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THE PROJECTIVE ANTECEDENT OF THE THREE REFLECTION THEOREM

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ABSTRACT. A complete answer is given to the question: *Under what circumstances is the product of three harmonic homologies in* PG(2, F) *again a harmonic homology*? This is the natural question to ask in seeking a generalization to projective geometry of the Three Reflection Theorem of metric geometry. It is found that apart from two familiar special cases, and with one curious exception, the necessary and sufficient conditions on the harmonic homologies produce exactly the Three Reflection Theorem.

1. **Introduction.** The Three Reflection Theorem [1, pp. 24, 36] states that the product of reflections in three lines of a given pencil is again a reflection in a line of the same pencil. The Theorem is true in any metric plane, whether it be Euclidean, Minkowski, elliptic or hyperbolic. Since the common geometry for all these systems is projective, we propose to look at the theorem in the projective context.

Any reflection is a harmonic homology (**hh** for short) in the underlying projective plane. That is to say, it is a collineation H with a *centre* O (a point of fixed lines) and a non-incident *axis* a (a line of fixed points) with the property that if P is any point distinct from O and not on a, and if $OP \cap a = M$, then P', the image of P under H, is the harmonic conjugate of P with respect to O and M (in symbols: H(OM; PP') [4, p. 22]). It follows that H is involutory. Conversely, any involutory projective collineation in the plane is a **hh** [4, p. 55].

A **hh** exists in a projective plane over a field F if and only if the characteristic of F is not 2. Accordingly we consider any plane PG(2, F) with char $f \neq 2$ [3, p. 237; 5, p. 168] and we ask the question:

Under what circumstances is the product of three harmonic homologies in PG(2, F) again a harmonic homology?

The answer is that apart from two familiar special cases, and with one curious exception, the necessary and sufficient conditions for the desired result produce exactly the three-reflection theorem. More precisely, solutions of the question posed imply the following theorems:

THEOREM 1. *The product of three harmonic homologies with the same axis a is a* **hh** *with axis a.*

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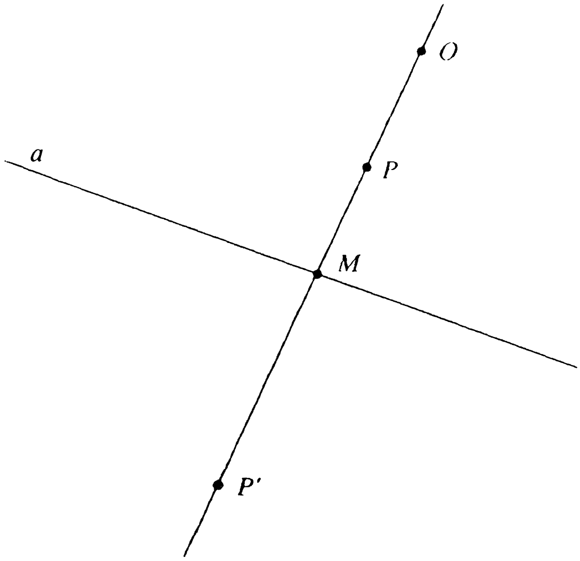

Figure 1.

THEOREM 2. *An elation with centre O and axis a may be expressed either as the product of two* **hh** *with common axis a whose centres are collinear with O, or as the product of two* **hh** *with common centre O, whose axes intersect on a.*

THEOREM 3. *The product of 3 harmonic homologies whose centres are distinct and collinear is again a* **hh** *if and only if there is a polarity π that interchanges the centres and axes of all four* **hh**.

Theorem 1 is well known in its affine context: *The product of three half turns in the affine plane is again a half-turn* [1, p. 202]. The dual of Theorem 1 implies a restricted version of the three reflection theorem in the Euclidean and Minkowski planes: *The product of reflections in the 3 lines of a parallel pencil is again a reflection in a line of that pencil.* Theorem 2 is also well known in its affine context: *A translation may be expressed either as the product of two half-turns or as the product of reflections in parallel lines* [3, pp. 41, 42]. Finally, Theorem 3 is a unified statement of all cases of the three reflection theorem not covered by the dual of Theorem 1.

Besides the configurations implied by the above theorems, the only other configuration which results when the product of 3 **hh** in PG(2,F) is again a **hh** occurs when the field F contains all three cube roots of unity. It is the only case in which simultaneously the four centres form a quadrangle while the four axes form a quadrilateral.

We assume that the reader is familiar with the rudiments of classical projective geometry. References [2; 3; 4; 5; 6] provide the basics, from different points of view. We emphasize the concepts of harmonic conjugates and polarities:

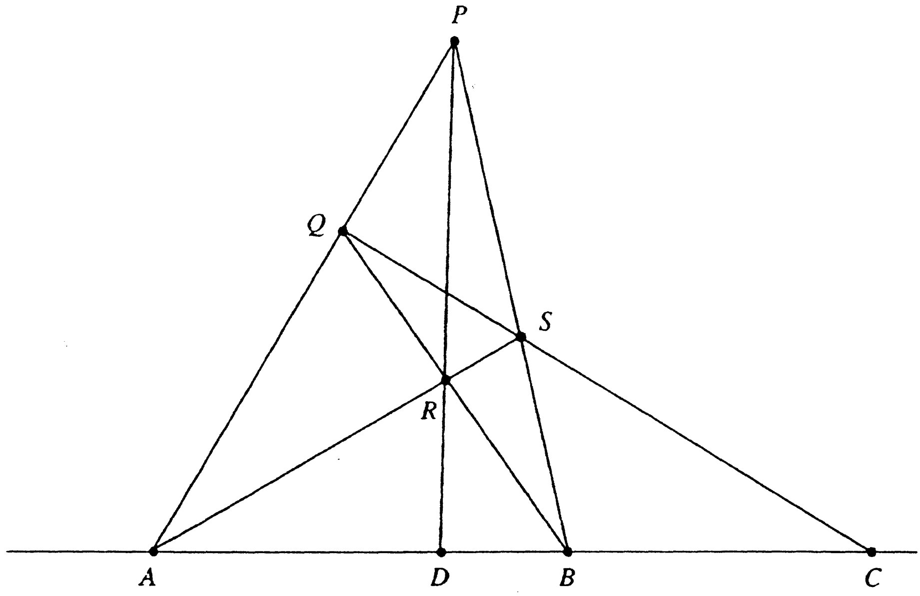

Figure 2.

Let A, B, C be three distinct collinear points in PG(2, F), the projective plane over a field F of characteristic $\neq 2$. Construct a quadrangle $PQRS$ such that $A = PQ \cap RS, B = PS \cap QR$, and C I QS. Then the *harmonic conjugate* of C with respect to A and B is $PR \cap AB = D$. In symbols we say H(AB; CD). D is independent of the choice of $PQRS$, and H(AB; CD) implies H(AB; DC), H(BA; CD) and H(CD; AB).

A *correlation* in PG(2, F) is a 1-1 mapping of points onto lines that preserves incidence [3, p. 248; 4, p. 57; 6, p. 262]. A *polarity* is a correlation of period 2. Any point P is the *pole* of its image p under a polarity, and p is called the *polar* of P. If a polarity π admits points lying on their own polars (called *self-conjugate* or *absolute* points), then π is said to be *hyperbolic*; otherwise, π is *elliptic*.

Elliptic (hyperbolic) geometry is the study of properties that remain invariant under the action of the group Γ_π, the centralizer of an elliptic (hyperbolic) polarity in the group Γ of collineations of PG(2, F). In particular, a *reflection* is a harmonic homology in Γ_π. Such a **hh** has its centre and axis interchanged by π.

For a full account of elliptic and hyperbolic geometry from the projective point of view, the reader is referred to [2], especially chapters VI and X.

The Euclidean and Minkowski planes are affine planes obtained from PG(2, F) by deleting some line, often denoted by ℓ_∞ and all points on it. A pairing of points on ℓ_∞, is established by fixing some involution (projectivity of period 2) on ℓ_∞. A *reflection* is a harmonic homology whose centre, on ℓ_∞, is the mate of the intersection of its axis with ℓ_∞, under the fixed involution.

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I am grateful to C. W. L. Garner for drawing my attention to this problem. He himself has been working on a unified treatment of the Three Reflection Theorem, but from a purely synthetic point of view. His findings as yet have not been published.

2. **Collineations and correlations in** PG(2, F). In answering the question that we have posed, we make free use of both synthetic and analytic methods. Regarding the latter, we remind the reader that a point of PG(2, F) is denoted by an ordered triple (x, y, z) of elements of F, not all zero, with the understanding that $(\lambda x, \lambda y, \lambda z)$ $(\lambda \in F, \lambda \neq 0)$ is the same point. Likewise a line is an ordered triple $[s, t, u]$ $(s, t, u$ not all $= 0)$ with the understanding that $[\lambda s, \lambda t, \lambda u]$ is the same line. $[s, t, u]$ and (x, y, z) are incident if and only if $sx + ty + uz = 0$ [4, pp. 111–118; 6, pp. 169–180].

Each projective collineation Γ is identified by a non-singular 3×3 matrix M over F, where λM and M represent the same collineation. Γ is a permutation of points: $(x, y, z) \longrightarrow (x', y', z') = (x, y, z)M$ (here, (x, y, z) is considered to be a matrix, and we have the indicated matrix product). Γ is also a permutation of lines: $[r, s, t] \longrightarrow [r', s', t'] = [r, s, t](M^{-1})^t$ ($(M^{-1})^t$ being the transpose of the inverse of M). For convenience of expression we identify Γ with its matrix M. The product of two collineations M_1 and M_2 is a collineation $M_{12} = M_1M_2$.

The harmonic homology H with centre (0, 0, 1) and axis [0, 0, 1] is

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Transforming H by the elation

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & k & 1 \end{pmatrix},$$

we get

$$H' = \tau^{-1}H\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2h & -2k & -1 \end{pmatrix},$$

which represents any **hh** with axis [0, 0, 1]. The centre of H$'$ is $(h, k, 1)$. Since H$'$ is involutory, the transpose of H$'$,

$$H_0 = \begin{pmatrix} 1 & 0 & -2h \\ 0 & 1 & -2k \\ 0 & 0 & -1 \end{pmatrix},$$

is a **hh** with centre (0, 0, 1) and axis $[h, k, 1]$. In what follows, we express any **hh** with a given centre O by transforming H$_0$ by some collineation Γ taking (0, 0, 1) onto O.

Let H$_i$ $(i = 1, 2, 3, 4)$ be a **hh** with centre O_i and axis a_i. Our problem is to investigate the nature of the set $\{H_i\}$ in the event that H$_1H_2H_3 = H_4$.

The above equation is equivalent to H$_1H_2 = H_4H_3$. When we use the analytic method, we express both H$_1$H$_2$ and H$_4H_3$ as matrices M_{12} and M_{43} respectively. Adjusting, if necessary, so that $\det(M_{43}) = \det(M_{12})$, we then have that $M_{43} = \delta M_{12}$, where δ is some cube root of 1. By equating entries, we find necessary and sufficient conditions on $\{H_i\}$ so that $H_1H_2H_3 = H_4$. Of course if M_{43} and M_{12} have corresponding entries which are the same non-zero element of F, then $\delta = 1$.

Any projective correlation γ in PG(2, F) can also be expressed by a matrix. As a point-to-line mapping it is simply $[x', y', z'] = (x, y, z)C$, where C is a non-singular 3×3 matrix, since γ is the product of the collineation $(x', y', z') = (x, y, z)C$ and the special correlation $(x, y, z) \longrightarrow [x, y, z]$. As a line-to-point mapping, γ is $(x', y', z') = [x, y, z](C^{-1})^t$. It follows that γ is a polarity if and only if C is symmetric [4, p. 123; 6, p. 280].

3. **The Product of Three Harmonic Homologies.** As we have just noted, we consider four **hh**'s $\{H_i\}$ with centres $\{O_i\}$ and axes $\{a_i\}$ under the following:

ASSUMPTION I. $H_1H_2H_3 = H_4$.

Our problem is to completely analyze the relationships existing in the set $\{O_i, a_i\}$. If $H_1 = H_2$, then $H_4 = H_3$, and if $H_1 = H_3$, then H_4 is the transform of H_2 by H_1. These, and similar obvious solutions are ignored in the following by adopting

ASSUMPTION II. H_1, H_2, H_3, H_4 are all distinct.

Our approach is to consider all possible arrangements of the four centres $\{O_i\}$, including cases in which they are not all distinct. The arrangements may be reduced to two cases:

(i) At least three of the $\{O_i\}$ are collinear.

(ii) The $\{O_i\}$ form a quadrangle.

We consider each case in turn.

CASE (i) At least three of the $\{O_i\}$ are collinear. By re-naming, if necessary, we may assume that O_1, O_2, O_3 are collinear.

(a) If $O_1 = O_2 = O_3$, then $H_4 = H_1H_2H_3$ also has centre O_1. The dual of this particular case leads immediately to Theorem 1. By a short synthetic argument, it follows that either the $\{a_i\}$ are concurrent in a point $\neq O_1$ or else the $\{a_i\}$ form a quadrilateral, one of whose diagonal points is O_1 (Figure 3).

(b) Assuming now that O_1, O_2, O_3 are not all identical, let ℓ be their common line. Since ℓ is fixed by $H_1H_2H_3 = H_4$, $\ell = a_4$ or $\ell I O_4$.

LEMMA 3.1. *If O_1, O_2, O_3 are collinear, then O_4 is also on their common line ℓ.*

PROOF. Assume to the contrary that $O_4 \not\mathrm{I}\ \ell$. Then $\ell = a_4$. If $a_3 \mathrm{I} O_4$, then H_4H_3 is easily seen to be a **hh** with centre $a_3 \cap a_4 = O$ and axis O_3O_4. However, if $O_1 = O_2$, then H_1H_2 is an elation [3, p. 247; 5, p. 120], while if $O_1 \neq O_2$, then H_1H_2 can fix only points on $O_1O_2 = a_4$; so $H_1H_2 \neq H_4H_3$, contrary to Assumption I. Therefore $a_3 \not\mathrm{I}\ O_4$, and H_4H_3 is a collineation fixing only two points: $O = a_3 \cap a_4$ and O_3. Since $H_1H_2 = H_4H_3$ and H_1H_2 fixes $a_1 \cap a_2$, then $a_1 \cap a_2 = O$ or O_3. But if $a_1 \cap a_2 = O$, then since H_1H_2 fixes O_3, $H(O_1O;\ O_3O_3')$ and $H(O_20;\ O_30_3')$ (for some $O_3' I \ell$), yielding $O_1 = O_2$. Similarly, $a_1 \cap a_2 = O_3$ implies $O_1 = O_2$. But if $O_1 = O_2$, then $H_1H_2 = H_4H_3$ is an elation, which is false. ∎

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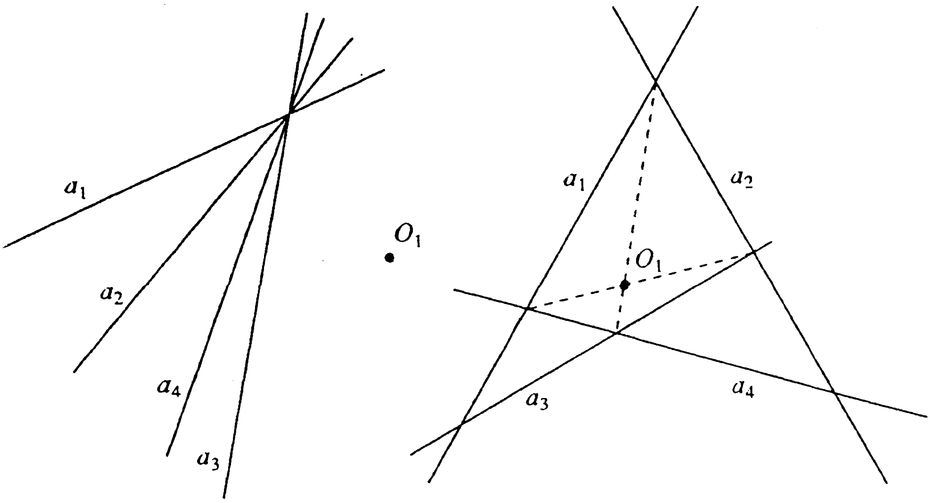

Figure 3.

LEMMA 3.2. *If O_1, O_2, O_3 are collinear and not all identical, then the collinear set $\{O_i\}$ $(1 \leqq i \leqq 4)$ consists of at least three distinct points.*

PROOF. If $\{O_i\}$ does not consist of at least three distinct points then it can only be two pairs of identical points. With re-lettering if necessary, this implies either $O_1 = O_2, O_3 = O_4$ or $O_1 = O_3, O_2 = O_4$. If $O_1 = O_2, O_3 = O_4$, then $H_1H_2(H_4H_3)$ is an elation with centre O_1 $(O_3 \neq O_1)$. Since $H_1H_2 = H_4H_3$ now has two different centres, $H_1H_2 = 1$, i.e. $H_1 = H_2$, contrary to Assumption II.

On the other hand, if $O_1 = O_3, O_2 = O_4$, let $\{P_j\}$ $(j = 1, 2)$ be two distinct points of $\{a_1\}$, both $\neq a_1 \cap O_1O_2$. Now $P_j \xrightarrow{H_1} P_j \xrightarrow{H_2} P'_j \xrightarrow{H_3} P''_j$; hence $P'_j \xrightarrow{H_3} P''_j$ and $P_j \xrightarrow{H_4} P''_j$, so $P''_j = P'_jO_3 \cap P_jO_4 = P'_j$. If $P'_j \neq P_j$, let M_j be such that $H(P_jP'_j; O_2M_j)$ while if $P'_j = P_j$, let $M_j = P_j$. Then $P_j \xrightarrow{H_2} P'_j$ and $P_j \xrightarrow{H_4} P'_j$ imply that $a_2 = a_4 = M_1M_2$. Since also $O_2 = O_4, H_2 = H_4$, which again is contrary to Assumption II. ∎

We are now ready to analyze case (i). In virtue of Lemmas 3.1 and 3.2, and relettering if necessary, we may assume that the points $\{O_i\}$ $(1 \leqq i \leqq 4)$ are collinear, and that O_1, O_2, O_3 are distinct. The group PGL(3, F) of projective collineations in PG(2, F) is sufficiently versatile to ensure that we lose no generality in letting $O_1 = (0, 1, 0)$, $O_2 = (1, 1, 0)$ and $O_3 = (1, 0, 0)$. Then

$$H_1 = \begin{pmatrix} 1 & -2s_1 & 0 \\ 0 & -1 & 0 \\ 0 & -2u_1 & 1 \end{pmatrix}, H_2 = \begin{pmatrix} 1+2s_2 & 2s_2 & 0 \\ -2(1+s_2) & -1-2s_2 & 0 \\ 2u_2 & 2u_2 & 1 \end{pmatrix}, H_3 = \begin{pmatrix} -1 & 0 & 0 \\ -2t_3 & 1 & 0 \\ -2u_3 & 0 & 1 \end{pmatrix},$$

$a_1 = [s_1,\ 1,\ u_1]$, $a_2 = [s_2, -1-s_2,\ u_2]$, $a_3 = [1,\ t_3,\ u_3]$,

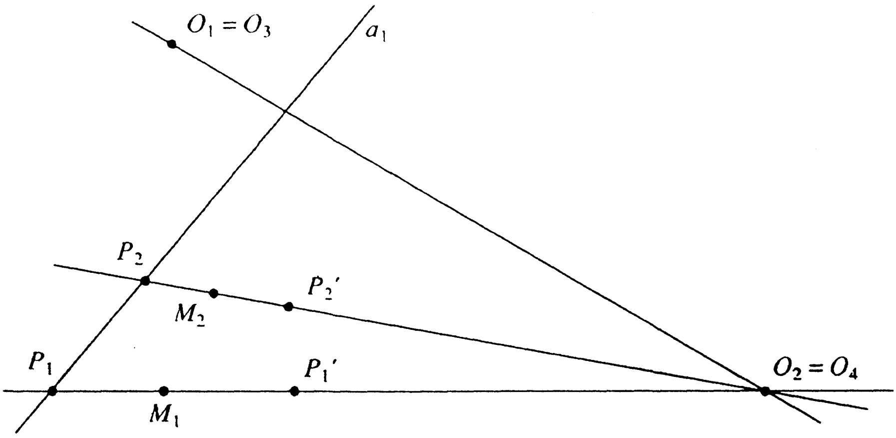

Figure 4.

and

$$H_1H_2 = \begin{pmatrix} 1+2s_2+4s_1(1+s_2) & 2s_2+2s_1(1+2s_2) & 0 \\ 2(1+s_2) & 1+2s_2 & 0 \\ 4u_1(1+s_2)+2u_2 & 2u_1(1+2s_2)+2u_2 & 1 \end{pmatrix}. \tag{3.1}$$

(a) We consider first a special possibility, namely $O_4 = (0,1,0) = O_1$. In this case,

$$H_4 = \begin{pmatrix} 1 & -2s_4 & 0 \\ 0 & -1 & 0 \\ 0 & -2u_4 & 1 \end{pmatrix}, \quad a_4 = [s_4,\ 1,\ u_4],$$

and

$$H_4H_3 = \begin{pmatrix} -1+4s_4t_3 & -2s_4 & 0 \\ 2t_3 & -1 & 0 \\ 4u_4t_3-2u_3 & -2u_4 & 1 \end{pmatrix} \tag{3.2}$$

Since the two matrices (3.1) and (3.2) have common corresponding entries of 1, they are identical. We equate corresponding entries to find possible solutions (the work is made easier by assuming without loss of generality that $a_2\,\mathrm{I}\,(0,\ 0,\ 1)$, i.e. that $u_2 = 0$). The only solution is $a_2 = a_3\ I\ O_1$, and $\{a_i\}$ are all concurrent (in a point not on [0,0,1]). Thus $\mathrm{H}_1H_4 = H_2H_3$ is an elation with centre O_1 and axis a_2, and we have Theorem 2.

(b) The general case is $O_4 = (1,y,0)$. We may assume without loss of generality that $a_4\,\mathrm{I}\,(0,0,1)$. This simplification gives

$$H_4 = \begin{pmatrix} -1+2yt & 2y(-1+yt) & 0 \\ -2t & 1-2yt & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_4 = [1-yt,\ t,\ 0],$$

and

$$H_4H_3 = \begin{pmatrix} 1-2yt+4yt_3(1-yt) & 2y(-1+yt) & 0 \\ 2t-2t_3(1-2yt) & 1-2yt & 0 \\ -2u_3 & 0 & 1 \end{pmatrix}. \tag{3.3}$$

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As in case (a), the two matrices (3.1) and (3.3) are identical. Equating corresponding entries, we get one special solution which is ruled out by Assumption II, three special solutions verifying Theorem 2, and the following:

1. $a_1 = [-1, 1, u_1]$, $a_2 = [-1, 0, u_1]$, $a_3 = [1, t_3, -u_1]$ $O_4 = (t_3, -1, 0)$, $a_4 = [0, 1, 0]$. $(t_3(t_3 + 1) \neq 0)$
2. $a_1 = [s_1, 1, u_1]$, $a_2 = [-s_2, 1 + s_2, u_1(1 + 2s_2)]$, $a_3 = [s_{12}, -s_1(1 + s_2), -u_1 s_{12}]$ $O_4 = (1 + s_2, -s_{12}, 0)$ $a_4 = [s_{12}, s_2, 0]$. $(s_1 s_2(1 + s_1)(1 + s_2)(1 + 2s_2) \neq 0)$ where $s_{12} = s_1 + s_2 + 2s_1 s_2$.

In Solution 1, the $\{a_i\}$ are concurrent in the point $(u_1, 0, 1)$, and the polarity

$$[s,\ t,\ u] = (x, y, z) \begin{pmatrix} -1 & -t_3 & u_1 \\ -t_3 & t_3 & u_1 t_3 \\ u_1 & u_1 t_3 & d \end{pmatrix}, \quad (d + u_1^2 \neq 0),$$

interchanges O_i with a_i for all i. Similarly, in Solution 2, the $\{a_i\}$ are concurrent in $(u_1 s_2, -u_1 s_{12}, s'_{12})$, where $s'_{12} = s_1 + s_2 + s_1 s_2$, and the polarity

$$[s,\ t,\ u] = (x,\ y,\ z) \begin{pmatrix} -s_{12} & s_1(1 + s_2) & u_1 s_{12} \\ s_1(1 + s_2) & 1 + s_2 & u_1(1 + s_2) \\ u_1 s_{12} & u_1(1 + s_2) & d \end{pmatrix}$$

$(s'_{12} d - s_{12} u_1^2 \neq 0)$ interchanges O_i with a_i for all i. Thus we have Theorem 3.

Finally, we turn to

CASE (ii) *The* $\{O_i\}$ *form a quadrangle.*

We may assume without loss of generality that $O_1 = (0, 0, 1)$, $O_2 = (0, 1, 0)$, $O_3 = (1, 0, 0)$, $O_4 = (1, 1, 1)$. Then

$$H_1 = \begin{pmatrix} 1 & 0 & -2s_1 \\ 0 & 1 & -2t_1 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & -2s_2 & 0 \\ 0 & -1 & 0 \\ 0 & -2u_2 & 1 \end{pmatrix},$$

$a_1 = [s_1, t_1, 1]$, $a_2 = [s_2, 1, u_2]$, and

$$H_1 H_2 = \begin{pmatrix} 1 & 2(2s_1 u_2 - s_2) & -2s_1 \\ 0 & -1 + 4t_1 u_2 & -2t_1 \\ 0 & 2u_2 & -1 \end{pmatrix} \tag{3.4}$$

Likewise,

$$H_4 = \begin{pmatrix} 1 - 2s_4 & -2s_4 & -2s_4 \\ -2t_4 & 1 - 2t_4 & -2t_4 \\ 2(s_4 + t_4 - 1) & 2(s_4 + t_4 - 1) & 2s_4 + 2t_4 - 1 \end{pmatrix}, \quad H_3 = \begin{pmatrix} -1 & 0 & 0 \\ -2t_3 & 1 & 0 \\ -2u_3 & 0 & 1 \end{pmatrix},$$

$a_4 = [s_4, t_4 1 - s_4 - t_4]$, $a_3 = [1, t_3, u_3]$, and

(3.5)

$$H_4 H_3 = \begin{pmatrix} -1 + 2s_4 + 4s_4 t_3 + 4s_4 u_3 & -2s_4 & -2s_4 \\ 2[t_4 + t_3(2t_4 - 1) + 2t_4 u_3] & 1 - 2t_4 & -2t_4 \\ -2[(s_4 + t_4 - 1)(1 + 2t_3) + u_3(2s_4 + 2t_4 - 1)] & 2(s_4 + t_4 - 1) & 2s_4 + 2t_4 - 1 \end{pmatrix}$$

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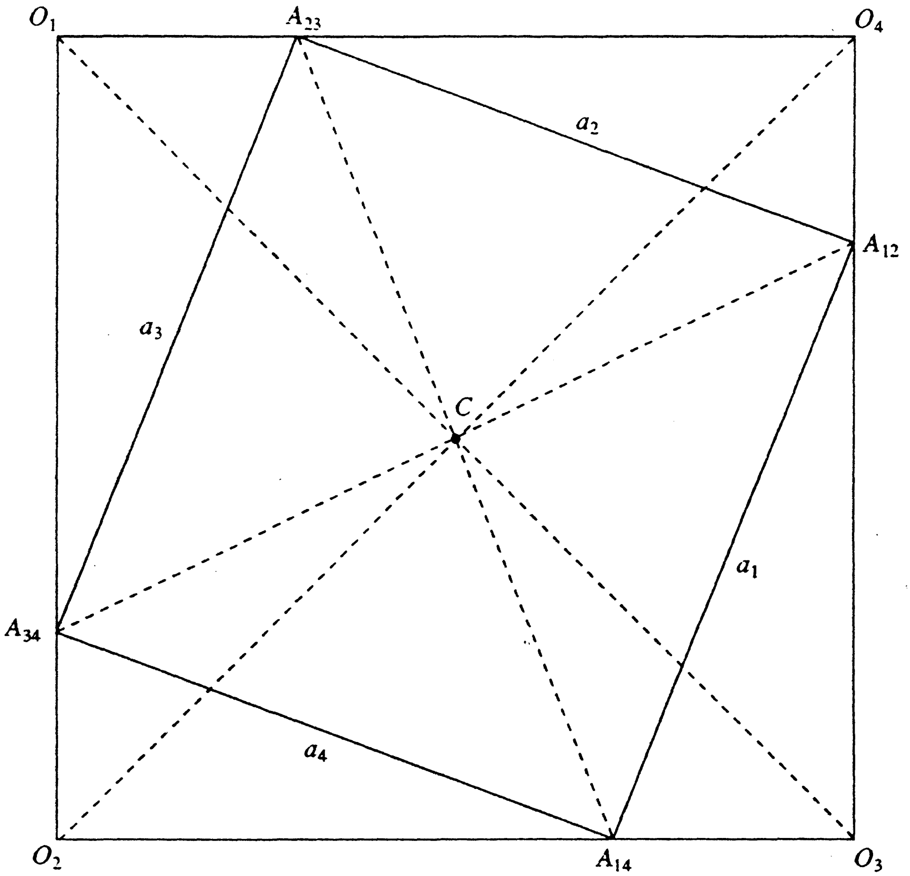

Figure 5.

The matrices in (3.4) and (3.5) both have determinant 1. We therefore get one solution to $H_1H_2 = H_4H_3$ by considering the two matrices to be identical. This solution is $a_1 = a_2 = a_3 = a_4 = [1, -1, 1]$. $[1, -1, 1]$ is a diagonal line of the quadrangle $O_1O_2O_3O_4$; this case therefore reduces to a special case of Theorem 1.

If the field F contains three cube roots of unity: ω, ω^2 and 1, then $H_1H_2 = H_4H_3$ has another solution. This is obtained by solving the matrix equation $H_4H_3 = \omega H_1H_2$. The solution is $a_1 = [-1, \omega^2, 2], a_2 = [\omega^2, 2, \omega]\, a_3 = [2, \omega, -1], a_4 = [\omega, -1, \omega^2] \cdot a_1a_2a_3a_4$ is a quadrilateral, and if we let $A_{ij} = a_i \cap a_j$ $(i \neq j)$, we can routinely verify that A_{12} I O_3O_4, A_{34} I O_1O_2, A_{14} I O_2O_3 and A_{23} I O_1O_4. Moreover, the four lines $A_{12}A_{34}$, $A_{14}A_{23}$, O_1O_3, and O_2O_4 are concurrent in a point C, while the four points A_{13}, A_{24}, $O_1O_2 \cap O_3O_4$, and $O_1O_4 \cap O_2O_3$ are collinear in a line $c \nmid C$. By means of the collineation Γ:

$$(x', y', z') = (x, y, z)\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix},$$

we take C onto $(0,0,1)$ and c onto $[0,0,1]$, which we identify as the line at infinity in the affine plane. The points $A_{12}, A_{23}, A_{34}, O_1, O_2, O_3$, and O_4 (more precisely their images under Γ), together with some joining lines, now form the affine configuration shown in Figure 5. In affine coordinates, $O_1 = (-1,1), O_2 = (-1,-1), O_3 = (1,-1), O_4 = (1,1)$, $A_{12} = (1, \omega^2/\omega - 1)$, $A_{23} = (\omega^2/1 - \omega, 1)$, $A_{34} = (-1, \omega^2/1 - \omega)$, and $A_{14} = (\omega^2/\omega - 1, -1)$. The parallelogram $a_1a_2a_3a_4$ is inscribed in the parallelogram $O_1O_2O_3O_4$. This configuration is somewhat reminiscent of the classical configuration of two mutually inscribed squares [4; p. 129, Ex. 6; 3, p. 237], especially since both configurations occur only over a field containing ω; however, they are not the same configurations, since $O_1O_2O_3O_4$ is not inscribed in $a_1a_2a_3a_4$. Also, in contrast to previous solutions of our problem, the correlation taking O_i onto a_i $(i = 1,2,3,4)$ is *not* a polarity.

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