection $q = \int_R r \, \mathrm{d}r$ is in $M(B \times_\alpha Q)$ and $B^R \times_\beta Q/R \cong q(B \times_\alpha Q)q$. This generalizes the result of [**13**].

We also show in Theorem 4.6 that under this correspondence the ideal

$$\overline{(B \times_\alpha G)p(B \times_\alpha G)}$$

is mapped to an ideal $I \times_\beta Q/R$, where I is a Q/R-invariant ideal of B^R.

In § 5, we assume that R is normal in Q, and (without loss of generality) that the pair (G, H) is equal to its Schlichting completion. The main result is Theorem 5.1, in which we take full advantage of the semi-direct-product decomposition to show that the Hecke C^*-algebra $p_H C^*(G) p_H$ is Morita–Rieffel equivalent to a crossed product $I \times_\beta Q/R$, where I is the ideal in $C^*(N)^R$ generated by $\{\alpha_s(p_M) : s \in Q\}$. We look briefly at the special case where the normal subgroup N is abelian.

Finally, in § 6 we give some examples to illustrate our results. Classical Hecke algebras have most commonly treated pairs of semi-simple groups such as $(\mathrm{GL}(n, \mathbb{Q}), \mathrm{SL}(n, \mathbb{Z}))$. The work of Bost and Connes [**1**] showed the importance of also studying Hecke pairs of solvable groups. In our examples we mostly deal with the following situation: K is either the field $\mathbb{Q}$ of rational numbers or the field $\mathbb{Z}[p^{-1}]$ of rational numbers with denominators of the form p^n; $N = K^2$; $M = \mathbb{Z}^2$; Q is a subgroup of $\mathrm{GL}(2, K)$ containing the diagonal subgroup, acting on N in the obvious way, and $R = Q \cap \mathrm{GL}(2, \mathbb{Z})$. It is not so difficult to see that the Schlichting completions are p-adic or adelic versions of the same groups.

As to specific examples we look at the algebra studied by Connes and Marcolli in [**2**] (see also [**8**]). Here R is not normal in Q, so the full results of § 5 do not apply. On the other hand, if R is normal in Q then Corollary 5.7 does apply, and as in [**7**] one can use the Mackey orbit method to study the ideal structure of the C^*-algebras involved. A particular example of this is the $ax+b$ group over a quadratic extension $K[\sqrt{d}]$ treated in [**9**], and we shall see that this example raises some interesting questions. We also look at a nilpotent example, i.e. one version of the Heisenberg group over the rationals.

After we had completed the research for this paper, we became aware of the recent preprint [**8**], which treats semi-direct-product Hecke pairs in a way quite similar to ours. The present paper and [**8**] were written independently, and the techniques have only incidental overlap. We should mention that we treat only the case where M is normal in N, while the context in [**8**] seems to be more general. Thus, for example, it would be difficult to adapt our results on inverse limits (see § 3.4) to the context of [**8**].

2. Preliminaries

We adopt the conventions of [**5**], which contains more references. A *Hecke pair* (G,H) comprises a group G and a *Hecke subgroup* H, i.e. one for which every double coset HxH is a finite union of left cosets $\{y_1H,\dots,y_{L(x)}H\}$. A good reference for the basic theory of Hecke pairs is [**6**]. A Hecke pair (G,H) is *reduced* if $\bigcap_{x\in G} xHx^{-1}=\{e\}$, and a reduced Hecke pair (G,H) is a *Schlichting pair* if G is locally compact Hausdorff and H is compact and open in G. In [**5**, Theorem 3.8], we gave a new proof of [**14**, Proposition 4.1], which says that every reduced Hecke pair (G,H) can be embedded in an essentially unique Schlichting pair $(\bar{G},\bar{H})$, which we call the *Schlichting completion* of (G,H). Specifically, $\bar{G}$ is the completion of G in the (two-sided uniformity defined by the) *Hecke topology* having a local sub-base $\{xHx^{-1}\mid x\in G\}$ of neighbourhoods of e, and $(\bar{G},\bar{H})$ is unique in the sense that if (L,K) is any Schlichting pair and $\sigma: G\to L$ is a homomorphism such that $\sigma(G)$ is dense and $H=\sigma^{-1}(K)$, then σ extends uniquely to a topological isomorphism $\bar{\sigma}:\bar{G}\to L$, and, moreover, $\bar{\sigma}(\bar{H})=K$.

The associated *Hecke algebra* is the vector subspace $\mathcal{H}$ of $\mathbb{C}^G$ spanned by the characteristic functions of double H-cosets, with operations defined by

$$f*g(x)=\sum_{yH\in G/H} f(y)g(y^{-1}x),$$

$$f^*(x)=\overline{f(x^{-1})}\Delta(x^{-1}),$$

where $\Delta(x)=L(x)/L(x^{-1})$ and $L(x)$ is the number of left cosets yH in the double coset HxH. Warning: some authors do not include the factor of Δ in the involution; for us it arises naturally when we embed $\mathcal{H}$ in $C_{\mathrm{c}}(\bar{G})$ (see [**5**, § 1]). One way to see how this embedding goes is the following: let $p=\chi_{\bar{H}}$, which is a projection in $C_{\mathrm{c}}(\bar{G})$ when the Haar measure on $\bar{G}$ has been normalized so that $\bar{H}$ has measure 1. Then the restriction map $f\mapsto f|_G$ gives a $*$-isomorphism of the convolution algebra $pC_{\mathrm{c}}(\bar{G})p$ onto $\mathcal{H}$.

2.1. Notation

$H < G$ means that H is a subgroup of G. $H \lhd G$ means H is a normal subgroup of G. If $N \lhd G$ and $Q < G$ such that $N \cap Q = \{e\}$ and $NQ = G$, then G is the (internal) semi-direct product of N by Q, and we write $G = N \rtimes Q$.

3. Groups

Here we describe the main properties of our group-theoretic set-up (1.1) for Hecke semi-direct products. We need to establish many elementary facts from group theory which are not standard, so we will give more detail than might seem necessary.

3.1. Generalities

We will be interested in subgroups of H of the form LS, where $L < M$ and $S < R$. Note that $LS < MR$ if and only if S normalizes L.

Lemma 3.1. *If A, B, C are subgroups of G with*

(i) $A \supset B$,

(ii) $A \cap C = \{e\}$,

(iii) $AC = CA$,

(iv) $BC = CB$,

then

$$[AC : BC] = [A : B].$$

Proof. The map $aB \mapsto aBC : A/B \to AC/BC$ is obviously well defined and surjective, and is injective because

$$a_1BC = a_2BC \implies a_2^{-1}a_1 \in BC \cap A = B.$$

□

Corollary 3.2. *Suppose that $L < M$ and $S < R$ and suppose that S normalizes L, so that LS is a subgroup of MR. Then*

$$[M : L][R : S] = [MR : LS].$$

Proof. We have

$$[MR : LS] = [MR : MS][MS : LS],$$

so the result follows from the above lemma. □

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Notation. For any subgroup K of G and $x \in G$, we define

$$K_x = K \cap xKx^{-1}.$$

Thus, K_x is precisely the stabilizer subgroup of the coset xK under the action of K on G/K by *left translation*, and

$$[K : K_x] = |KxK/K|. \tag{3.1}$$

If T is another subgroup of G, we let

$$T_{x,K} = \{t \in T \mid txKt^{-1} = xK\}$$

denote the stabilizer subgroup of xK under the action of T by *conjugation* on the set of all subsets of G; thus,

$$[T : T_{x,K}] = |\{txKt^{-1} \mid t \in T\}|. \tag{3.2}$$

Note that if T normalizes K, then the conjugation action of T descends to G/K.

For $E \subset G$, we further define

$$K_E = \bigcap_{x \in E} K_x \quad \text{and} \quad T_{E,K} = \bigcap_{x \in E} T_{x,K}.$$

It will also be useful to observe that if $\{M_i\}_{i \in I}$ is a family of subgroups of N and $\{R_i\}_{i \in I}$ is a family of subgroups of Q such that R_i normalizes M_i for each $i \in I$, then, because $N \cap Q = \{e\}$, we have

$$\bigcap_{i \in I} M_i R_i = \biggl(\bigcap_{i \in I} M_i\biggr)\biggl(\bigcap_{i \in I} R_i\biggr). \tag{3.3}$$

Lemma 3.3. *Let L be a subgroup of N which is normalized by R. For any $r \in R$ and $n \in N$, the following are equivalent:*

(i) $r \in R_{n,L}$;

(ii) $rnr^{-1} \in nL$;

(iii) $r \in nLRn^{-1}$.

Sketch of the proof. (i) $\implies$ (ii) $\implies$ (iii) is clear. (iii) $\implies$ (ii) uses $N \cap Q = \{e\}$. (ii) $\implies$ (i) because R normalizes L. $\square$

Taking $L = M$ in Lemma 3.3 and using $H = MR$, we have

$$R_{n,M} = R \cap nMRn^{-1} = R \cap nHn^{-1} \supset R \cap nRn^{-1} = R_n. \tag{3.4}$$

From this we deduce the following.

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Lemma 3.4. *For any $n \in N$ and $q \in Q$,*

(i) $H_n = MR_{n,M}$,

(ii) $H_q = M_q R_q$,

(iii) $H_{qn} \cap H_q = M_q(qR_{n,M}q^{-1} \cap R)$.

Proof. (i) Suppose that $h = mr \in H_n$ for $m \in M$ and $r \in R$. Then

$$r \in m^{-1}nMRn^{-1} = n(n^{-1}m^{-1}nMR)n^{-1} = nMRn^{-1} = nHn^{-1},$$

so (using (3.4))

$$mr \in m(R \cap nHn^{-1}) \subset MR_{n,M}.$$

Thus, $H_n \subset MR_{n,M}$. Conversely, also using (3.4),

$$MR_{n,M} = M(R \cap nHn^{-1}) \subset MR \cap MnHn^{-1} = H \cap nHn^{-1} = H_n.$$

(ii) By (3.3) we have

$$\begin{aligned} H_q &= H \cap qHq^{-1} \\ &= MR \cap (qMq^{-1})(qRq^{-1}) \\ &= (M \cap qMq^{-1})(R \cap qRq^{-1}) \\ &= M_q R_q. \end{aligned}$$

(iii) Using part (i) and (3.3) we have

$$\begin{aligned} H_{qn} \cap H_q &= H \cap qHq^{-1} \cap qnHn^{-1}q^{-1} \\ &= H \cap q(H_n)q^{-1} \\ &= MR \cap (qMq^{-1})(qR_{n,M}q^{-1}) \\ &= M_q(R \cap qR_{n,M}q^{-1}). \end{aligned}$$

□

3.2. Hecke pairs

Since $[H : H_x] = |HxH/H|$ for any $x \in G$, the pair (G, H) is Hecke if and only if each subgroup H_x has finite index in H. Applying this to the pair $(N \rtimes Q, Q)$, we see that $(N \rtimes Q, Q)$ is Hecke if and only if $[Q : Q_n] = [Q : Q_{n,\{e\}}] < \infty$ for each $n \in N$. The next proposition extends this observation to our more general context.

Proposition 3.5. *The following are equivalent:*

(i) *(G, H) is a Hecke pair;*

(ii) *$[R : R_q]$, $[M : M_q]$ and $[R : R_{n,M}]$ are all finite for each $q \in Q$ and $n \in N$;*

(iii) *(Q, R), (G, M) and $(N/M \rtimes R, R)$ are Hecke pairs;*

(iv) *(Q, R), (G, M) and (NR, H) are Hecke pairs.*

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Proof. If (G, H) is a Hecke pair, then for all $q \in Q$ and $n \in N$ we have

$$[M : M_q][R : R_q] = [MR : M_q R_q] = [H : H_q] < \infty$$

and

$$[R : R_{n,M}] = [MR : MR_{n,M}] = [H : H_n] < \infty,$$

so (i) $\implies$ (ii). Conversely, assuming (ii), for any $q \in Q$ and $n \in N$, Lemma 3.4 gives

$$\begin{aligned} [H : H_{qn}] \leqslant [H : H_{qn} \cap H_q] &= [MR : M_q(qR_{n,M}q^{-1} \cap R)] \\ &= [M : M_q][R : qR_{n,M}q^{-1} \cap R] \\ &= [M : M_q][R : R_q][R_q : qR_{n,M}q^{-1} \cap R], \end{aligned}$$

which is finite because, for any subgroups $S \supset T$ of G, we have $[R \cap S : R \cap T] \leqslant [S : T]$ and $[qSq^{-1} : qTq^{-1}] = [S : T]$. Thus, (ii) $\implies$ (i).

If $q \in Q$ and $n \in N$ then $qnMn^{-1}q^{-1} = qMq^{-1}$, so $[M : M_q] < \infty$ for all $q \in Q$ if and only if (G, M) is Hecke. As observed above, R is a Hecke subgroup of $N/M \rtimes R$ if and only if, for each $nM \in N/M$, the stabilizer subgroup of nM in R (acting by conjugation) has finite index in R. Since this subgroup is precisely $R_{n,M}$, we have $[R : R_{n,M}] < \infty$ for all $n \in N$ if and only if $(N/M \rtimes R, R)$ is Hecke. Therefore, (ii) $\iff$ (iii).

Finally, if $n \in N$ and $r \in R$ then $nrHr^{-1}n^{-1} = nHn^{-1}$, so

$$[H : H_{nr}] = [H : H_n] = [R : R_{n,M}].$$

Therefore (iii) $\iff$ (iv). □

Proposition 3.6. *Suppose that (G, H) is a Hecke pair. Then the following are equivalent:*

(i) *(G, H) is reduced;*

(ii) *$M_Q = \{e\}$ and $R_{N,\{e\}} \cap R_Q = \{e\}$.*

Proof. Since (G, H) is reduced if and only if $H_G = \{e\}$, the proposition will follow easily from the identity

$$H_G = M_Q(R_{N,M_Q} \cap R_Q). \tag{3.5}$$

To establish (3.5), we first use Lemma 3.4 (iii) and Corollary 3.2 to get

$$\begin{aligned} H_G &= \bigcap_{x \in G} H_x \\ &= \bigcap_{q \in Q,\, n \in N} H_{qn} \\ &= \bigcap_{q \in Q,\, n \in N} (H_q \cap H_{qn}) \end{aligned}$$

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$$= \bigcap_{q\in Q,\, n\in N} M_q(qR_{n,M}q^{-1} \cap R)$$
$$= \Big(\bigcap_{q\in Q} M_q\Big)\Big(\bigcap_{q\in Q,\, n\in N} qR_{n,M}q^{-1} \cap R\Big).$$

Furthermore,

$$\begin{aligned}\bigcap_{q\in Q,\, n\in N} qR_{n,M}q^{-1} \cap R &= \bigcap_{q\in Q,\, n\in N} R_{qnq^{-1},qMq^{-1}} \cap R_q \\ &= \bigcap_{q\in Q,\, n\in N} R_{n,qMq^{-1}} \cap \bigcap_{q\in Q} R_q \\ &= R_{N,M_Q} \cap R_Q.\end{aligned}$$

□

Note that $R_{N,\{e\}}$ consists of those elements of R which commute element-wise with N.

3.3. Hecke topology

In addition to our semi-direct product set-up (1.1), we now assume that (G, H) is a reduced Hecke pair. Let $(\bar{G}, \bar{H})$ denote its Schlichting completion.

Proposition 3.7. *The relative Hecke topologies of the relevant subgroups have the following sub-bases at the identity.*

(i) *For both N and M, $\{M_q \mid q \in Q\}$.*

(ii) *For Q, $\{qR_{n,M}q^{-1} \mid q \in Q,\ n \in N\}$.*

(iii) *For R, $\{R \cap qR_{n,M}q^{-1} \mid q \in Q,\ n \in N\}$.*

Proof. (i) This follows from the computation

$$N \cap qnHn^{-1}q^{-1} = qn(N \cap H)n^{-1}q^{-1} = qnMn^{-1}q^{-1} = qMq^{-1}$$

and its immediate consequence, $M \cap qnHn^{-1}q^{-1} = M_q$.

For (ii), we have

$$Q \cap nHn^{-1} = Q \cap nMRn^{-1} \subset Q \cap MNRN = Q \cap NR = R,$$

so

$$Q \cap qnHn^{-1}q^{-1} = q(Q \cap nHn^{-1})q^{-1} = q(R \cap nHn^{-1})q^{-1} = qR_{n,M}q^{-1}.$$

Finally, (iii) follows from (ii). □

The following corollary should be compared with [**8**, Theorem 2.9 (ii)]; the extra hypothesis therein is satisfied in our special case ($M \lhd N$), but it would be complicated to verify that our result follows from theirs because their construction is significantly different from ours.

Corollary 3.8. *If (G, H) as in (1.1) is a reduced Hecke pair with Schlichting completion $(\bar{G}, \bar{H})$, then*

$$\bar{G} = \bar{N} \rtimes \bar{Q} \quad \text{and} \quad \bar{H} = \bar{M} \rtimes \bar{R},$$

where the closures are all taken in $\bar{G}$.

Proof. First, in order to show that $\bar{G}$ is the semi-direct product $\bar{N} \rtimes \bar{Q}$ of its subgroups $\bar{N}$ and $\bar{Q}$, we require that

(i) $\bar{N} \lhd \bar{G}$,

(ii) $\bar{G} = \bar{N}\bar{Q}$,

(iii) $\bar{N} \cap \bar{Q} = \{e\}$,

(iv) $\bar{G}$ has the product topology of $\bar{N} \times \bar{Q}$.

Item (i) is obvious. To see (ii), note that the subgroup $\bar{N}\bar{Q}$ contains both $G = NQ$ and $\bar{M}\bar{R}$. Since $\bar{M}$ is compact, the subgroup $\bar{M}\bar{R}$ is closed, and it follows that $\bar{H} = \bar{M}\bar{R}$. This implies (ii), since every coset in $\bar{G}/\bar{H}$ can be expressed in the form $x\bar{H}$ for $x \in G$.

For (iii), note that the quotient map $\psi : G \to Q \subset \bar{Q}$ is continuous for the Hecke topology of G and the relative Hecke topology of Q, because a typical sub-basic neighbourhood of e in Q is of the form $qR_{n,M}q^{-1}$ for $q \in Q$ and $n \in N$, and

$$\psi^{-1}(qR_{n,M}q^{-1}) = NqR_{n,M}q^{-1}$$

contains the neighbourhood

$$H_{qn} \cap H_q = M_q(R \cap qR_{n,M}q^{-1})$$

of e in G. Since $\bar{Q}$ is a complete topological group, ψ extends uniquely to a continuous homomorphism $\bar{\psi} : \bar{G} \to \bar{Q}$. Because ψ takes N to e and agrees with the inclusion map on Q, by density and continuity $\bar{\psi}$ takes $\bar{N}$ to e and agrees with the inclusion map on $\bar{Q}$. Therefore, $\bar{N} \cap \bar{Q} = \{e\}$.

To see how (iv) follows, note that the multiplication map $(n, q) \mapsto nq$ of $\bar{N} \times \bar{Q}$ onto $\bar{G}$ is continuous by definition, and its inverse $x \mapsto (x\bar{\psi}(x)^{-1}, \bar{\psi}(x))$ is also continuous because $\bar{\psi}$ is, as shown above.

It only remains to show that $\bar{H} = \bar{M} \rtimes \bar{R}$, but this follows immediately: we have $\bar{M} \cap \bar{R} = \{e\}$, and the subgroup $\bar{M}\bar{R}$ has the product topology since $\bar{N}\bar{Q}$ does. □

3.4. Inverse limits

Here we again assume that (G, H) is a reduced Hecke pair. For each of our groups M, N, R, H and Q we want to describe the closure as an inverse limit of groups, so that we capture both the algebraic and the topological structure. From [**5**, Proposition 3.10], we know that the closure is topologically the inverse limit of the coset spaces of finite intersections of stabilizer subgroups. To get the algebraic structure we need enough of these intersections to be normal subgroups. In the case of M and N, we already have what we need, since each M_q is normal in N, and hence also in M. However, for R we need to do more work.

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Lemma 3.9. *Suppose $L < M$ and $S < R$. Then $LS \lhd MR$ if and only if*

(i) $L \lhd MR$,

(ii) $S \lhd R$ *and*

(iii) $S \subset R_{M,L}$.

Moreover, in this case

$$MR/LS \cong (M/L) \rtimes (R/S).$$

Proof. First assume $LS \lhd MR$. Then

$$S = R \cap LS \lhd R \cap MR = R,$$

and since $M \lhd MR$ we also have

$$L = M \cap LS \lhd MR.$$

For (iii), fix $s \in S$ and $m \in M$. Then $m^{-1}sm \in LS$ because $LS \lhd MR$, so $m^{-1}sms^{-1} \in LS$. On the other hand, $m^{-1}sms^{-1} \in M$ because $S \subset R$ and R normalizes M. Thus,

$$m^{-1}sms^{-1} \in LS \cap M = L,$$

so $s \in R_{m,L}$.

Conversely, assume (i)–(iii). It then suffices to show that M conjugates S into LS: for $m \in M$ and $s \in S$ we have $m^{-1}sms^{-1} \in L$ by Lemma 3.3 (ii), and hence $m^{-1}sm \in LS$.

For the 'moreover' statement, it is routine to verify that the map

$$mrLS \mapsto (mL, rS) \quad \text{for } m \in M,\ r \in R,$$

gives a well-defined isomorphism. □

Notation. For $E \subset Q$ and $F \subset N$ set

$$R_F^E = \bigcap_{q \in E} qR_{F,M}q^{-1} \cap R = \bigcap_{q \in E} \bigcap_{n \in F} qR_{n,M}q^{-1} \cap R.$$

Note that the families

$$\{M_E : E \subset Q \text{ finite}\} \quad \text{and} \quad \{R_F^E : \text{ both } E \subset Q \text{ and } F \subset N \text{ finite}\}$$

are neighbourhood bases at e in the relative Hecke topology of M and R, respectively.

Notation. Let $\mathcal{E}$ be the family of all subsets $E \subset Q$ such that

(i) E is a finite union of cosets in Q/R,

(ii) $e \in E$,

(iii) $RE = E$,

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and let $\mathcal{F}$ be the family of all pairs (E, F) such that

(iv) $E \in \mathcal{E}$,

(v) F is a finite union of cosets in N/M,

(vi) $q^{-1}Mq \subset F$ for all $q \in E$.

Lemma 3.10. *For all* $(E, F) \in \mathcal{F}$,

(i) $R_F^E \lhd R$,

(ii) $[R : R_F^E] < \infty$.

Proof. R_F^E is a subgroup of R because $R_{F,M}$ is a subgroup of R. For $r \in R$ we have

$$rR_F^E r^{-1} = \bigcap_{q\in E} r(qR_{F,M}q^{-1} \cap R)r^{-1} = \bigcap_{q\in E} rqR_{F,M}q^{-1}r^{-1} \cap R = R_F^E$$

since $rE = E$. This proves (i).

For (ii), first note that $[R : R_{F,M}] < \infty$ because $|F/M| < \infty$ and $R_{n,M}$ only depends upon the coset nM. Thus,

$$R_0 := \bigcap_{r\in R} rR_{F,M}r^{-1}$$

has finite index in R. For each coset tR contained in E we have

$$\bigcap_{q\in tR} qR_{F,M}q^{-1} = \bigcap_{r\in R} trR_{F,M}r^{-1}t^{-1} = tR_0t^{-1}.$$

Thus,

$$\bigcap_{q\in tR} qR_{F,M}q^{-1} \cap R$$

has finite index in R. Letting $E = \{t_1R, \ldots, t_kR\}$, it follows that

$$\bigcap_{q\in E} qR_{F,M}q^{-1} \cap R = \bigcap_{i=1}^{k} \left(\bigcap_{q\in t_iR} qR_{F,M}q^{-1} \cap R \right)$$

has finite index in R. □

Lemma 3.11. *For all* $E \in \mathcal{E}$,

(i) $M_E \lhd N$,

(ii) $M_E \lhd M$,

(iii) $M_E \lhd H$,

(iv) $[M : M_E] < \infty$.

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Proof. Part (i) holds because $M_q \lhd N$ for each q, and (ii) follows since $M_E \subset M$.

(iii) For $r \in R$ we have

$$rM_E r^{-1} = \bigcap_{q \in E} r(qMq^{-1} \cap M)r^{-1} = \bigcap_{q \in E} rqMq^{-1}r^{-1} \cap M = M_E$$

since $rE = E$. Thus, $M_E \lhd MR = H$ by (ii).

(iv) For each coset tR contained in E we have

$$\bigcap_{q \in tR} qMq^{-1} = \bigcap_{r \in R} trMr^{-1}t^{-1} = tMt^{-1}.$$

Thus, $\bigcap_{q \in tR} M_q = M_t$ has finite index in M, and it follows that $M_E = \bigcap_{q \in E} M_q$ has finite index in M as well. $\square$

Lemma 3.12. *For all $(E, F) \in \mathcal{F}$ we have*

$$R_F^E \subset R_{M,M_E}.$$

Proof. Fix $s \in R_F^E$ and $m \in M$; we need to show that $s \in R_{m,M_E}$. Thus, for $q \in E$, we must show that

$$m^{-1}sms^{-1} \in qMq^{-1}.$$

We have $q^{-1}mq \in F$, so $s \in qR_{q^{-1}mq,M}q^{-1}$. It follows that

$$q^{-1}m^{-1}sms^{-1}q = (q^{-1}m^{-1}q)(q^{-1}sq)(q^{-1}mq)(q^{-1}s^{-1}q) \in M,$$

and hence $m^{-1}sms^{-1} \in qMq^{-1}$, as desired. $\square$

Lemmas 3.10–3.12 yield the following.

Proposition 3.13. *For all $(E, F) \in \mathcal{F}$ we have*

$$M_E R_F^E \lhd H \quad \text{and} \quad [H : M_E R_F^E] < \infty.$$

Theorem 3.14. *With the above notation, we have*

(i) $\bar{M} = \varprojlim_{E \in \mathcal{E}} M/M_E$,

(ii) $\bar{N} = \varprojlim_{E \in \mathcal{E}} N/M_E$,

(iii) $\bar{R} = \varprojlim_{(E,F) \in \mathcal{F}} R/R_F^E$,

(iv) $\bar{H} = \varprojlim_{(E,F) \in \mathcal{F}} M/M_E \rtimes R/R_F^E$,

all as topological groups.

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Proof. By the preceding results, it suffices to show that for all finite subsets

$$E' \subset Q \quad \text{and} \quad F' \subset N$$

there exists $(E, F) \in \mathcal{F}$ such that

$$M_E \subset M_{E'} \quad \text{and} \quad R_F^E \subset R_{F'}^{E'}.$$

Set

$$E'' = (E' \cup \{e\})R \quad \text{and} \quad F'' = (F' \cup \{e\})M.$$

Since (Q, R) is Hecke, $E := RE''$ is a finite union of cosets in Q/R, and it follows that $E \in \mathcal{E}$. We have $M_E \subset M_{E'}$ since $E \supset E'$.

Let M_0 be the subgroup of N generated by the conjugates $q^{-1}Mq$ for $q \in E$. Then $M_0 \lhd N$ since $q^{-1}Mq \lhd N$ for each q. Since E is a finite union of double cosets of R in Q, and since (Q, R) is Hecke, E is a finite union of right cosets of R in Q. Thus, the family $\{q^{-1}Mq : q \in E\}$ is finite. Since M_0 is the product of the subgroups $q^{-1}Mq$ (because they are normal in N), it follows that $[M_0 : M] < \infty$. Thus, setting $F = M_0F''$, we have $(E, F) \in \mathcal{F}$ and, moreover, $R_F^E \subset R_{F'}^{E'}$ since $E \supset E'$ and $F \supset F'$. □

As a topological space, $\bar{Q} = \varprojlim_{E,F} Q/R_F^E$, but since the subgroups R_F^E are not in general normal in Q, the group structure of $\bar{Q}$ is more complicated. For details on this, we refer the reader to [**5**, Remark 3.11]. In the special case where Q is abelian, we do have $R_F^E \lhd Q$, so

$$\bar{Q} = \varprojlim_{E,F} Q/R_F^E$$

as topological groups.

4. Crossed products

In this section we prove a few results concerning crossed products, subgroups and projections. We state these results in somewhat greater generality than we require, since they might be useful elsewhere and no extra work is required.

4.1. Compact subgroups

Let R be a compact normal subgroup of a locally compact group Q. We identify Q and $C_c(Q)$ with their canonical images in $M(C^*(Q))$ and $C^*(Q)$, respectively. Normalize the Haar measure on R so that R has measure 1. Then $q := \chi_R$ is a central projection in $M(C^*(Q))$, and the map $\tau : Q/R \to M(C^*(Q))$ defined by

$$\tau(sR) = sq \quad \text{for } s \in Q \tag{4.1}$$

integrates to give an isomorphism of $C^*(Q/R)$ with the ideal $C^*(Q)q$ of $C^*(Q)$.

Let α be an action of Q on a C^*-algebra B. We identify B and $C^*(Q)$ with their canonical images in $M(B \times_\alpha Q)$. Thus, q is a projection in $M(B \times_\alpha Q)$, and we may regard τ as a homomorphism of Q/R into $M(B \times_\alpha Q)$.

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Let

$$\Phi(b) = \int_R \alpha_r(b)\,\mathrm{d}r$$

be the faithful conditional expectation of B onto the fixed-point algebra B^R. Then an elementary calculation shows that

$$qbq = \Phi(b)q = q\Phi(b) \quad \text{for } b \in B.$$

Thus, $qBq = B^R q$, and q commutes with every element of B^R. Thus, the formula

$$\sigma(b) = bq \tag{4.2}$$

defines a homomorphism σ of B^R onto the C^*-subalgebra $B^R q$ of $M(B \times_\alpha Q)$. We will deduce from Proposition 4.1 that σ is in fact an isomorphism.

Let β be the action of Q/R on B^R obtained from α. It is easy to see that the maps σ and τ from Equations (4.2) and (4.1) combine to form a covariant homomorphism (σ, τ) of the action $(B^R, Q/R, \beta)$, and that the integrated form

$$\theta := \sigma \times \tau : B^R \times_\beta Q/R \to q(B \times_\alpha Q)q \tag{4.3}$$

is surjective.

In the special case $R = Q$, the following is the main result of [**13**].

Proposition 4.1. *Let (B, Q, α) be an action, let R be a compact normal subgroup of Q, let $(B^R, Q/R, \beta)$ be the associated action and let $q = \chi_R$. Then the map $\theta : B^R \times_\beta Q/R \to q(B \times_\alpha Q)q$ from (4.3) is an isomorphism.*

Proof. In light of the discussion preceding the statement of the proposition, it remains to verify that θ is injective, and we do this by showing that, for every covariant representation (π, U) of $(B^R, Q/R, \beta)$ on a Hilbert space V, there exists a representation ρ of $q(B \times_\alpha Q)q$ on V such that $\rho \circ \theta = \pi \times U$.

Recall from the theory of Rieffel induction [**12**] that the conditional expectation $\Phi : B \to B^R$ gives rise to a B^R-valued inner product

$$\langle b, c\rangle_{B^R} = \Phi(b^* c)$$

on B, so the completion X is a Hilbert B^R-module. Moreover, B acts on the left of X by adjointable operators, so we can use X to induce π to a representation $\tilde{\pi}$ of B on $\tilde{V} := X \otimes_{B^R} V$. An easy computation shows that the formula

$$\tilde{U}_s(b \otimes \xi) = \alpha_s(b) \otimes U_{sR}\xi \quad \text{for } s \in Q,\ b \in B,\ \xi \in V$$

determines a representation $\tilde{U}$ of Q on $\tilde{V}$ such that $(\tilde{\pi}, \tilde{U})$ is a covariant representation of (B, Q, α).

Thus, $\tilde{\pi} \times \tilde{U}$ is a representation of the crossed product $B \times_\alpha Q$ on $\tilde{V}$; let ρ_1 be its restriction to the corner $q(B \times_\alpha Q)q$. We have $\rho_1(q)\tilde{V} = B^R \otimes_{B^R} V$, because if $b \in B$

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and $\xi \in V$, then

$$\begin{aligned}\rho_1(q)(b \otimes \xi) &= \int_R \tilde{U}_r(b \otimes \xi)\,\mathrm{d}r \\ &= \int_R (\alpha_r(b) \otimes U_{rR}\xi)\,\mathrm{d}r \\ &= \int_R \alpha_r(b)\,\mathrm{d}r \otimes \xi.\end{aligned}$$

The subspace $B^R \otimes_{B^R} V$ is invariant for the representation ρ_1; let ρ_2 denote the associated subrepresentation of $q(B \times_\alpha Q)q$. A routine computation shows that

$$W(b \otimes \xi) = \pi(b)\xi \quad \text{for } b \in B^R,\ \xi \in V$$

determines a unitary map W of $B^R \otimes_{B^R} V$ onto V which implements an equivalence between the representations $\rho_2 \circ \theta$ and $\pi \times U$. Thus, we can take $\rho = \operatorname{Ad} W \circ \rho_2$. □

Corollary 4.2. *Let (B, Q, α) be an action, let R be a compact normal subgroup of Q, let $(B^R, Q/R, \beta)$ be the associated action and let $q = \chi_R$. Then the map $\sigma : B^R \to B^R q$ from (4.2) is an isomorphism.*

Proof. It remains to observe that σ is faithful, being the composition of the injective homomorphism θ with the canonical embedding of B^R into $M(B^R \times_\beta Q/R)$. □

4.2. Two projections

If A is a C^*-algebra and p is a projection in $M(A)$, then one of the most basic applications of Rieffel's theory [**12**] is that the ideal $\overline{ApA}$ is Morita–Rieffel equivalent to the corner pAp via the $\overline{ApA} - pAp$ imprimitivity bimodule Ap. For later purposes, we will need the following slightly more subtle variant.

Lemma 4.3. *Let A be a C^*-algebra, and let $p, q \in M(A)$ be projections with $p \leqslant q$. Then $q\overline{ApA}q$ is Morita–Rieffel equivalent to pAp.*

Proof. Just apply the above Morita–Rieffel equivalence $\overline{ApA} \sim pAp$ with A replaced by qAq. □

4.3. Central projection

Let β be an action of a locally compact group T on a C^*-algebra C, and let $d \in M(C)$ be a central projection. Then d may also be regarded as a multiplier of the crossed product $C \times_\beta T$, and it generates the ideal

$$\overline{(C \times_\beta T)d(C \times_\beta T)}.$$

Proposition 4.4. *With the above notation, we have*

(i) $\overline{(C \times_\beta T)d(C \times_\beta T)} = I \times_\beta T$, *where I is the T-invariant ideal of C generated by d,*

(ii) $I = \overline{\operatorname{span}}\{\beta_t(d)C : t \in T\} = \{c \in C : p_\infty c = c\}$, *where $p_\infty = \sup\{\beta_t(d) : t \in T\}$.*

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Proof. (i) This follows from [**4**, Propositions 11 (ii) and 12 (i)].

(ii) The first equality holds because d is a central projection. For the second, note that the projections $\{\beta_t(d) : t \in T\}$ are central, so their supremum p_∞ is an open central projection in C^{**}, and the desired equality follows from, for example, [**11**, Proposition 3.11.9]. To make this part of the proof self-contained, we include the argument: set

$$J = \{c \in C : p_\infty c = c\}.$$

For any $t \in T$ and $c \in C$ we have $\beta_t(d) \leqslant p_\infty$, so

$$p_\infty \beta_t(d) c = \beta_t(d) c.$$

Thus, $I \subset J$. Suppose that $a \in J$ but $a \notin I$. There then exists a non-degenerate representation π of C such that $\pi(a) \neq 0$ but $I \subset \ker \pi$. Extend π to a weak*-weak-operator continuous representation $\bar{\pi}$ of C^{**}. Enlarge the set $\{\beta_t(d) : t \in T\}$ to an upward-directed set P of central projections in $M(C)$, so that there is an increasing net $\{p_i\}$ in P converging weakly* to p_∞. Then $p_i a \to p_\infty a$ weakly*, so $\pi(p_i a) \to \bar{\pi}(p_\infty a)$. We have $\bar{\pi}(p_\infty a) = \pi(a)$ because $a \in J$, and $\pi(p_i a) = 0$ for all i, so we deduce that $\pi(a) = 0$, which is a contradiction. □

Question 4.5. *When will p_∞ be a multiplier of B^R? (Example 6.10 shows that it is not always so.)*

4.4. Combined results

With the notation and assumptions of Proposition 4.1, set

$$A = B \times_\alpha Q.$$

Also let $d \in M(B)$ be an R-invariant central projection, so that d is also a central projection in $M(B^R)$. Set

$$p_\infty = \sup\{\alpha_s(d) : s \in Q\}.$$

Then p_∞ is an open central projection in $(B^R)^{**}$. Let I be the Q/R-invariant ideal of B^R generated by d. We have $dq = qd \in M(A)$, and we denote this projection by p.

The following theorem combines the previous results in this section.

Theorem 4.6. *With the above notation, we have the following:*

(i) $\theta(I \times_\beta Q/R) = q\overline{ApA}q$;

(ii) $I = \overline{\operatorname{span}}\{\alpha_s(d)B^R : s \in Q\} = \{b \in B^R : p_\infty b = b\}$;

(iii) $\sigma(I) = \overline{\operatorname{span}}\{sds^{-1}qBq : s \in Q\} = \overline{\operatorname{span}}\{sqdBqs^{-1} : s \in Q\}$;

(iv) *pAp is Morita–Rieffel equivalent to $I \times_\beta Q/R$.*

Proof. The only part that still requires proof is (iii). We have

$$\sigma(I) = \overline{\operatorname{span}}_{s \in Q} \theta \circ \alpha_s(dB^R)$$

because $\alpha_s(B^R) = B^R$. For each $s \in Q$ we have

$$\begin{aligned}
\sigma \circ \alpha_s(dB^R) &= \sigma \circ \beta_{sR}(dB^R) \\
&= \tau_{sR}\sigma(dB^R)\tau_{sR}^* \quad && \text{(covariance)} \\
&= (sq)dB^R q(sq)^* \\
&= sqdB^R qs^{-1} \\
&= sqdqBqs^{-1} \\
&= sqdBqs^{-1} && (dq = qd) \\
&= sds^{-1}qBq && (sq = qs,\ sB = Bs),
\end{aligned}$$

and (iii) follows. □

5. Hecke crossed products

In this section our main object of study is a Schlichting pair (G, H) which has the semi-direct-product decomposition of (1.1), with the additional condition that R be normal in Q. We shall obtain crossed-product C^*-algebras which are Morita–Rieffel equivalent to the completion of the Hecke algebra inside $C^*(G)$, giving results which are similar to certain results of [**5**]. At the end of the section we shall briefly indicate how our results can be applied if the Hecke pair is incomplete.

Set $A = C^*(G)$ and $B = C^*(N)$, and let α denote the canonical action of Q on B determined by conjugation of Q on N. Then A is isomorphic to the crossed product $B \times_\alpha Q$, and we identify these two C^*-algebras.

Normalize the Haar measures on N and Q so that M and R each have measure 1. Then the product measure is a Haar measure on G, and H has measure 1. Thus, $p_M := \chi_M$ is a central projection in B, and hence is a projection in $M(A)$. Similarly, $p_R := \chi_R$ is a central projection in $C^*(Q)$, and hence also a projection in $M(A)$, and we have

$$p_H := \chi_H = p_M p_R = p_R p_M \in A.$$

By [**5**, Corollary 4.4] the Hecke algebra of the pair (G, H) is $\mathcal{H} = p_H C_c(G) p_H$, whose closure in A is the corner $p_H A p_H$. From § 4 we obtain the isomorphisms

$$\begin{aligned}
\theta = \sigma \times \tau : B^R \times_\beta Q/R &\xrightarrow{\cong} p_R A p_R, \\
\sigma : B^R &\xrightarrow{\cong} B^R p_R, \\
\tau : C^*(Q/R) &\xrightarrow{\cong} C^*(Q) p_R,
\end{aligned}$$

and an ideal

$$I = \{b \in B^R : p_\infty b = b\} \lhd B^R,$$

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where

$$p_\infty = \sup\{\alpha_s(p_M) : s \in Q\} \in (B^R)^{**}.$$

Theorem 4.6 quickly gives the following analogue of [**5**, Theorem 8.2].

Theorem 5.1. *With the above notation,*

(i) $\theta(I \times_\beta Q/R) = p_R\overline{Ap_H A}p_R$,

(ii) $I = \overline{\operatorname{span}}\{\alpha_s(p_M)B^R : s \in Q\}$,

(iii) *we have*

$$\begin{aligned}\sigma(I) &= \overline{\operatorname{span}}\{sp_M s^{-1}p_R B p_R : s \in Q\}\\ &= \overline{\operatorname{span}}\{sp_R p_M B p_R s^{-1} : s \in Q\}\\ &= \overline{\operatorname{span}}\{sp_M s^{-1} p_R n p_R : s \in Q,\ n \in N\},\end{aligned}$$

and

(iv) $p_H A p_H$ *is Morita–Rieffel equivalent to* $I \times_\beta Q/R$.

Proof. The only thing left to prove is the last equality of part (iii), and this follows from Theorem 4.6, because M is compact open in N, and hence

$$p_M B = \overline{\operatorname{span}}\{p_M n : n \in N\}$$

(note that the projection d from Theorem 4.6 is p_M here). □

Remark 5.2. Note that if R is non-trivial, then p_H is never full in A: since N is normal in G with $Q = G/N$, there is a natural homomorphism $C^*(G) \to C^*(Q)$ which maps p_H to p_R. Thus, p_R is a non-trivial projection, which, being central, is not full in $C^*(Q)$.

We say that the family $\{sMs^{-1} : s \in Q\}$ of conjugates of M is *downward-directed* if the intersection of any two of them contains a third.

Proposition 5.3. *If* $\{sMs^{-1} : s \in Q\}$ *is downward-directed, then*

$$p_R\overline{Ap_H A}p_R = p_R A p_R \cong B^R \times_\beta Q/R.$$

Proof. Because the pair (G, H) is reduced we have

$$\bigcap_{s\in Q} sMs^{-1} = \{e\},$$

so the upward-directed set $\{sp_M s^{-1} : s \in Q\}$ of projections has supremum $p_\infty = 1$ in $(B^R)^{**}$. Therefore, the ideal I from Theorem 5.1 coincides with B^R, and the result follows. □

Remark 5.4. In the above proposition, we have

$$p_R\overline{Ap_HA}p_R = p_RAp_R,$$

although the ideal $\overline{Ap_HA}$ of A is proper if R is non-trivial.

As in [**5**, § 7] we specialize to the case where N is abelian. Taking Fourier transforms, the action α of Q on B becomes an action α' on $C_0(\hat{N})$:

$$\alpha'_s(f)(\phi) = f(\phi \circ \alpha_s) \quad \text{for } s \in Q,\ f \in C_0(\hat{N}),\ \phi \in \hat{N}.$$

The smallest Q-invariant subset of $\hat{N}$ containing $M^\perp$ is

$$\Omega = \bigcup_{s \in Q} (sMs^{-1})^\perp.$$

The Fourier transform of the fixed-point algebra B^R is isomorphic to $C_0(\hat{N}/R)$, where $\hat{N}/R$ is the orbit space under the action of R. The smallest Q/R-invariant subset of $\hat{N}/R$ containing $M^\perp/R$ is Ω/R. Thus, the Fourier transform of the ideal I of B^R is $C_0(\Omega/R)$. Let γ be the associated action of Q/R on $C_0(\Omega/R)$. The following corollary is analogous to [**5**, Corollary 7.1].

Corollary 5.5. *With the assumptions and notation of Proposition 5.3, if N is abelian, then p_HAp_H is Morita–Rieffel equivalent to the crossed product $C_0(\Omega/R) \times_\gamma Q/R$.*

We finish this section with a brief indication of how the above general theory can be used when (G, H) is the Schlichting completion of a reduced Hecke pair (G_0, H_0). More precisely, we assume that $G_0 = N_0 \rtimes Q_0$, $M_0 \lhd N_0$, $R_0 \lhd Q_0$, R_0 normalizes M_0 and that (G_0, H_0) is a reduced Hecke pair (and Propositions 3.5 and 3.6 give conditions under which the latter happens). By Corollary 3.8, the closures N, Q, M and R of N_0, Q_0, M_0 and R_0, respectively, satisfy the conditions of the current section. The action (B, Q, α) restricts to an action (B, Q_0, α_0), and by density we have $B^R = B^{R_0}$. The map $sR_0 \mapsto sR$ for $s \in R_0$ gives an isomorphism $Q_0/R_0 \cong Q/R$ of discrete groups, and the action β of Q/R on B^R corresponds to an action β_0 of Q_0/R_0 on B^{R_0}. Thus, we have a natural isomorphism

$$B^R \times_\beta Q/R \cong B^{R_0} \times_{\beta_0} Q_0/R_0.$$

Again by density, for all $s \in Q$ there exists $s_0 \in Q_0$ such that $p_Rs = p_Rs_0$, and similarly for all $n \in N$ there exists $n_0 \in N$ such that $np_M = n_0p_M$. We deduce the following.

Corollary 5.6. *Using the above isomorphisms and identifications, we have the following:*

(i) *I is the Q_0/R_0-invariant ideal of B^{R_0} generated by p_M;*

(ii) $I \times_{\beta_0} Q_0/R_0 \cong p_R\overline{Ap_HA}p_R$;

(iii) $p_\infty = \sup\{sp_Ms^{-1} : s \in Q_0\}$;

(iv) $I \cong \overline{\mathrm{span}}\{sp_Ms^{-1}p_Rnp_R : s \in Q_0,\ n \in N_0\}$;

(v) *p_HAp_H is Morita–Rieffel equivalent to $I \times_{\beta_0} Q_0/R_0$.*

As explained in [**5**], many of the nice properties of the Hecke algebra in [**1**] hold because the family $\{xHx^{-1} \mid x \in G\}$ of conjugates of H is downward-directed; in particular this implies that the projection p is full. In our situation we can only have p full if $R = \{e\}$, but we do have the following.

Corollary 5.7. *Suppose the conjugates $\{sMs^{-1} \mid s \in Q\}$ of M are downward-directed. Then $I = B^{R_0}$ and $p_H A p_H$ is Morita–Rieffel equivalent to $B^{R_0} \times_{\beta_0} Q_0/R_0$.*

Proof. We have $sp_M s^{-1} = p_{sMs^{-1}}$, so by the assumptions $p_\infty = 1$. □

Continuing with (G, H) being the Schlichting completion of (G_0, H_0) as above, we again consider the special case where N, equivalently N_0, is abelian. Fourier transforming, by density we have

$$\Omega = \bigcup_{s \in Q_0} (sMs^{-1})^\perp,$$

and there is an associated action γ_0 of Q_0/R_0 on $C_0(\Omega/R)$, giving the following result.

Corollary 5.8. *With the above notation, $p_H A p_H$ is Morita–Rieffel equivalent to $C_0(\Omega/R) \times_{\gamma_0} Q_0/R_0$.*

6. Examples

We shall here illustrate the results from the preceding sections with a number of examples. Some arguments are only sketched.

First note that the case $R = \{e\}$ is treated in [**5**, §§ 7 and 8].

Example 6.1. The situation with $M = \{e\}$ and $R \lhd Q$ is also interesting. From § 3 we see that (NQ, R) is Hecke if and only if $R_{n,\{e\}} = \{r \in R \mid rnr^{-1} = n\}$ has finite index in R for all n. The pair is reduced if and only if $\bigcap_n R_{n,\{e\}} = \{e\}$, i.e. if the map $R \to \operatorname{Aut} N$ is injective. Here $\bar{N} = N$, $p := p_H = p_R$ and Theorem 5.1 gives Morita–Rieffel equivalences among $\overline{ApA}$, pAp and $C^*(N)^R \times Q/R$. Example 10.1 of [**5**] is a special case of this situation.

We shall next study 2×2 matrix groups (and leave it to the reader to see how this generalizes to $n \times n$ matrices). For any ring J we let $\mathrm{M}(2, J)$ denote the set of all 2×2 matrices with entries in J; we let $\mathrm{GL}(2, J)$ denote the group of invertible elements of $\mathrm{M}(2, J)$; $\mathrm{SL}_\pm(2, J)$ denotes the subgroup of $\mathrm{GL}(2, J)$ consisting of those matrices with determinant ± 1, and $\mathrm{SL}(2, J)$ is the subgroup of $\mathrm{GL}(2, J)$ of matrices with determinant 1.

Proposition 6.2. *Suppose that $N = \mathbb{Q}^2$ and $M = \mathbb{Z}^2$, that Q is a subgroup of $\mathrm{GL}(2, \mathbb{Q})$ containing the diagonal subgroup*

$$D = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \,\middle|\, \lambda \in \mathbb{Q}^\times \right\}$$

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and that $R = Q \cap \mathrm{GL}(2,\mathbb{Z})$. Then (NQ, MR) is a reduced Hecke pair, and the Schlichting completion is given by

$$\bar{N} = \mathcal{A}_f^2, \quad \bar{M} = \mathcal{Z}^2, \quad \bar{R} = \varprojlim R/R(s) \quad \text{and} \quad \bar{Q} = \bigcup_{q \in Q/R} q\bar{R},$$

where $\bar{Q}$ has the topology from $\bar{R}$, i.e. $q_i \to e$ if and only if $q_i \in \bar{R}$ eventually and $q_i \to e$ in $\bar{R}$.

Proof. Given $q \in Q$ there is an integer matrix $k \in D$ such that kq^{-1} is an integer matrix. From this it follows that $kq^{-1}\mathbb{Z}^2 \subset \mathbb{Z}^2$ and therefore $kMk^{-1} \subset qMq^{-1}$. This implies that the sets $\{k\mathbb{Z}^2\}$ are downward-directed and form a base at e for the Hecke topologies of M and N, by Proposition 3.7. We also note that $\bigcap_k kMk^{-1} = \bigcap_k k\mathbb{Z}^2 = \{e\}$, by Proposition 3.6. Thus, $\bar{N} = \mathcal{A}_f^2$ and $\bar{M} = \mathcal{Z}^2$, with $\mathcal{A}_f$ the finite adeles and $\mathcal{Z}$ the integers in $\mathcal{A}_f$.

Next, if $n \in N$ there exists $s \in \mathbb{Z}$ such that $sn \in M$. Take

$$n_1 = \begin{pmatrix} 1/s \\ 0 \end{pmatrix} \quad \text{and} \quad n_2 = \begin{pmatrix} 0 \\ 1/s \end{pmatrix}.$$

By definition $r \in R_{n,M}$ if and only if $(r - I)n \in \mathbb{Z}^2$. One checks that $R_{n_1,M} \cap R_{n_2,M} \subset R_{n,M}$ and that

$$R_{n_1,M} \cap R_{n_2,M} = \{r \in R \mid r - I \in \mathrm{M}(2, s\mathbb{Z})\}.$$

Call this subgroup $R(s)$; it is clearly a normal subgroup of finite index in R.

Suppose that

$$q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Q,$$

and without loss of generality we may assume $q \in \mathrm{M}(2,\mathbb{Z})$. Setting $t = \det(q) = ad - bc$, for $r \in R(t)$ we have

$$q^{-1}(r - I)q = t^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} (r - I) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}(2,\mathbb{Z}),$$

and it follows that $q^{-1}rq \in \mathrm{M}(2,\mathbb{Z})$. The same argument holds for r^{-1}, so both $q^{-1}rq$ and $q^{-1}r^{-1}q$ are integer matrices in Q. Thus,

$$q^{-1}rq \in Q \cap \mathrm{GL}(2,\mathbb{Z}) = R.$$

From this it follows that

$$R(t) \subset R \cap qRq^{-1} \quad \text{for } t = \det(q),$$

and we have just observed that $[R : R(t)] < \infty$, so $[R : R_q] < \infty$.

The same argument also shows that $R(st) \subset R \cap qR(s)q^{-1}$ for any s, and therefore for any given finite sets $E \subset Q$ and $F \subset N$ there exists $s \in \mathbb{N}$ such that $R(s) \subset R_F^E$. Combining all this with Proposition 3.7 we see that the family $\{R(s) \mid s \in \mathbb{N}\}$ is a base at e for the Hecke topology restricted to R or Q.

Finally, note that $\bigcap_s R(s) = \{e\}$. □

A similar result holds when $\mathbb{Q}$ is replaced by other number fields, e.g. $\mathbb{Z}[p^{-1}]$ for a prime number p (not to be confused with the projection p). We now state it without proof.

Proposition 6.3. *Suppose that $N = \mathbb{Z}[p^{-1}]^2$ and $M = \mathbb{Z}^2$, that Q is a subgroup of $\mathrm{GL}(2, \mathbb{Z}[p^{-1}])$ containing the diagonal subgroup*

$$D = \left\{ \begin{pmatrix} p^n & 0 \\ 0 & p^n \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\},$$

and that $R = Q \cap \mathrm{GL}(2, \mathbb{Z})$. Then (NQ, MR) is a reduced Hecke pair, and the Schlichting completion is given by

$$\bar{N} = \mathbb{Q}_p^2, \quad \bar{M} = \mathcal{Z}_p^2, \quad \bar{R} = \varprojlim R/R(p^n) \quad \text{and} \quad \bar{Q} = \bigcup_{q \in Q/R} q\bar{R},$$

where, as above, $\bar{Q}$ has the topology from $\bar{R}$.

Example 6.4. Let us first consider the maximal p-adic case with $Q = \mathrm{GL}(2, \mathbb{Z}[p^{-1}])$ and $R = \mathrm{GL}(2, \mathbb{Z})$.

Proposition 6.5. *Let*

$$T = \left\{ \begin{pmatrix} p^m & 0 \\ c & p^n \end{pmatrix} \;\middle|\; m, n \in \mathbb{Z},\ c \in \mathbb{Z}[p^{-1}] \right\}.$$

Then $T\,\mathrm{SL}_{\pm}(2, \mathcal{Z}_p) = \{g \in \mathrm{GL}(2, \mathbb{Q}_p) \mid \det(g) \in \pm p^{\mathbb{Z}}\}$.

Proof. Clearly, the left-hand side is included in the right-hand side. For the 'reverse inclusion' it suffices to show that every $g \in \mathrm{M}(2, \mathcal{Z}_p)$ with $\det(g) \in p^{\mathbb{N}}$ is a member of the left-hand side. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Case 1. Suppose that $b = 0$ and $ad = p^m$. If $a = p^n u$ with u a unit in $\mathcal{Z}_p$, we must have $d = u^{-1}p^{m-n}$. So

$$g = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} p^n & 0 \\ 0 & p^{m-n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$$

with $x = cu^{-1}p^{n-m}$. Now $x = y + z$ with $y \in \mathbb{Z}[1/p]$ and $z \in \mathcal{Z}_p$, and since

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in \mathrm{SL}(2, \mathcal{Z}_p),$$

it follows that

$$g = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in T\,\mathrm{SL}(2, \mathcal{Z}_p).$$

Case 2. Suppose that $a = 0$ and $b \neq 0$. Then

$$g = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & -c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in T\,\mathrm{SL}(2, \mathcal{Z}_p).$$

Case 3. Suppose $a = p^m u$ and $b = p^n v$ with u, v units in $\mathcal{Z}_p$. We may assume $m \geqslant n$, if not we multiply by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ as in case 2. So $p^{-n}a \in \mathcal{Z}_p$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p^n & 0 \\ v^{-1}d & p^{-n}ad - vc \end{pmatrix} \begin{pmatrix} p^{-n}a & v \\ -v^{-1} & 0 \end{pmatrix}.$$

The second matrix on the right-hand side is in $\mathrm{SL}(2, \mathcal{Z}_p)$, while the first has determinant equal to $ad - bc$, which by assumption is in $p^{\mathbb{N}}$, so by case 1 this matrix is in $T\,\mathrm{SL}(2, \mathcal{Z}_p)$. □

Theorem 6.6. *Let* $Q = \mathrm{GL}(2, \mathbb{Z}[p^{-1}])$ *and* $R = \mathrm{GL}(2, \mathbb{Z})$. *Then*

(i) $\bar{R} = \varprojlim R/R(p^n) = \mathrm{SL}_{\pm}(2, \mathcal{Z}_p)$,

(ii) $\bar{Q} = \bigcup_{q \in Q/R} q\,\mathrm{SL}_{\pm}(2, \mathcal{Z}_p) = \{g \in \mathrm{GL}(2, \mathbb{Q}_p) \mid \det(g) \in \pm p^{\mathbb{Z}}\}$,

where $\bar{Q}$ *has the topology from* $\bar{R} = \mathrm{SL}_{\pm}(2, \mathcal{Z}_p)$.

Proof. Since $Q = TR$ we get $\bar{Q} = T\bar{R}$, which by Proposition 6.5 equals the right-hand side. That [**5**, Theorem 3.8] applies to the pair $(\bar{N}\bar{Q}, \bar{M}\bar{R})$ now follows from Propositions 6.3 and 6.5, and density of $\mathrm{GL}(2, \mathbb{Z})$ in $\mathrm{SL}_{\pm}(2, \mathcal{Z}_p)$ (see [**6**, Proposition IV.6.3]). □

Now we look at the case where $Q = \mathrm{GL}(2, \mathbb{Q})$ and $R = \mathrm{GL}(2, \mathbb{Z})$. We first need the following version of Proposition 6.5.

Proposition 6.7. *Let*

$$T = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \,\middle|\, a, c, d \in \mathbb{Q},\ ad \neq 0 \right\}.$$

Then $T\,\mathrm{SL}(2, \mathcal{Z}) = \{g \in \mathrm{GL}(2, \mathcal{A}_f) \mid \det(g) \in \mathbb{Q}\}$.

Proof. Again one inclusion is obvious, so suppose that

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathcal{A}_f) \quad \text{with } \det g \in \mathbb{Q};$$

in fact, without loss of generality we may assume that $\det g = 1$. For each prime p let

$$g_p = \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix}$$

be the corresponding matrix in $\mathrm{GL}(2, \mathbb{Q}_p)$. For all but finitely many p we will have $g_p \in \mathrm{SL}(2, \mathcal{Z}_p)$. In these cases take $k_p = g_p$.

In the other cases we cannot have both a_p and b_p zero, so by Proposition 6.5 there is a matrix $k_p \in \mathrm{SL}(2, \mathcal{Z}_p)$ such that $g_p k_p^{-1} \in T \cap \mathrm{GL}(2, \mathbb{Z}[1/p])$. So $k = (k_p) \in \mathrm{SL}(2, \mathcal{Z})$ and $gk^{-1} \in T$ as claimed. □

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Theorem 6.8. *Let $Q = \mathrm{GL}(2,\mathbb{Q})$ and $R = \mathrm{GL}(2,\mathbb{Z})$. Then*

(i) $\bar{R} = \mathrm{SL}_{\pm}(2,\mathcal{Z})$,

(ii) $\bar{Q} = \bigcup_{q\in Q/R} q\,\mathrm{SL}_{\pm}(2,\mathcal{Z}) = \{g \in \mathrm{GL}(2,\mathcal{A}_f) \mid \det(g) \in \mathbb{Q}\}$,

where $\bar{Q}$ has the topology from $\bar{R} = \mathrm{SL}_{\pm}(2,\mathcal{Z})$.

Proof. From [**6**, Proposition IV.6.3] (the hard part is hidden there) it follows that

$$\bar{R} = \varprojlim R/R(s) = \varprojlim \mathrm{SL}_{\pm}(2,\mathbb{Z}_s) = \mathrm{SL}_{\pm}(2,\mathcal{Z}).$$

Since $\bar{Q} = T\bar{R}$, part (ii) follows from Proposition 6.7. □

Note that the topology on $\bar{Q}$ is not the relative topology from $\mathrm{GL}(2,\mathcal{A}_f)$, in contrast with Theorem 6.6.

This is essentially the same result as [**8**, Proposition 2.5]. Since R is not normal in Q we cannot use Theorem 5.1, but it would be interesting to get a description of the C^*-algebra $p_R A p_H A p_R$ in these cases (see [**2**]). However, note that we are not using exactly the same algebra, since in both [**2**] and [**8**] the action of Q is by left multiplication on $\mathrm{M}(2,\mathbb{Q})$.

Example 6.9. Much recent work on Hecke algebras started with the study of the affine group over $\mathbb{Q}$ in [**1**]. Other number fields have also been extensively studied, as in, for example, [**2**, **9**]. For a survey, see [**2**, § 1.4]. We shall here illustrate how our approach works for a quadratic extension of $\mathbb{Q}$. For details about the number theory used here we refer the reader to [**10**].

Let d be a square-free integer such that $d \not\equiv 1 \mod 4$, and let $N = \mathbb{Q}(\sqrt{d})$, $M = \mathbb{Z}[\sqrt{d}]$, $Q = \mathbb{Q}(\sqrt{d})^\times$, and $R = \{r \in Q \mid r, r^{-1} \in M\}$.*

So

$$R = \{m + n\sqrt{d} \mid m, n \in \mathbb{Z},\ m^2 - dn^2 = \pm 1\}$$

is the group of units in the field N. An alternative matrix description is as follows:

$$\begin{aligned}
N &= \mathbb{Q}^2,\\
M &= \mathbb{Z}^2,\\
Q &= \left\{ \begin{pmatrix} a & db \\ b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{Q},\ a^2 - db^2 \neq 0 \right\},\\
R &= \left\{ \begin{pmatrix} m & dn \\ n & m \end{pmatrix} \,\middle|\, m, n \in \mathbb{Z},\ m^2 - dn^2 = \pm 1 \right\}.
\end{aligned}$$

So we get $\bar{N} = \mathcal{A}_f^2$ and $\bar{M} = \mathcal{Z}^2$.

Here Theorem 5.1 applies, so

$$pAp \sim_{MR} C_0(\mathcal{A}_f^2/\bar{R}) \rtimes Q/R.$$

* If, for instance, $d = 5$, one should instead use $M = \mathbb{Z}[(1+\sqrt{5})/2]$, etc. (see [**10**, Theorem 9.20]).

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In this way we obtain [**9**, Proposition 3.2] for the field $\mathbb{Q}(\sqrt{d})$ without using the theory of semigroup crossed products, and this will also work in greater generality.

The structure of these crossed products can be studied by the Mackey–Takesaki orbit method as in [**7**]; note that the orbit closures in $\bar{N}/\bar{R}$ under the action of Q/R are basically the same as the orbit closures in $\bar{N}$ under the action of Q.

To determine $\bar{R}$ and its topology we need some more information. First, if $d < 0$ then R is finite (of order 2 or 4). So let us concentrate on the case with $d > 1$. We then, by [**10**, Theorem 7.26], have $R \cong \{\pm 1\} \times \mathbb{Z}$, and in fact there exists $r_0 \in R$ such that $R = \{\pm r_0^n \mid n \in \mathbb{Z}\}$. For instance, if $d = 2$, one can take $r_0 = 1 + \sqrt{2}$.

Let us look at $R(s)$. There is a smallest integer $n_s > 0$ such that $r_0^{n_s} \equiv 1 \bmod s$. From this we get $\bar{R} = \varprojlim R/R(s) = \{\pm 1\} \times \varprojlim \mathbb{Z}/\mathbb{Z}_{n_s}$. However, examples show that the behaviour of the numbers n_s is complicated, so a more exact description of $\bar{R}$ is difficult.

Perhaps counter-intuitively, in general it turns out that

$$\bar{R} \subsetneq \{m + n\sqrt{d} \mid m, n \in \mathcal{Z},\ m^2 - dn^2 = \pm 1\}.$$

This is because under the homomorphism $\mathbb{Z}[\sqrt{d}] \mapsto \mathbb{Z}_s[\sqrt{d}]$ the units R in $\mathbb{Z}[\sqrt{d}]$ are in general mapped onto a proper subgroup of the units in $\mathbb{Z}_s[\sqrt{d}]$. For instance, 4 is a unit in $\mathbb{Z}_{17}[\sqrt{2}]$, but $\pm(1 + \sqrt{2})^n \not\equiv 4 \bmod 17$ for all n.

Example 6.10. We shall here give a slightly different treatment of the Heisenberg group from that in [**5**]. Take

$$N = \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}, \qquad M = \{0\} \times \mathbb{Z},$$
$$Q = \left\{ \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \middle|\ q \in \mathbb{Q} \right\}, \qquad R = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \middle|\ r \in \mathbb{Z} \right\},$$

with the obvious action of Q on N. If

$$x = \begin{pmatrix} 1 & 1/n \\ 0 & 1 \end{pmatrix} \quad \text{with } n \in \mathbb{N},$$

one checks that $M \cap xMx^{-1} = \{0\} \times n\mathbb{Z}$. So we have

$$\bar{N} = \mathbb{Q}/\mathbb{Z} \times \mathcal{A}_f = \mathcal{A}_f/\mathcal{Z} \times \mathcal{A}_f \quad \text{and} \quad \bar{M} = \{0\} \times \mathcal{Z}.$$

If

$$n = \begin{pmatrix} a \\ b/m \end{pmatrix} \quad \text{with } b, m \in \mathbb{Z} \quad \text{and} \quad r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix},$$

then $rnr^{-1} - n \in M$ if and only if $rb \in m\mathbb{Z}$. Thus,

$$\bar{Q} = \left\{ \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \middle|\ q \in \mathcal{A}_f \right\} \quad \text{and} \quad \bar{R} = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \middle|\ r \in \mathcal{Z} \right\}.$$

We have $\hat{\bar{N}} = \mathcal{Z} \times \mathcal{A}_f$ and $\bar{M}^\perp = \mathcal{Z} \times \mathcal{Z}$. Moreover, the dual action of $\bar{Q}$ on $\hat{\bar{N}}$ is given by

$$(z, w)\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = (z, qz + w).$$

Lemma 6.11. *Set* $\Omega = \bigcup_{q \in \bar{Q}} q\bar{M}^\perp$. *Then*

$$\begin{aligned} \Omega &= \{(z, qz + w) \mid z, w \in \mathcal{Z},\ q \in \mathcal{A}_f\} \\ &= \{(z, u) \in \mathcal{Z} \times \mathcal{A}_f \mid z_p = 0 \implies u_p \in \mathcal{Z}_p\}. \end{aligned}$$

Proof. Clearly, if $(z, w) \in \Omega$ and $z_p = 0$, then $w_p \in \mathcal{Z}_p$.

Conversely, suppose (z, u) is an element of the right-hand side. If $u_p \in \mathcal{Z}_p$, take $q_p = 1$ and $w_p = u_p - z_p \in \mathcal{Z}_p$. For the finitely many p with $u_p \notin \mathcal{Z}_p$, we have $u_p = x_p + v_p$ with $x_p \in \mathbb{Q}^\times$ and $v_p \in \mathcal{Z}_p$ and, by assumption, $z_p \neq 0$. Take $q_p = z_p^{-1} x_p \in \mathbb{Q}_p$, so $q_p z_p + w_p = u_p$. Thus, with $q := (q_p) \in \mathcal{A}_f$ and $w := (w_p) \in \mathcal{Z}$, we have $qz + w = u$. $\square$

So here Ω is open but not closed; hence, the projection p_∞ defined in § 5 is not in $M(B^R)$.

The orbits under the action of R can be described as follows: $(0, w)$ is always a fixed point. If $z \neq 0$, then the R-orbit of (z, w) is $(z, w + z\mathcal{Z})$.

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