might be approximated to; the mathematical or physical treatment pointing out the way in which the subject might be treated.

Kötter's Synthetic Geometry of Algebraic Curves—Part III., Involution Nets, and Involutions of 2nd, 3rd, ... Rank.

By Rev. Norman Fraser, B.D.

[See Index.]

Sixth Meeting, 12th April 1889.

A. Y. Fraser, Esq., M.A., Vice-President, in the Chair.

On Vortex Motion in a rotating fluid.

By C. Chree, M.A.

The object of the following paper is to consider the motion of one or more vortices in a compressible fluid, which is rotating as a whole with uniform angular velocity ω about an axis, taken as axis of z. To save space I shall when possible refer for results to a previous paper in the *Proceedings*, distinguishing the equations of that paper, Vol. V., pp. 52-59, by the suffix a.

Formulæ for fluid motion relative to rotating axes are given by Greenhill in the article "Hydrodynamics" in the *Encyclopaedia Britannica*, and by Basset in his "Treatise on Hydrodynamics," Vol. I., § 23. The velocities appearing however in these equations are partly at least absolute velocities, while equations containing only velocities relative to the moving axes seem most suitable for our purpose. Such equations may be obtained shortly as follows, confining our attention to the case when there is no velocity parallel to the axis of rotation.

Let u, v denote the velocity components relative to the moving axes ox, oy in the fluid at the point x, y at the time t, and let u', v'

denote the velocity components relative to fixed axes with which the moving axes coincide at the instant considered. Then we have

$$u' = u - \omega y,\ v' = v + \omega x \quad \dots \quad \dots \quad \dots \quad (1);$$

whence

$$\left.\begin{aligned} \frac{du'}{dx} &= \frac{du}{dx}, \quad \frac{dv'}{dx} = \frac{dv}{dx} + \omega \\ \frac{du'}{dy} &= \frac{du}{dy} - \omega, \quad \frac{dv'}{dy} = \frac{dv}{dy} \end{aligned}\right\} \quad \dots \quad \dots \quad (2).$$

At the end of an indefinitely short interval τ the volocity components at the point x, y relative to the moving axes are

$$u + \frac{du}{dt}\tau, \text{ and } v + \frac{dv}{dt}\tau \quad \dots \quad \dots \quad \dots \quad (3);$$

where $\frac{d}{dt}$ denotes partial differentiation and is employed when there is no variation in the co-ordinates x, y. At the end of this interval the axes of x and y make angles $\omega\tau$ with the fixed axes ox', oy' with which they coincided at the time t. Thus the co-ordinates of the point, which we denote above by x, y, referred to the fixed axes are

$$\left.\begin{aligned} x' &= x\cos\omega\tau - y\sin\omega\tau = x - \omega\tau y \text{ in limit} \\ y' &= y\cos\omega\tau + x\sin\omega\tau = y + \omega\tau x \text{ in limit} \end{aligned}\right\} \quad \dots \quad \dots \quad (4).$$

Thus the velocity component in the direction ox' at the above point is

$$u' + \frac{du'}{dt}\tau + \frac{du'}{dx}(x' - x) + \frac{du'}{dy}(y' - y) \quad \dots \quad \dots \quad (5);$$

where $\frac{du'}{dt}$ denotes partial differentiation in the sense that is usual in the equations of motion, and signifies the variation that occurs in the velocity at a point absolutely fixed in space. In the limit when τ is very small (5) transforms into

$$u' + \tau\left(\frac{du'}{dt} - \omega y\frac{du'}{dx} + \omega x\frac{du'}{dy}\right)$$

and from (1) and (2) this is identical with

$$u - \omega y + \tau\left(\frac{du'}{dt} - \omega y\frac{du}{dx} + \omega x\frac{du}{dy} - \omega^2 x\right) \quad \dots \quad \dots \quad (5')$$

But again from (1) we have for the velocity relative to the fixed axis ox' at time $t + \tau$ the expression

$$\left(u + \frac{du}{dt}\tau - \omega y\right)\cos\omega\tau - \left(v + \frac{dv}{dt}\tau + \omega x\right)\sin\omega\tau;$$

which becomes in the limit

$$u - \omega y + \tau\left(\frac{du}{dt} - \omega v - \omega^2 x\right) \quad \dots \quad \dots \quad (6).$$

Equating (5′) and (6), we find

$$\frac{du'}{dt}=\frac{du}{dt}-\omega v+\omega y\frac{du}{dx}-\omega x\frac{du}{dy} \quad \ldots \quad \ldots \quad (7).$$

Again, from (1) and (2) we have

$$u'\frac{du'}{dx}+v'\frac{du'}{dy}=(u-\omega y)\frac{du}{dx}+(v+\omega x)\left(\frac{du}{dy}-\omega\right) \quad \ldots \quad (8).$$

If we use p, ρ, X, Y in their usual sense, the equations of motion referred to the fixed axes ox', oy' are

$$\frac{du'}{dt}+u'\frac{du'}{dx}+v'\frac{du'}{dy}=\mathrm{X}-\frac{1}{\rho}\frac{dp}{dx} \quad \ldots \quad \ldots \quad (9),$$

$$\frac{dv'}{dt}+u'\frac{dv'}{dx}+v'\frac{dv'}{dy}=\mathrm{Y}-\frac{1}{\rho}\frac{dp}{dy} \quad \ldots \quad \ldots \quad (10).$$

Now, adding (7) and (8) we transform (9) into

$$\frac{\delta u}{\delta t}-2\omega v-\omega^2 x=\mathrm{X}-\frac{1}{\rho}\frac{dp}{dx} \quad \ldots \quad \ldots \quad \ldots \quad (11),$$

where

$$\frac{\delta u}{\delta t}\equiv\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)u \quad \ldots \quad \ldots \quad (12).$$

Remembering the meaning that now attaches to $\frac{du}{dt}$ we see that the operator $\frac{\delta}{\delta t}$ denotes differentiation following the fluid, and so has exactly the same meaning as in my former papers.

Similarly (10) transforms into

$$\frac{\delta v}{\delta t}+2\omega u-\omega^2 y=\mathrm{Y}-\frac{1}{\rho}\frac{dp}{dy} \quad \ldots \quad \ldots \quad \ldots \quad (13).$$

If it be assumed that p is a function of ρ and that X, Y are derivable from a potential, or vanish, we get by differentiating (13) with respect to x and subtracting (11) differentiated with respect to y

$$\frac{\delta\zeta}{\delta t}+(\zeta+\omega)\left(\frac{du}{dx}+\frac{dv}{dy}\right)=0 \quad \ldots \quad \ldots \quad (14),$$

where $2\zeta\equiv\frac{dv}{dx}-\frac{du}{dy}$.

We next must find the form of the equation of continuity containing u and v. Let us suppose dx, dy to form adjacent sides of the section of a rectangular prism, infinite in the direction of z, which is fixed in space so that its sides dx, dy coincide in direction with the instantaneous positions of the axes ox, oy at time t. The fluid velocities normal to the faces dx, dy are respectively $v+\omega x$ and $u-\omega y$. Thus if $\frac{d'\rho}{dt}$ indicates that the point where the variation in density is being measured is absolutely fixed in space, the ordinary equation of continuity for motion in two dimensions is

$$\frac{d.}{dx}\rho(u-\omega y)+\frac{d.}{dy}\rho(v+\omega x)+\frac{d'\rho}{dt}=0 \quad \ldots \quad \ldots \quad (15).$$

But if $\frac{d\rho}{dt}$ indicates that the point where the density is being measured is fixed relative to the moving axes, we get

$$\frac{d\rho}{dt}\tau=\frac{d'\rho}{dt}\tau+(x'-x)\frac{d\rho}{dx}+(y'-y)\frac{d\rho}{dy},$$

where x', y' are given by (4). Proceeding to the limit, we get

$$\frac{d'\rho}{dt}=\frac{d\rho}{dt}+\omega y\frac{d\rho}{dx}-\omega x\frac{d\rho}{dy}.$$

Thus (15) simply transforms into

$$\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d\rho}{dt}=0 \quad \ldots \quad \ldots \quad \ldots \quad (15').$$

This is identical in *form* with the ordinary equation when the axes are fixed; but the difference in the meaning of the symbols, particularly of $\frac{d\rho}{dt}$, must be carefully attended to.

Using $\frac{\delta}{\delta t}$ in the sense of equation (12), we can transform (15') into

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{1}{\rho}\frac{\delta\rho}{\delta t}=0 \quad \ldots \quad \ldots \quad \ldots \quad (15'').$$

Combining (14) and (15'') we get

$$\frac{\delta}{\delta t}\left(\frac{\zeta+\omega}{\rho}\right)=0 \quad \ldots \quad \ldots \quad \ldots \quad (16).$$

Thus the apparent vorticity ζ of an element remains constant if the fluid be incompressible, but in a gas the apparent vorticity will vary unless the density remain constant. If we suppose the density to be the same over the cross section σ of the vortex, then since $\sigma\rho$ is necessarily constant in virtue of the principle of continuity, we may replace (16) by

$$\sigma(\zeta+\omega)=m'(a \text{ constant}) \quad \ldots \quad \ldots \quad \ldots \quad (17).$$

If $m\equiv\sigma\zeta$ denote the apparent strength of the vortex, then it follows that

$$m=m'-\omega\sigma \quad \ldots \quad \ldots \quad \ldots \quad (18).$$

Thus while the absolute strength m' is constant, the apparent strength m will vary unless the cross section, and so the density, remains constant. The relation between the rates of variation of the apparent strength and the cross section, or the density, is by (18)

$$\frac{\delta m}{\delta t}=-\omega\frac{\delta\sigma}{\delta t}=\frac{\omega\sigma}{\rho}\frac{\delta\rho}{\delta t} \quad \ldots \quad \ldots \quad \ldots \quad (19).$$

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For brevity we may term a vortex a *cyclone* or an *anticyclone* according as its vorticity is of the same or opposite sign to ω. We see from (19) that the apparent strength of a cyclone increases or diminishes according as its density is increasing or diminishing. The reverse is the case with the numerical value of the strength of an anticyclone.

From (19) it follows as a particular case than an element of fluid originally devoid of apparent vorticity cannot alter in density without developing vorticity. The vorticity developed will be cyclonic or anticyclonic according as the density increases or diminishes. It must be borne in mind that the density here referred to is that of a particular element of the fluid, and not the density found at a point fixed relative to the rotating axes.

The remarks made in my previous papers on the case of a fluid bounded by an infinite plane, and acted on by forces at right angles to, and functions only of the distance from, that plane, apply equally to the case of a rotating fluid, provided the infinite plane be perpendicular to the axis of rotation. The remarks on the applicability of our formulæ, in cases where ρ and ζ vary over the cross section of a vortex, require no modification.

From (15″) we see that the formulæ for the velocity components relative to the moving axes, due to the variation of density in the rotating fluid, must be the same as those for the velocity components in the ordinary case of fixed axes and a non-rotating fluid. Thus the velocity components in the case of a rotating fluid and axes are given by the formulæ (4_a) and (5_a), viz.:—

$$u = -\frac{my}{\pi r^2} + \frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\frac{x}{r^2}, \qquad v = \frac{mx}{\pi r^2} + \frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\frac{y}{r^2},$$

remembering that m is no longer a constant, but equal to $m' - \omega\sigma$ where m' is a constant.

If ϕ be an angle measured from the plane of xz, the stream lines due to the vortex are given by the differential equation

$$\frac{1}{r^2}\frac{dr^2}{dt} = \frac{1}{m' - \omega\sigma}\frac{\delta\sigma}{\delta t} \times \frac{d}{dt}(\tan^{-1}y/x) \quad \ldots \quad \ldots \quad (20).$$

This admits of immediate integration if

$$\frac{1}{m' - \omega\sigma}\frac{\delta\sigma}{\delta t} = \nu, \text{ (a constant)} \quad \ldots \quad \ldots \quad (21);$$

or

$$m' - \omega\sigma = (m' - \omega_0\sigma)e^{-\omega\nu t} \ldots \quad \ldots \quad \ldots \quad (22)$$

where ${}_0\sigma$ is the value of σ when $t = 0$.

In this case the integral of (20) is

$$r = ae^{\nu\phi/2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

where a is the value of r when $\phi = 0$.

When ν is very small we observe that so long as t is not very large, (22) is approximately of the form

$$\sigma = {}_0\sigma + \sigma' t + \sigma'' t^2 + \quad \ldots \quad \ldots \quad \ldots \quad (24),$$

where σ', σ'', &c., are constants of which σ'' is small compared to σ'. Thus when considering what happens within a comparatively short interval of a given instant, we might in such a case neglect all but the first two terms.

For the action on each other of two vortices, distinguished by the suffixes 1 and 2, we have equations $(7_a) - (10_a)$, with m representing apparent vorticity. Whence for the distance of the vortices at time t, and the inclination of the line joining their centres to the axis of x, we get

$$r^2 = a^2 + \frac{1}{\pi}(\sigma_1 + \sigma_2 - {}_0\sigma_1 - {}_0\sigma_2) \quad \ldots \quad \ldots \quad \ldots \quad (25),$$

$$\phi = \frac{1}{\pi}\int_0^t \frac{m_1' + m_2' - \omega(\sigma_1 + \sigma_2)}{a^2 + \frac{1}{\pi}(\sigma_1 + \sigma_2 - {}_0\sigma_1 - {}_0\sigma_2)} dt \quad \ldots \quad (26).$$

If the variations of the cross sections be given by equations of the type (24) with $\sigma'' = 0$, we find

$$r^2 = a^2 + \frac{1}{\pi}(\sigma_1' + \sigma_2')t \quad \ldots \quad \ldots \quad \ldots \quad (27),$$

$$\phi = \frac{m_1' + m_2' - \omega({}_0\sigma_1 + {}_0\sigma_2) + \omega\pi a^2}{\sigma_1' + \sigma_2'} \log\left\{1 + \frac{(\sigma_1' + \sigma_2')t}{\pi a^2}\right\} - \omega t \quad (28),$$

$$= \frac{{}_0m_1 + {}_0m_2 + \omega\pi a^2}{\sigma_1' + \sigma_2'} \log \frac{r^2}{a^2} - \frac{\omega\pi(r^2 - a^2)}{\sigma_1' + \sigma_2'} \quad \ldots \quad \ldots \quad (28'),$$

where as usual the suffix 0 denotes values at the instant $t = 0$.

If we suppose $(\sigma_1' + \sigma_2')/\pi a^2$ to be very small, then, so long as t is not very large, we have: as a first approximation

$$\phi = ({}_0m_1 + {}_0m_2)t/\pi a^2 = \frac{{}_0m_1 + {}_0m_2}{\sigma_1' + \sigma_2'}\left(\frac{r^2}{a^2} - 1\right) \quad \ldots \quad \ldots \quad (28''),$$

and as a second approximation

$$\phi = ({}_0m_1 + {}_0m_2)t/\pi a^2 - ({}_0m_1 + {}_0m_2 + \omega\pi a^2)(\sigma_1' + \sigma_2')t^2/2\pi^2 a^4 \qquad (28''').$$

So long as terms in t^3 are neglected, the result obtained from the more general form (24), retaining σ'', would not differ from (28''').

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In cases where (28‴) applies, we see that the angular velocity of the line joining the vortices increases or diminishes as t increases, according as $(\sigma_1' + \sigma_2')\{1 + \omega\pi a^2/({}_0m_1 + {}_0m_2)\}$ is negative or positive. If ${}_0m_1 + {}_0m_2$ have the same sign as ω, the angular velocity thus increases when the sum of the cross sections diminishes, and conversely. The same law holds good when ${}_0m_1 + {}_0m_2$ is of opposite sign to ω and numerically greater than $\omega\pi a^2$; but it must be reversed when ${}_0m_1 + {}_0m_2$ is numerically less than $\omega\pi a^2$.

Returning to the general equations $(7_a) - (10_a)$, and putting

$$\left.\begin{aligned} m_1'x_1 + m_2'x_2 &= (m_1' + m_2')X \\ m_1'y_1 + m_2'y_2 &= (m_1' + m_2')Y \end{aligned}\right\} \quad \dots \quad \dots \quad (29),$$

we find, precisely, as we found (19a),

$$(m_1' + m_2')\frac{\delta X}{\delta t} = \omega(m_2'\sigma_1 - m_1'\sigma_2)\frac{y_2 - y_1}{\pi r^2} + \frac{x_2 - x_1}{2\pi r^2}\left(m_2'\frac{\delta\sigma_1}{\delta t} - m_1'\frac{\delta\sigma_2}{\delta t}\right) \dots (30),$$

$$(m_1' + m_2')\frac{\delta Y}{\delta t} = -\omega(m_2'\sigma_1 - m_1'\sigma_2)\frac{x_2 - x_1}{\pi r^2} + \frac{y_2 - y_1}{2\pi r^2}\left(m_2'\frac{\delta\sigma_1}{\delta t} - m_1'\frac{\delta\sigma_2}{\delta t}\right) (31).$$

Whether σ_1 and σ_2 vary or not, it follows that X and Y are constants provided $m_2'/\sigma_2 = m_1'/\sigma_1$.

If ζ_1, ζ_2 be the apparent mean vorticities, this leads to $\zeta_2 = \zeta_1$. Thus two vortices which have at every instant the same mean vorticity or "concentration," and whose vorticities are in the same direction—whether coinciding or not with that of ω—have a fixed point answering to a centre of gravity about which they describe similar orbits. The distances of the vortices from this centre are inversely as their strengths real or apparent. This result, combined with the values already found for r and ϕ, determines in this case the motion of the vortices. If ω and the rates of variation of σ_1 and σ_2 be small, X and Y remain nearly constant, and approximate values for them can be easily deduced from (30) and (31). A general solution of these equations seems scarcely likely to be obtainable.

The case where the apparent strengths of the vortices are equal but opposite in sign presents certain peculiarities.

In every case we have

$$m_1 + m_2 = m_1' + m_2' - \omega(\sigma_1 + \sigma_2).$$

Now m_1', m_2' are constant, and so when $m_1 + m_2$ is zero we must have

$$\sigma_1 + \sigma_2 = \text{constant} \dots \quad \dots \quad \dots \quad (32).$$

Supposing, initially, $x_2 - b = -(x_1 - b) = a/2$, $y_2 = y_1 = 0$, where a and b are constants, we get in this case

$$\left.\begin{aligned} x_2 - b &= \frac{1}{2}a + \frac{1}{2\pi a}(\sigma_1 - {}_0\sigma_1) \\ x_1 - b &= -\frac{1}{2}a + \frac{1}{2\pi a}(\sigma_1 - {}_0\sigma_1) \\ y_2 = y_1 &= \frac{1}{\pi a}m_1't - \frac{\omega}{\pi a}\int_0^t \sigma_1 dt \end{aligned}\right\} \quad \ldots \quad \ldots \quad (33).$$

The vortices thus remain a constant distance apart, and the line joining them retains a fixed direction relative to the rotating axes.

If the fluid be incompressible, or more generally if the density remain constant, this will solve the case of a vortex in presence of an infinite plane boundary parallel to the axis of rotation and rotating with the fluid. For the velocity is everywhere tangential to the plane $x = b$, and this may accordingly be supposed to become a fixed boundary.

If, however, the density of the vortex vary, the velocity in the case above would not be wholly tangential to the plane $x = b$, unless we had $\frac{\delta\sigma_1}{\delta t} = \frac{\delta\sigma_2}{\delta t}$.

This is obviously reconcileable with (32) only when σ_1 and σ_2 both remain constant.

The case of a vortex of varying density in a rotating fluid, in presence of an infinite wall, might thus, at first sight, seem insoluble by the method of images. For if we suppose m_1 to be the real vortex, its image m_2 would have to satisfy the condition $\frac{\delta\sigma_2}{\delta t} = \frac{\delta\sigma_1}{\delta t}$ to ensure the velocity being tangential to the wall, and simultaneously the condition $\frac{\delta\sigma_2}{\delta t} = -\frac{\delta\sigma_1}{\delta t}$ in order to retain a constant absolute strength. This last condition, however, though necessary in a vortex of real fluid, is in no way necessary in the case of a vortex image which has no material existence, and is merely a mathematical device for obtaining velocities which at every instant satisfy the equations in the interior and at the surface of a fluid. When the image does not satisfy all the conditions that a true material existence would demand, it might be as well to term it an *instantaneous* image, as merely satisfying the mathematical conditions at the instant considered.

The instantaneous image of a vortex of section σ and apparent strength m in presence of an infinite plane $x=b$, has a section σ and apparent strength $-m$; and the velocity of the vortex at any instant is thus given by

$$\frac{dx}{dt}=-\frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\frac{1}{2(b-x)},$$

$$\frac{dy}{dt}=\frac{m}{\pi}\frac{1}{2(b-x)}.$$

Supposing $x=b-a$ and $\sigma={}_0\sigma$ when $t=0$, we find

$$\left.\begin{aligned} x&=b-\sqrt{a^2+\frac{1}{2\pi}(\sigma-{}_0\sigma)}\\ y&=\frac{1}{2\pi}\int_0^t\frac{(m'-\omega\sigma)dt}{\sqrt{a^2+\frac{1}{2\pi}(\sigma-{}_0\sigma)}}\end{aligned}\right\}\quad\ldots\quad\ldots\quad(34).$$

The motion of the vortex is thus completely determined if the law of variation of σ be known.

The case of a straight vortex in a fluid bounded by a right circular cylinder, whose axis is parallel to the vortex, and coincides with the axis of rotation if the fluid be rotating, can also be solved by the method of images. Let the radius of the cylinder be c, and at time t let m and σ denote the apparent strength and cross section of the vortex, and r and ϕ the polar co-ordinates of the vortex centre referred to the axis of rotation, and an initial plane moving with the fluid. It is easily proved that the instantaneous image consists of a vortex, whose apparent strength is $-m$ and rate of variation of cross section $\frac{\delta\sigma}{\delta t}$, situated at the point $(c^2/r,\ \phi)$, and of a vortex of apparent strength 0 and rate of variation of cross section $-\frac{\delta\sigma}{\delta t}$ coinciding with the axis of rotation.

We thus get for the motion of the vortex the two equations

$$\frac{\delta r}{\delta t}=-\frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\left(\frac{1}{r}+\frac{r}{c^2-r^2}\right)\quad\ldots\quad\ldots\quad\ldots\quad(35),$$

$$r\frac{\delta\phi}{\delta t}=\frac{mr}{\pi(c^2-r^2)}\quad\ldots\quad\ldots\quad\ldots\quad(36).$$

Supposing that when $t=0$, $r=r_0$ and $\sigma={}_0\sigma$, we find

$$r^2=c^2-\sqrt{(c^2-r_0^{\,2})^2+2c^2(\sigma-{}_0\sigma)/\pi}\quad\ldots\quad\ldots\quad(37),$$

$$\phi=\int_0\frac{mdt}{\sqrt{\pi^2(c^2-r_0^{\,2})^2+2\pi c^2(\sigma-{}_0\sigma)}}\quad\ldots\quad\ldots\quad(38).$$

When c/r_0 is very large, approximate values are

$$\left.\begin{aligned} r^2 &= r_0^2 - (\sigma - {}_0\sigma)/\pi \\ \frac{\delta\phi}{\delta t} &= \frac{m}{\pi c^2}\left(1 + \frac{r^2}{c^2}\right) \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (39).$$

The vortex will thus approach to or recede from the axis of rotation according as its density is diminishing or increasing; and it will revolve about the axis of rotation in the same direction as the moving axes or in the opposite, according as it is a cyclone or an anticyclone. This rotation, it will be observed, takes place relative to the moving axes, and is in addition to the common angular velocity ω; for a vortex of given strength it will be greater the greater the distance of the vortex from the axis of rotation.

The question next arises as to how the density of the vortex compares with the density which would be found in the fluid at the same distance from the axis of rotation in the absence of all apparent vorticity.

Referring to (11) and (13) we see that when the fluid is undisturbed by the existence of vortices, or of external forces causing motion relative to the moving axes, we have, since $u = 0 = v$,

$$\frac{1}{\rho'}\frac{dp}{dx} = \omega^2 x, \quad \frac{1}{\rho'}\frac{dp}{dy} = \omega^2 y,$$

where ρ' denotes the density in the undisturbed state.

Supposing a constant force $Z = -g$ parallel to the axis of rotation, we also have

$$\frac{1}{\rho'}\frac{dp}{dz} = -g.$$

If the relation between the pressure and density be $p = k\rho$ where k is constant, we thence obtain at a distance r from the axis of rotation

$$\log(\rho'/\rho'_0) = -gz/k + \tfrac{1}{2}\omega^2 r^2/k \quad \ldots \quad \ldots \quad \ldots \quad (40),$$

where ρ_0' is the density where the axis of rotation cuts the plane of xy. The density thus increases with the distance from the axis of rotation, and to a body moving along a radius vector the relation between the rate of variation in the density of the surrounding fluid and the velocity of the body is given by

$$\frac{1}{\rho'}\frac{\delta\rho'}{\delta t} = \frac{\omega^2}{k} r \frac{\delta r}{\delta t} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (41).$$

Referring to (35) we see that the rate of change of density in a single vortex in a cylinder of radius c and its radial velocity are connected by the relation

$$\frac{1}{\rho}\frac{\delta\rho}{\delta t} = \frac{2\pi}{\sigma}(1 - r^2/c^2) r \frac{\delta r}{\delta t} \quad \ldots \quad \ldots \quad \ldots \quad (42).$$

In any given case, by comparing (41) and (42), we get the information desired.

Suppose, for example, the fluid atmospheric air, ω the earth's angular velocity, and r/c small. Then, if the imaginary body of equation (41) and the vortex be at the same distance from the axis, and possessed of the same velocity, we find approximately

$$\frac{\delta\rho}{\delta t}\Big/\frac{\delta\rho'}{\delta t}=\frac{2k\rho}{\omega^2e^2\rho'},$$

where e is the radius of the vortex, supposed circular.

If R denote the earth's radius, and H the height of the homogeneous atmosphere, this leads to

$$\frac{\delta\rho}{\delta t}\Big/\frac{\delta\rho'}{\delta t}=\frac{578\text{RH}}{e^2}\,\frac{\rho}{\rho'},\quad \text{approximately.}$$

Thus $\dfrac{\delta\rho}{\delta t}>\text{or}<\dfrac{\delta\rho'}{\delta t}$, according as $e<\text{or}>\sqrt{578\text{RH}\rho/\rho'}$.

The critical value of e would be over 3000 miles unless the difference between ρ and ρ' exceeded the differences occuring in the earth's atmosphere.

Our formulæ would cease to be at all accurate if e/c ceased to be small, or if e became comparable with the distance of the vortex from the cylindrical boundary. So in the case considered, when our formulae are applicable $\dfrac{\delta\rho}{\delta t}$ would be certainly greater than $\dfrac{\delta\rho'}{\delta t}$ unless c were very much greater than R.

In the case of the earth's atmosphere there would appear a fair probability that the motion of a vortex in comparatively high latitudes would resemble, in its general features, the motion we have found for a vortex in a cylinder. The radius of the cylinder would be of the same order of magnitude as an earth's quadrant. The two main reasons for this statement are, 1°, that as the earth's atmosphere is presumably not being thrown off, the motion at every point on its limiting surface must be very approximately at right angles to the perpendicular on the axis of rotation, and, 2°, that the winds in the northern and southern hemispheres are on the whole independent of one another. The conclusions so deduced should certainly answer better than those derived from the hypothesis of a fluid extending to infinity in all directions at right angles to the axis of rotation. In this latter case we should have finite radial velocities at all distances from the axis of rotation, and also at great distances from this axis an atmosphere of enormous density.

The conclusions derived from the motion of a single vortex in a cylinder would lead to the following laws for the motion of a solitary *cyclone or anti-cyclone in the earth's northern hemisphere in high* latitudes.

1^0. Motion in longitude:—

(α). The vortex moves from west to east if a cyclone, and from east to west if an anti-cyclone.

(β). The velocity in longitude is proportional to the (apparent) strength of the vortex, and for a vortex of given strength is nearly proportional to the polar distance. The rate of variation of its degrees of longitude is thus nearly uniform.

2^0. Motion in latitude:—

(γ). The vortex approaches or recedes from the pole according as its density is diminishing or increasing; or, what is the same thing, according as its area (cross section) is increasing or diminishing.

(δ). The velocity in latitude is in general greater the smaller the distance from the pole.

(ϵ). A cyclone is diminishing in strength or increasing according as it is approaching the pole or receding from it. Exactly the opposite holds of an anti-cyclone.

(ζ). If a vortex approach the pole its density will fall more rapidly than that of the surrounding air, and if it be receding from the pole its density will rise faster than that of the surrounding air.

3^0, combining 1^0 and 2^0. The direction of motion of a vortex is more nearly due north and south, the closer the vortex is to the pole.

Of course, in general, we do not find a solitary vortex, but a series of them scattered over the hemisphere at once, and their mutual action must be considered in determining their motion. Thus, if there were n vortices in a circular cylinder, we must have n images at the n inverse points outside the cylinder, and at the centre a composite image, the rate of variation of whose cross section, with its sign changed, equals the algebraic sum of the rates of variation of the cross sections of the n real vortices. The components of the velocity at any point due to such a system can be at once obtained from the principles laid down.

Before leaving the subject, I would remark that our theory asserts that a cyclone could travel from east to west only if a strong anti-cyclone were to the north of it, or a second cyclone to the south of it.

On the expression of a symmetric function in terms of the elementary symmetric functions.

By R. E. Allardice, M.A.

The theorem that any rational symmetric function of n variables $x_1, x_2, \ldots x_n$ is expressible as a rational function of the n elementary symmetric functions, Σx_1, $\Sigma x_1 x_2$, $\Sigma x_1 x_2 x_3$, etc., is usually proved by means of the properties of the roots of an equation. It is obvious, however, that the theorem has no necessary connection with the properties of equations; and the object of this paper is to give an elementary proof of the theorem, based solely on the definition of a symmetric function.

It is obvious that only integral symmetric functions need be considered.

Let ${}_np_1, {}_np_2, {}_np_3, \ldots$ stand for $\Sigma x_1, \Sigma x_1 x_2, \Sigma x_1 x_2 x_3 \ldots$, when there are n variables. If x_n vanishes, ${}_np_1, {}_np_2, {}_np_3, \ldots$ evidently become ${}_{n-1}p_1, {}_{n-1}p_2, {}_{n-1}p_3, \ldots$

Now assume that all integral symmetric functions involving not more than $(n-1)$ variables can be expressed rationally in terms of the elementary symmetric functions.

Let $f(x_1, x_2, \ldots x_n)$ be any integral symmetric function of n variables. Then $f(x_1, x_2, \ldots x_{n-1}, 0)$ is a symmetric function of $(n-1)$ variables, and, by supposition, may be expressed in terms of ${}_{n-1}p_1, {}_{n-1}p_2 \ldots$. Let its expression be $\phi({}_{n-1}p_1, {}_{n-1}p_2, \ldots {}_{n-1}p_{n-1})$.

Assume now

$$f(x_1, x_2, \ldots x_n) = \phi({}_np_1, {}_np_2 \ldots {}_np_{n-1}) + \psi(x_1, x_2, \ldots x_n),$$

where ψ is obviously a symmetric function.

Put $x_n = 0$, on both sides of this identity; then

$$f(x_1, x_2, \ldots x_{n-1}, 0) = \phi({}_{n-1}p_1, {}_{n-1}p_2, \ldots {}_{n-1}p_{n-1}) + \psi(x_1, x_2, \ldots x_{n-1}, 0);$$

and hence $\psi(x_1, x_2, \ldots x_{n-1}, 0) = 0,$

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