M. Hanamura and M. Yoshida
Nagoya Math. J.
Vol. 154 (1999), 123–139

HODGE STRUCTURE ON TWISTED COHOMOLOGIES AND TWISTED RIEMANN INEQUALITIES I

MASAKI HANAMURA AND MASAAKI YOSHIDA

Dedicated to Professor Kazuhiko Aomoto on the occasion of his sixtieth birthday

Abstract. We show the twisted cohomology on $\mathbb{P}^1$ has a natural polarized Hodge structure and hence derive the analogues of Riemann's equality and inequailty.

§0. Introduction

Hypergeometric-type integrals, for example the Euler integral representation of the hypergeometric function

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^a(1-t)^{c-a}(1-tx)^{-b} \frac{dt}{t(1-t)},$$

can be considered as the dual pairing between twisted homologies and twisted cohomologies. Various vanishing theorems, structure theorems and intersection theories are established (cf. [AK], [KY], [CM], [M1], [M2]) for twisted (co)homologies. These theories imply, in particular, a twisted analogues of the Riemann equality for periods of algebraic curves. For example, the quadratic relation (due to Gauss)

$$\begin{aligned} &F(a,b,c;x)F(1-a,1-b,2-c;x) \\ &\quad -F(a+1-c,b+1-c,2-c;x)F(c-a,c-b,c;x) = 0 \end{aligned}$$

for hypergeometric functions can be obtained systematically in this frame work.

In this paper, we study the Hodge structure for twisted cohomologies on curves, from which we derive a twisted analogue of the Riemann *inequality.*

Received November 28, 1997.

Our key tool is Zucker's theorem ([Zuc]) for variations of Hodge structures. Examples are given in the end of the paper (The case of a base of higher dimension will be studied in our paper in preparation.)

Here we would like to explain briefly the meaning of *twisted* theories. Let C be a Riemann surface defined by

$$s^d = \prod_1^n (t - x_j)^{n_j},$$

where d and n_j are natural numbers. Since the covering surface C is uniquely determined by the data $\{x_j, d, n_j\}$ downstairs, every happening upstairs should be described in terms of those downstairs — This is the heart of the twisted theories. For example, though the genus of C can be very high, $H^*(C, Z)$ and $H_*(C, Z)$ can be understood in terms of twisted (co)homologies downstairs, i.e., on the t-space (cf. [CY]). Our twisted Riemann inequality is essentially equivalent to that for the covering Riemann surface.

§1. Twisted (co)homology; complex conjugates

Let $x_1, \dots, x_n$ be distinct points in the complex projective line $\mathbb{P}^1$; for simplicity, we assume none of these points is the point at infinity. For each point x_j, we give a *real* number j, called the exponent at x_j. Throughout this paper, we assume

$$j \notin \mathbb{Z}, \quad 1 + \cdots + n = 0.$$

Consider the multi-valued function

$$u := \prod_{j=1}^n (t - x_j)^j \quad \text{on} \quad U := \mathbb{P}^1 - \{x_1, \dots, x_n\}.$$

Let $\mathcal{L} = \mathcal{L}_u$ be the local system on U determined by the function u; the stalk at $x \in U$ is $\mathbb{C}v$, where v is a branch of u, and the monodromy around x_j is $\exp 2\pi i j$. We are interested in the cohomology groups $H^*(U, \mathcal{L})$ coefficients in the sheaf $\mathcal{L}$. Let us define a connection ∇ by

$$\nabla = \nabla_u := d - d\log u\wedge\,,$$

and fix some (standard) notation:

$\Gamma(U, \mathcal{S})$: Space of sections of a sheaf $\mathcal{S}$ on U,

$\mathcal{E}^p$: Sheaf of p-forms,

$\mathcal{E}^p(\mathcal{S})$: Sheaf of p-forms with coefficients in $\mathcal{S}$.

Then the cohomology groups can be expressed as follows:

$$H^*(U, \mathcal{L}) \cong H^*(\Gamma(U, \mathcal{E}^\bullet(\mathcal{L})), d \otimes 1) \cong H^*(\Gamma(U, \mathcal{E}^\bullet), \nabla).$$

The second isomorphism is given by

$$\varphi \otimes v \longrightarrow \varphi v, \quad \phi v^{-1} \otimes v \longleftarrow \phi,$$

where v is a(ny) branch of the multi-valued function u. Compatibility of the two derivations comes from the identity

$$(d\varphi)v = \nabla(\varphi v), \quad \varphi \in \mathcal{E}.$$

In the intersection theory of twisted cohomology and homology (see [AK], [KY] for details) there appear cohomology groups $H^1(U, \mathcal{L})$ and $H^1(U, \check{\mathcal{L}})$ and homology groups $H_1(U, \mathcal{L})$ and $H_1(U, \check{\mathcal{L}})$. There are four duality parings

$$\begin{array}{ccc} H^1(U, \mathcal{L}) & \longleftrightarrow & H^1(U, \check{\mathcal{L}}) \\ \updownarrow & & \updownarrow \\ H_1(U, \check{\mathcal{L}}) & \longleftrightarrow & H_1(U, \mathcal{L}) \end{array}$$

that are compatible. We will review the definitions of the pairings later.

In this paper we will also consider $\overline{\mathcal{L}}$, the complex conjugate of $\mathcal{L}$, and its cohomology and homology. If α_j's are real, then $\overline{\mathcal{L}} \cong \check{\mathcal{L}}$ canonically, so $H^1(U, \overline{\mathcal{L}}) \cong H^1(U, \check{\mathcal{L}})$ and $H_1(U, \overline{\mathcal{L}}) \cong H_1(U, \check{\mathcal{L}})$. Despite apparent redundancy we will mostly use $\overline{\mathcal{L}}$ instead of $\check{\mathcal{L}}$ for the following reasons. First it will keep track of complex conjugation which is part of Hodge structure. Second the polarization can be expressed as an interesting integral (1.1) in terms of representing forms.

By definition $\overline{\mathcal{L}} = \mathcal{L}_{\overline{u}}$ is the local system determined by $\overline{u}$. If α_j's are real, there is a canonical isomorphism

$$\check{\mathcal{L}} \cong \overline{\mathcal{L}}, \quad \alpha v^{-1} \longmapsto \alpha \overline{v} \quad (\alpha \in \mathbb{C}).$$

The map is well-defined because another branch of u is of the form λv, $|\lambda| = 1$. (That there is an isomorphism of local systems can be seen by looking at the monodromies of $\overline{\mathcal{L}}$ and $\check{\mathcal{L}}$, which are respectively $\overline{e^{2\pi i \alpha_j}}$ and $e^{-2\pi i \alpha_j}$. In the above we have specified an isomorphism.)

This isomorphism can be extended to an isomorphism of twisted de Rham complexes:

Proposition 1.1. *Through the quasi isomorphisms* $\check{\mathcal{L}} \simeq (\mathcal{E}^\bullet, \nabla_{1/u})$ *and* $\overline{\mathcal{L}} \simeq (\mathcal{E}^\bullet, \nabla_{\overline{u}})$ *the isomorphism* $\check{\mathcal{L}} \cong \overline{\mathcal{L}}$ *can be expressed by*

$$(\mathcal{E}^\bullet, \nabla_{1/u}) \ni \phi \longleftrightarrow \phi|u|^2 \in (\mathcal{E}^\bullet, \nabla_{\overline{u}}).$$

Indeed, we have

$$\begin{aligned} d(\phi|u|^2) - d\log\overline{u} \wedge \phi|u|^2 &= d(\phi|u|^2) - \frac{d\overline{u}}{\overline{u}} \wedge \phi|u|^2 \\ &= |u|^2\Big\{d\phi + \frac{du}{u} \wedge \phi\Big\} = |u|^2\Big\{d\phi - d\log\frac{1}{u} \wedge \phi\Big\}. \end{aligned}$$

The non-degenerate pairing

$$\langle\ ,\ \rangle : H^1(U, \mathcal{L}) \otimes H^1(U, \check{\mathcal{L}}) \longrightarrow \mathbb{C}$$

is defined as

$$\langle[\phi], [\psi]\rangle = \int_{\mathbb{P}^1} \phi \wedge \psi$$

where either ϕ or ψ is compactly supported. (Note $H^1_c(U, \mathcal{L}) = H^1(U, \mathcal{L})$; for a proof see Proposition 2.2.) Via the isomorphism $H^1(U, \check{\mathcal{L}}) \to H^1(U, \overline{\mathcal{L}})$ the pairing

$$Q : H^1(U, \mathcal{L}) \otimes H^1(U, \overline{\mathcal{L}}) \longrightarrow \mathbb{C}, \quad Q([\phi], [\psi]) = \int_{\mathbb{P}^1} \frac{\phi \wedge \psi}{|u|^2} \tag{1.1}$$

is obtained. The Q will be part of a polarization of a Hodge structure, see §2.

An element of the homology $H_1(U, \mathcal{L})$ can be represented by

$$\sum_j a_j \rho_j \otimes v_j\,,$$

where $a_j \in \mathbb{C}$, ρ_j are smooth paths in U and v_j are branches of u defined on ρ_j. Similarly elements of $H_1(U, \overline{\mathcal{L}})$ has representatives $\sum_j a_j \rho_j \otimes \overline{v_j}$.

There is a conjugate linear isomorphism (the complex conjugation)

$$\overline{} : H_1(U, \mathcal{L}) \longrightarrow H_1(U, \overline{\mathcal{L}})$$

given by

$$\overline{\Big[\sum_j a_j \rho_j \otimes v_j\Big]} = \Big[\sum_j \overline{a_j} \rho_j \otimes \overline{v_j}\Big]$$

(complex conjugation acting just on coefficients and v_j's; ρ_j *not* to be replaced by $\overline{\rho}_j$). There is also a $\mathbb{C}$-linear isomorphism induced from $\check{\mathcal{L}} \cong \overline{\mathcal{L}}$:

$$\begin{aligned} H_1(U, \overline{\mathcal{L}}) &\longrightarrow H_1(U, \check{\mathcal{L}}) \\ \Big[\sum_j a_j \rho_j \otimes \overline{v}_j\Big] &\longmapsto \Big[\sum_j a_j \rho_j \otimes v_j^{-1}\Big]. \end{aligned}$$

The non-degenerate pairing $H_1(U, \mathcal{L}) \otimes H_1(U, \check{\mathcal{L}}) \to \mathbb{C}$ is defined as follows:

$$\Big[\sum a_j \rho_j \otimes v_j\Big] \cdot \Big[\sum b_k \sigma_j \otimes {v'}_k^{-1}\Big] = \sum_{j,k} a_j b_k \sum_{p \in \rho_j \cap \sigma_k} I_p(\rho_j, \sigma_k) v_j(p) {v'}_k^{-1}(p).$$

Here the intersections of ρ_j and σ_k are transversal and $I_p(\rho_j, \sigma_k)$ is the topological intersection number. Via the isomorphism $H_1(U, \overline{\mathcal{L}}) \to H_1(U, \check{\mathcal{L}})$ induced is the pairing $H_1(U, \mathcal{L}) \otimes H_1(U, \overline{\mathcal{L}}) \longrightarrow \mathbb{C}$,

$$\Big[\sum a_j \rho_j \otimes v_j\Big] \cdot \Big[\sum b_k \sigma_j \otimes \overline{v'_k}\Big] = \sum_{j,k} a_j b_k \sum_{p \in \rho_j \cap \sigma_k} I_p(\rho_j, \sigma_k) v_j(p) {v'}_k^{-1}(p).$$

Similarly there is $H_1(U, \check{\mathcal{L}}) \otimes H_1(U, \overline{\check{\mathcal{L}}}) \to \mathbb{C}$.

We recall that a basis of $H_1(U, \mathcal{L})$ is given as follows. Assume just for simplicity $x_1, \ldots, x_n$ are all real and $x_1 < x_2 < \cdots < x_n$. Let $\rho_j = \overrightarrow{x_j\, x_{j+1}}$ and v a branch of u defined on the lower half plane. Then

$$\gamma_j = \rho_j \otimes v \in H_1(U, \mathcal{L}) \quad (j = 1, \ldots, n-2)$$

is for example a basis. Similarly one has a basis

$$\check{\gamma}_j = \rho_j \otimes v^{-1} \in H_1(U, \check{\mathcal{L}})$$

Taking complex conjugates one obtains bases

$$\begin{aligned} \overline{\gamma}_j &= \rho_j \otimes \overline{v} \in H_1(U, \overline{\mathcal{L}}); \\ \overline{\check{\gamma}}_j &= \rho_j \otimes \overline{v}^{-1} \in H_1(U, \overline{\check{\mathcal{L}}}). \end{aligned}$$

Under the isomorphism $H_1(U, \mathcal{L}) \to H_1(U, \overline{\check{\mathcal{L}}})$ there correspond γ_j and $\overline{\check{\gamma}}_j$ so

(1.2) $$\check{\gamma}_i \cdot \overline{\check{\gamma}}_j = \check{\gamma}_i \cdot \gamma_j\,.$$

https://doi.org/10.1017/S0027763000025344 Published online by Cambridge University Press

Similarly $\gamma_i \cdot \overline{\gamma}_j = \gamma_i \cdot \check{\gamma}_j$.

The pairing between cohomology and homology is the map

$$\begin{aligned} H^1(U, \mathcal{L}) \otimes H_1(U, \check{\mathcal{L}}) &\longrightarrow \mathbb{C}\,, \\ [\phi] \otimes [\Delta \otimes v^{-1}] &\longmapsto \int_{\Delta \otimes v^{-1}} \phi = \int_{\Delta} v^{-1}\phi \end{aligned}$$

(either ϕ or Δ is compactly supported). Similarly one may define

$$\begin{aligned} H^1(U, \overline{\mathcal{L}}) \otimes H_1(U, \check{\overline{\mathcal{L}}}) &\longrightarrow \mathbb{C}\,, \\ [\phi] \otimes [\Delta \otimes \overline{v}^{-1}] &\longmapsto \int_{\Delta \otimes \overline{v}^{-1}} \phi = \int_{\Delta} \overline{v}^{-1}\phi\,. \end{aligned}$$

From the definitions one has for $[\phi] \in H^1(U, \mathcal{L})$ and $\gamma' \in H_1(U, \check{\mathcal{L}})$

$$\overline{\int_{\gamma'} \phi} = \int_{\overline{\gamma'}} \overline{\phi}\,. \tag{1.3}$$

Remark. If $\alpha_j \notin \mathbb{R}$, the local systems $\check{\mathcal{L}}$ and $\overline{\mathcal{L}}$ are unrelated, and the integral (1.1) is not defined ($|u|^2$ is multi-valued). This paper will have nothing to say in that case. (The period relation in the first half in §5 is the only exception.)

§2. Hodge structure on twisted cohomologies

We recall some basic definitions from Hodge theory. See [G] for more information.

Recall that for a projective complex algebraic variety X its cohomology $H^m(X, \mathbb{C})$ decomposes as $\bigoplus_{p+q=m} H^{p,q}$ where $H^{p,q}$ is the subspace represented by harmonic (p,q)-forms. In other words, $H^m(X, \mathbb{C})$ has $\mathbb{R}$-Hodge structure (in fact $\mathbb{Z}$-Hodge structure). If $L \in H^2(X, \mathbb{C})$ is the class of a hyperplane section divisor and $d = \dim X$ then the kernel of the map $L^{d-m+1} : H^m(X, \mathbb{C}) \to H^{2d-m+2}(X, \mathbb{C})$ is by definition the primitive cohomology $H^m_{\mathrm{prim}}(X, \mathbb{C})$. It is not only Hodge structure but also a polarized one.

A *polarized* $\mathbb{R}$*-Hodge structure* of weight m is a finite dimensional $\mathbb{C}$-vector space H together with

(1) an $\mathbb{R}$-structure on H, i.e., an $\mathbb{R}$-subspace $H_{\mathbb{R}} \subset H$ such that $H_{\mathbb{R}} \otimes \mathbb{C} = H$ (so there is complex conjugation on H),

(2) a direct sum decomposition $H = \bigoplus_{p+q=m,\, p,q \geq 0} H^{p,q}$ such that $\overline{H^{p,q}} = H^{q,p}$, and

(3) a $(-1)^m$-symmetric $\mathbb{C}$-bilinear pairing (called the polarization) $Q : H \otimes H \to \mathbb{C}$ defined over $\mathbb{R}$ (i.e., $Q(\overline{x}, \overline{y}) = \overline{Q(x,y)}$) satisfying (called the Riemann-Hodge bilinear relations)

(a) $Q(H^{p,q}, H^{p',q'}) = 0$ unless $p + p' = q + q' = m$, and

(b) for any non-zero $x \in H^{p,q}$,

$$i^{p-q} Q(x, \overline{x}) > 0\,.$$

In the example $H = H^m_{\mathrm{prim}}(X, \mathbb{C})$, the polarization is given by

$$Q(x,y) = (-1)^{m(m-1)/2} x \cdot y \cdot L^{d-m}\,.$$

The following generalization is due to Griffiths.

A *variation of polarized* $\mathbb{R}$*-Hodge structure* of weight m on a smooth complex algebraic variety S is a $\mathbb{C}$-local system of finite rank $\mathcal{M}$ with the data:

(1) An $\mathbb{R}$-local subsystem $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ such that $\mathcal{M}_{\mathbb{R}} \otimes \mathbb{C} = \mathcal{M}$.

(2) A direct sum decomposition of the C^∞-vector bundle $\mathcal{M} \otimes \mathcal{E}$ ($\mathcal{E} = C^\infty$-functions on S) into C^∞-subbundles

$$\mathcal{M} \otimes \mathcal{E} = \bigoplus_{p+q=m} \mathcal{M}^{p,q}$$

satisfying the conditions:

(a) One has $\overline{\mathcal{M}^{p,q}} = \mathcal{M}^{q,p}$.

(b) If $F^p(\mathcal{M} \otimes \mathcal{E}) := \bigoplus_{p' \geq p} \mathcal{M}^{p',q'}$, they are holomorphic subbundles of $\mathcal{M} \otimes \mathcal{E}$.

(c) If $\nabla : \mathcal{M} \otimes_{\mathbb{C}} \mathcal{E} \to \mathcal{M} \otimes_{\mathbb{C}} \mathcal{E}^1$ is the flat connection, then $\nabla(F^p) \subset F^{p-1} \otimes \mathcal{E}^1$ (*the Griffiths transversality*).

(3) a $(-1)^m$-symmetric bilinear pairing $Q : \mathcal{M} \otimes \mathcal{M} \to \mathbb{C}_S$ of local systems defined over $\mathbb{R}$, which satisfies the condition for polarization fiberwise.

We make use of

THEOREM 2.1. ([Zuc]) *Let a* $\mathbb{C}$*-local system* $\mathcal{M}$ *over an algebraic curve* S *is a variation of polarized* $\mathbb{R}$*-Hodge structure of weight* n*, and* $j : S \overset{\to}{\subset} \overline{S}$ *be the completion. Then* $H^1(\overline{S}, j_*\mathcal{M})$ *has a polarized* $\mathbb{R}$*-Hodge structure of weight* $n+1$*. Moreover, each cohomology class has a unique* L_2*-harmonic representative. Here* $\psi \otimes m \in \mathcal{E}(\mathcal{M})$ *is harmonic if and only if* ψ *is holomorphic when* $\psi \in \mathcal{E}^{10}$*, and* ψ *is anti-holomorphic when* $\psi \in \mathcal{E}^{01}$*. The*

https://doi.org/10.1017/S0027763000025344 Published online by Cambridge University Press

polarization on $H^1(\overline{S}, j_*\mathcal{M})$ is the bilinear form induced naturally by the polarization on $\mathcal{M}$ and a Kähler metric on S which is equivalent to Poincaré metric at each point of $\overline{S} - S$.

PROPOSITION 2.2. *Let S be an algebraic curve, $U = S - \{x_1, \ldots, x_N\}$ and $j : U \to S$ the open immersion. If $\mathcal{M}$ be a $\mathbb{C}$-local system of finite rank on U whose monodromy at each x_j has no eigenvalue 1, then we have*

$$H^1_c(U, \mathcal{M}) = H^1(U, \mathcal{M}) = H^1(S, j_*\mathcal{M})\,.$$

Proof. We have $H^1(U, \mathcal{M}) = H^1(S, \mathbb{R}j_*\mathcal{M})$ and $H^1_c(U, \mathcal{M}) = H^1(S, j_!\mathcal{M})$. Since $\mathbb{R}^0 j_*\mathcal{M} = j_*\mathcal{M}$ it suffices to show $\mathbb{R}^q j_*\mathcal{M} = 0$ for $q > 0$ and $j_!\mathcal{M} = j_*\mathcal{M}$.

The question is local. $(R^q j_*\mathcal{M})_{x_j} = H^q(S_1, \mathcal{M})$ where S^1 is a small circle around x_j. So it is zero if $q \geq 2$. If $q = 1$, $H^1(S, \mathcal{M})$ is dual to $H^0(S^1, \mathcal{M}^*)$, which is zero by assumption. Moreover $(j_*\mathcal{M})_{x_j} = H^0(S^1, \mathcal{M})$ $= 0$ so $j_!\mathcal{M} = j_*\mathcal{M}$.

Our key idea of studying the cohomology groups $H^*(U, \mathcal{L})$ is to consider the cohomology groups

$$H^*(U, \mathcal{L} \oplus \overline{\mathcal{L}}) \quad (= H^*(U, \mathcal{L}) \oplus H^*(U, \overline{\mathcal{L}})).$$

Our goal is

THEOREM 2.3. *$H^1(U, \mathcal{L} \oplus \overline{\mathcal{L}})$ has a polarized $\mathbb{R}$-Hodge structure of weight 1. The Hodge decomposition is compatible with the decomposition $H^1(U, \mathcal{L}) \oplus H^1(U, \overline{\mathcal{L}})$, namely*

$$H^1(U, \mathcal{L}) = H^{10} \cap H^1(U, \mathcal{L}) \oplus H^{01} \cap H^1(U, \mathcal{L})\,,$$

and similarly for $H^1(U, \overline{\mathcal{L}})$. The polarization is the skew-symmetric bilinear form

$$Q : (H^1(U, \mathcal{L}) \oplus H^1(U, \overline{\mathcal{L}})) \otimes (H^1(U, \mathcal{L}) \oplus H^1(U, \overline{\mathcal{L}})) \longrightarrow \mathbb{C},$$

where

$$H^1(U, \mathcal{L}) \otimes H^1(U, \mathcal{L}) \longrightarrow 0, \quad H^1(U, \overline{\mathcal{L}}) \otimes H^1(U, \overline{\mathcal{L}}) \longrightarrow 0$$

and $H^1(U,\mathcal{L})\otimes H^1(U,\overline{\mathcal{L}})\to\mathbb{C}$ *is the pairing*

$$\begin{aligned} H^1(U,\mathcal{L})\otimes H^1(U,\overline{\mathcal{L}}) &\longrightarrow \mathbb{C} \\ (\psi,\phi) &\longmapsto i\int_{\mathbb{P}^1}\frac{\psi\wedge\phi}{|u|^2}, \end{aligned}$$

where ψ, ϕ *are certain representatives for which the integral converges.* (*Via the isomorphism* $\overline{\mathcal{L}}$ *it is compatible with the pairing* $H^1(U,\mathcal{L})\otimes H^1(U,\check{\mathcal{L}})\to\mathbb{C}$ *up to the factor of* i.)

We apply Theorem 2.1 for $S=U,\overline{S}=\mathbb{P}^1$ and $\mathcal{M}=\mathcal{L}\oplus\overline{\mathcal{L}}$. Let us briefly see that $\mathcal{L}\oplus\overline{\mathcal{L}}$ is a polarized variation of $\mathbb{R}$-hodge structure of weight 0:

1) Complex conjugation $\mathcal{L}\to\overline{\mathcal{L}}$ is given by $v\mapsto\overline{v}$. The real structure $\mathcal{M}_{\mathbb{R}}$ is the fixed part of $\mathcal{M}$ under the complex conjugation. This is a 2-dimensional $\mathbb{R}$-local system; indeed, its local monodromy around the point x_j is given by

$$\begin{pmatrix} v+\overline{v} \\ i(v-\overline{v}) \end{pmatrix} \looparrowright \begin{pmatrix} \cos 2\pi\alpha_j & \sin 2\pi\alpha_j \\ -\sin 2\pi\alpha_j & \cos 2\pi\alpha_j \end{pmatrix}\begin{pmatrix} v+\overline{v} \\ i(v-\overline{v}) \end{pmatrix}.$$

2) Let us define a $\mathbb{C}$-bilinear form

$$Q:(\mathcal{L}\oplus\overline{\mathcal{L}})\otimes(\mathcal{L}\oplus\overline{\mathcal{L}})\longrightarrow\mathbb{C}_U$$

by

$$(\alpha v+\beta\overline{v})\otimes(\gamma v+\delta\overline{v})\longmapsto\alpha\delta+\beta\gamma,\quad \alpha,\beta,\gamma,\delta\in\mathbb{C}.$$

This is induced by the natural pairing $\mathcal{L}\otimes\check{\mathcal{L}}\to\mathbb{C}$ through the isomorphism $\overline{\mathcal{L}}\cong\check{\mathcal{L}}$ above. One can readily check that this is well-defined (i.e., single-valued), defined over $\mathbb{R}$ (i.e., $Q(\overline{x},\overline{y})=\overline{Q(x,y)}$), and $Q(x,\overline{x})>0, x\neq 0$.

The other conditions for $\mathcal{M}$ to be a polarized variation of Hodge structure are just trivial. Thus Theorem 2.1 tells us that $H^1(\mathbb{P}^1,j_*(\mathcal{L}\oplus\overline{\mathcal{L}}))$ has a polarized Hodge structure of weight 1, where $j:U\to\mathbb{P}^1$ is the inclusion.

Under the condition $\alpha_j\notin\mathbb{Z}$, by Proposition 2.2 we have the isomorphism

$$H^1(\mathbb{P}^1,j_*(\mathcal{L}\oplus\overline{\mathcal{L}}))=H^1(U,\mathcal{L}\oplus\overline{\mathcal{L}}).$$

We verify the last statement about the polarization. Suppose F is a Hermitian vector bundle over a Riemann surface, ϕ and ψ are 1-forms with values in F and a Kähler matric on the base is given. Let $\{u_1, \ldots, u_r\}$ be a local frame of F and write

$$\phi = \sum \phi_i \otimes u_i \,,$$

$$\psi = \sum \psi_i \otimes u_i \,,$$

where ϕ_i, ψ_i are 1-forms. Then the bilinear pairing is given by

$$(\phi, \psi) = i \sum_{i,j} \int \phi_i \wedge \psi_j (u_i, u_j)$$

where (u_i, u_j) is the bilinear form giving the Hermitian metric. In particular it is independent of the choice of a Kähler matric. From this it is clear that the paring among $H^1(U, \mathcal{L})$'s and $H^1(U, \overline{\mathcal{L}})$'s are zero. If $\psi \in H^1(U, \mathcal{L})$ and $\phi \in H^1(U, \overline{\mathcal{L}})$, then writing them locally

$$\psi = \psi v^{-1} \otimes v, \quad \phi = \phi \overline{v}^{-1} \otimes \overline{v},$$

since $Q(v, \overline{v}) = 1$, one has

$$Q(\psi, \phi) = i \int \psi \wedge \phi |v|^{-2} \,.$$

§3. L_2-harmonic basis

Let

$$\Big(H^1(U, \mathcal{L}) \oplus H^1(U, \overline{\mathcal{L}}) = \Big) \; H^1(U, \mathcal{L} \oplus \overline{\mathcal{L}}) = H^{10} \oplus H^{01}$$

be the Hodge decomposition of the cohomology group in question.

Let us represent a cohomology class of $H^1(U, \mathcal{L})$ by a form $\phi \in \Gamma(U, \mathcal{E}^1)$. Then ϕ is harmonic if and only if ϕ is holomorphic when $\phi \in \mathcal{E}^{10}$, and ϕu^{-1} is anti-holomorphic, i.e., $\phi |u|^{-2}$ is anti-holomorphic when $\phi \in \mathcal{E}^{01}$. Moreover ϕ is L_2 if and only if

$$Int := \int \frac{\phi \wedge \overline{\phi}}{|u|^2}$$

is convergent. That is, we have the isomorphisms

$$H^{10} \cap H^1(U, \mathcal{L}) \cong \left\{ [\phi] \;\middle|\; \phi \text{ is a 1-form holomorphic on } U \text{, and} \right.$$

https://doi.org/10.1017/S0027763000025344 Published online by Cambridge University Press

$$\text{meromorphic on } \mathbb{P}^1\text{, such that } \int \frac{\phi \wedge \overline{\phi}}{|u|^2} \text{ converges}\Big\},$$

$$H^{01} \cap H^1(U, \mathcal{L}) \cong \Big\{ \Big[\overline{\phi}|u|^2\Big] \Big| \ \phi \text{ is a 1-form holomorphic on } U\text{, and}$$

$$\text{meromorphic on } \mathbb{P}^1\text{, such that } \int \phi\wedge\overline{\phi}|u|^2 \text{ converges}\Big\}.$$

Indeed, it can be easily seen that if one of the above integrals converges, ϕ can not have an essential singularity at x_j.

Let us examine the convergence conditions above. Let z be a local coordinate around the point x_j, and express ϕ as $f(z)dz$. Let n be the order of zeros of f; the order of $|u|$ is α_j. The integral *Int* converges locally at x_j if and only if $2n - 2\alpha_j > -2$, i.e., $n > \alpha_j - 1$.

We rename the points x_j as follows: For $i \in \mathbb{Z}$, if there are p_i exponents α_k in the interval $(i, i+1)$, we name them as

$$\alpha_{ij}, \quad j = 1, \dots, p_i,$$

and call the corresponding points x_k as x_{ij}. By the definition of the α_{ij} and the p_i, we have

$$ip_i < \sum_j \alpha_{ij} < (i+1)p_i.$$

Thus the identity $\sum_{i,j} \alpha_{ij} = 0$ (recall the assumption $\sum \alpha_j = 0$) leads to

LEMMA 3.1.

$$-\sum ip_i - 1 \geq 0, \quad \sum (i+1)p_i - 1 \geq 0.$$

Let n_{ij} be the order of zeros of $\phi = f(t)\,dt$ at x_{ij}. Then the integral *Int* converges if and only if $n_{ij} > \alpha_{ij} - 1$, i.e., $n_{ij} \geq i$. Since we assumed that non of the x_{ij} is at infinity, the 1-form ϕ is regular at infinity, i.e., f has at least double zero at infinity. So we have the expression

$$\phi = \prod_i \prod_{j=1}^{p_i} (t - x_{ij})^i P\, dt,$$

where P is a polynomial of degree not greater than $-\sum ip_i - 2$. Thus we have

Proposition 3.2.

$$H^{10} \cap H^1(U, \mathcal{L}) = \left\{ \left[\prod_i \prod_{j=1}^{p_i} (t - x_{ij})^i P \, dt \right] \;\middle|\; \deg P \leq -\sum i p_i - 2 \right\}.$$

In particular,

$$\dim H^{10} \cap H^1(U, \mathcal{L}) = -\sum i p_i - 1.$$

Now we turn to (01)-type. Let $\overline{\phi}|u|^2$ represents a class of $H^{01} \cap H^1(U, \mathcal{L})$, where ϕ is a holomorphic 1-form on U. Let n_{ij} be the order of zero of ϕ at x_{ij} as before. Then $\overline{\phi}|u|^2$ is of L_2-class, i.e.,

$$\int \frac{\overline{\phi}|u|^2 \wedge \phi|u|^2}{|u|^2} = \int \overline{\phi} \wedge \phi |u|^2$$

is finite if and only if $n_{ij} > -\alpha_{ij} - 1$, that is, $n_{ij} \geq -i - 1$. Thus we have

Proposition 3.3.

$$H^{01} \cap H^1(U, \mathcal{L}) = \left\{ \left[\overline{\phi}|u|^2 \right] \;\middle|\; \phi = \prod_i \prod_{j=1}^{p_i} (t - x_{ij})^{-i-1} P \, dt, \right.$$
$$\left. \deg P \leq \sum (i+1) p_i - 2 \right\}.$$

In particular,

$$\dim H^{01} \cap H^1(U, \mathcal{L}) = \sum (i+1) p_i - 1.$$

Remark. The propositions above yield the well known fact

$$\dim H^1(U, \mathcal{L}) = n - 2,$$

where n is the number of points x_j.

§4. Generalization to Riemann surfaces of arbitrary genus

Let X be a compact Riemann surface of genus g, $x_1, \ldots, x_n$ distinct points in X and $U = X - \{x_1, \ldots, x_n\}$. Let $\alpha_j \in \mathbb{R} - \mathbb{Z}$ $(j = 1, \ldots, n)$ be exponents such that $\sum \alpha_j = 0$. Suppose given a multivalued holomorphic function u on U such that du/u has only logarithmic singularities at x_j's and $\mathrm{Res}_{x_j} du/u = \alpha_j$ (the reader may verify such u exists uniquely up to multiples by nowhere vanishing multi-valued homomorphic functions on X).

Denote by $\mathcal{L}$ the local system determined by u; similarly for $\check{\mathcal{L}}$ and $\overline{\mathcal{L}}$. There is a canonical isomorphism $\check{\mathcal{L}} \to \overline{\mathcal{L}}$. The first half of the following theorem is proved as for the case $X = \mathbb{P}^1$.

THEOREM 4.1. *The $\mathbb{C}$-vector space $H^1(U,\mathcal{L}) \oplus H^1(U,\overline{\mathcal{L}})$ has polarized $\mathbb{R}$-Hodge structure of weight 1.*

We have

$$H^{10} \cap H^1(U,\mathcal{L}) \cong H^0\Big(X, \Omega^1\Big(-\sum_{i,j} i x_{i,j}\Big)\Big)$$

where $\{x_{i,1},\ldots,x_{i,p_i}\}$ are the singular points with exponents in the interval $(i,i+1)$, and

$$\dim H^{10} \cap H^1(U,\mathcal{L}) = g - 1 - \sum i p_i\,.$$

Similarly

$$H^{01} \cap H^1(U,\mathcal{L}) \cong \overline{H^0\Big(X, \Omega^1\Big(\sum_{i,j}(i+1)x_{i,j}\Big)\Big)}$$

and

$$\dim H^{01} \cap H^1(U,\mathcal{L}) = g - 1 + \sum (i+1) p_i\,.$$

In particular $\dim H^1(U,\mathcal{L}) = 2g - 2 + n$.

Proof. By the same argument as in §3, we have

$$\begin{aligned} H^{10} \cap H^1(U,\mathcal{L}) &= \Big\{[\phi] \;\Big|\; \phi \text{ holomorphic on } U \text{ and } \operatorname*{ord}_{x_j}(\phi) > \alpha_j - 1\Big\} \\ &\cong H^0\Big(X, \Omega^1\Big(-\sum_{i,j} i x_{i,j}\Big)\Big)\,. \end{aligned}$$

Since $\deg(\sum -i x_{i,j}) > 0$ by Lemma 3.1 in §3 one has $H^1(X,\Omega^1(-\sum_{i,j} i x_{i,j})) = 0$ and

$$\begin{aligned} \dim H^{10} \cap H^1(U,\mathcal{L}) &= \chi\Big(X, \Omega^1\Big(-\sum_{i,j} i x_{i,j}\Big)\Big) \\ &= 1 - g + (2g-2) - \sum i p_i \end{aligned}$$

by Riemann-Roch formula.

Similarly

$$H^{01} \cap H^1(U,\mathcal{L}) \cong \overline{H^0\Big(X, \Omega^1\Big(\sum_{i,j}(i+1)x_{i,j}\Big)\Big)}$$

and again by using $\deg(\sum (i+1)x_{i,j}) > 0$ and Riemann-Roch, we get the result.

https://doi.org/10.1017/S0027763000025344 Published online by Cambridge University Press

§5. Period relations and Riemann's (in)equalities

In this section we discuss period relations and Riemann's (in)equalities. The former has nothing to do with Hodge structure, and (with obvious modifications) the argument goes through if $\alpha_j \notin \mathbb{R}$.

To formulate the period relations in a general form, let V be a finite dimensional $\mathbb{C}$-vector space equipped with a non-degenerate bilinear pairing (not necessarily symmetric or skew-symmetric)

$$Q : V \otimes V \longrightarrow \mathbb{C}\,.$$

If V^* denotes the dual of V, there is an isomorphism

$$f : V \longrightarrow V^*$$

such that $f(v)(w) = Q(v,w)$. Denote by $Q^* : V^* \otimes V^* \to \mathbb{C}$ the pairing that Q induces via the isomorphism f.

Let $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ ($r = \dim V$) be a basis of V^* and $v, w \in V$. Then one has

$$\left(\int_{\alpha_1} v, \int_{\alpha_2} v, \dots, \int_{\alpha_r} v\right) \, {}^t(\alpha_i \cdot \alpha_j)^{-1} \begin{pmatrix} \int_{\alpha_1} w \\ \vdots \\ \int_{\alpha_r} w \end{pmatrix} w = Q(v,w)\,.$$

Here $\int_\alpha v = \alpha(v)$. To derive this write ($\{\alpha'_i\}$ is the dual basis of $\{\alpha_i\}$)

$$v = \sum \left(\int_{\alpha_i} v\right) \alpha'_i, \quad w = \sum \left(\int_{\beta_i} v\right) \beta'_i$$

and use $(\alpha'_i \cdot \beta'_j) = {}^t(\alpha_i \cdot \beta_j)^{-1}$.

We apply this to

$$V = H^1(U, \mathcal{L}) \oplus H^1(U, \overline{\mathcal{L}})\,, \quad V^* = H_1(U, \check{\mathcal{L}}) \oplus H_1(U, \check{\overline{\mathcal{L}}})\,,$$

and the basis $\{\check{\gamma}_j, \check{\overline{\gamma}}_j\}$ of V^*. ($\dim V = 2(n-2)$.) The matrix ${}^t(\alpha_i \cdot \alpha_j)^{-1}$ equals

$${}^t\begin{pmatrix} 0 & (\check{\gamma}_i \cdot \check{\overline{\gamma}}_j) \\ (\check{\overline{\gamma}}_i \cdot \check{\gamma}_j) & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & {}^tI_h^{-1} \\ -I_h^{-1} & 0 \end{pmatrix}$$

where by (1.2)

$$I_h := (\check{\gamma}_i \cdot \check{\overline{\gamma}}_j) = (\check{\gamma}_i \cdot \gamma_j)\,.$$

For $[\phi] \in H^1(U, \mathcal{L})$ and $[\psi] \in H^1(U, \overline{\mathcal{L}})$ we have

$$\left(\int_{\check{\gamma}_1} \phi, \dots, \int_{\check{\gamma}_r} \phi, 0, \dots, 0\right) \begin{pmatrix} 0 & {}^t I_h^{-1} \\ -I_h^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \int_{\check{\bar{\gamma}}_1} \psi \\ \vdots \\ \int_{\check{\bar{\gamma}}_r} \psi \end{pmatrix} = Q([\phi], [\psi])$$

namely *the period relations*

$$\left(\int_{\check{\gamma}_1} \phi, \dots, \int_{\check{\gamma}_r} \phi\right) {}^t I_h^{-1} \begin{pmatrix} \int_{\check{\bar{\gamma}}_1} \psi \\ \vdots \\ \int_{\check{\bar{\gamma}}_r} \psi \end{pmatrix} = Q([\phi], [\psi])\,. \tag{5.1}$$

This is the twisted Riemann's equality in [CM]. (Precisely speaking the result in [CM] — which holds without the assumption $\alpha_j \in \mathbb{R}$ — is obtained by the above argument applied to $V = H^1(U, \mathcal{L}) \oplus H^1(U, \check{\mathcal{L}})$.) It has nothing to do with Hodge structure.)

There is another equality resulting from Hodge structure: if $[\phi] \in H^{10} \cap H^1(U, \mathcal{L})$ and $[\psi] \in H^{10} \cap H^1(U, \overline{\mathcal{L}})$ then $Q([\phi], [\psi]) = 0$.

If $[\phi] \in H^1(U, \mathcal{L})$ then $\int_{\check{\bar{\gamma}}_j} \overline{\phi} = \overline{\int_{\check{\gamma}_j} \phi}$ by (1.3). Thus we obtain *twisted Riemann's inequality*:

THEOREM 5.1. *For any non-zero* $[\phi] \in H^{10} \cap H^1(U, \mathcal{L})$,

$$\sqrt{-1}\left(\int_{\check{\gamma}_1} \phi, \dots, \int_{\check{\gamma}_r} \phi\right) {}^t I_h^{-1} \begin{pmatrix} \overline{\int_{\check{\gamma}_1} \phi} \\ \vdots \\ \overline{\int_{\check{\gamma}_r} \phi} \end{pmatrix} > 0\,.$$

§6. Applications

We follow the argument in the preceding section by setting

$$V = H^1(U, \mathcal{L}) \oplus H^1(U, \overline{\mathcal{L}}) = H^{10} \oplus H^{01}, \quad r = 2(n-2),$$

$$\delta_i \in H^1(U, \mathcal{L}),\ \text{basis dual to } \check{\gamma}_i \in H_1(U, \check{\mathcal{L}}),$$

and $[\phi] \in H^{10} \cap H^1(U, \mathcal{L})$. Then there is an L_2-harmonic element $\phi_0 \in [\phi]$, and we have

$$\sqrt{-1} Q([\phi], [\overline{\phi}]) = \sqrt{-1} \int \frac{\phi_0 \wedge \overline{\phi}_0}{|u|^2} > 0.$$

If we write

$$[\phi] = \sum_{i=1}^{g} \lambda_i \delta_i, \quad \lambda_i = \int_{\check{\gamma}_i} [\phi],$$

then the above inequality becomes

$$\sqrt{-1} \sum_{i-1}^{g} \lambda_i \overline{\lambda}_j Q(\delta_i, \overline{\delta}_j) > 0.$$

Example 1. We consider four points $x_1, \dots, x_4$, exponents $\alpha_1, \alpha_2 \in (0,1)$, $\alpha_3, \alpha_4 \in (-1,0)$ — so $p_0 = p_1 = 2, n = 4$ — and

$$u = (t - x_1)^{\alpha_1} \cdots (t - x_4)^{\alpha_4}.$$

We have

$$H^{01} \cap H^1(U, \mathcal{L}) = \mathbb{C} \frac{dt}{(t - x_3)(t - x_4)}.$$

Take

$$\phi = \frac{dt}{(t - x_3)(t - x_4)} \in H^{01} \cap H^1(U, \mathcal{L})$$

and the twisted cycles γ_j $(j = 1, 2)$ with supports on the edge joining x_j and x_{j+1} as in [Y, p. 94], and put

$$\lambda_j = \int_{\gamma_j} u^{-1} \phi.$$

The inverse matrix of the intersection matrix $((Q(\delta_i, \overline{\delta_j}))$ is given ([Y, p. 102]) as

$$H := -\begin{pmatrix} d_{23}/d_2 d_3 & 1/d_2 \\ c_2/d_2 & d_{12}/d_1 d_2 \end{pmatrix},$$

where $d_j = c_j - 1, d_{jk} = c_j c_k - 1, c_j = \exp 2\pi i \alpha_j$. Then we have the inequality

$$\sqrt{-1}(\lambda_1, \lambda_2) H^t(\overline{\lambda}_1, \overline{\lambda}_2) > 0.$$

If one normalizes, for example, $x_1 = 0, x_2 = 1, x_3 = \infty, x_4 = x$, and put $\alpha_1 = -a, \alpha_2 = a - c, \alpha_3 = b, \alpha_4 = c - b$, then λ_j can be expressed by the hypergeometric series $F(a, b, c; x)$.

https://doi.org/10.1017/S0027763000025344 Published online by Cambridge University Press

EXAMPLE 2. If one specializes further by putting $\alpha_1 = \alpha_2 = 1/2, \alpha_3 = \alpha_4 = -1/2$, then $H = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, and λ_1 and λ_2 turn out to be two periods of the elliptic curve defined as the double cover of $\mathbb{P}^1$ branching at the four points x_j. The inequality obtained implies that the imaginary part of the ratio of the two periods is positive; this is a classical fact.

REFERENCES

[AK] K. Aomoto and M. Kita, Hypergeometric functions, Springer-Verlag, Tokyo, 1994, (Japanese).

[CM] K. Cho and K. Matsumoto, *Intersection theory for twisted cohomologies and twisted Riemann's period relation* I, Nagoya Math. J., **139** (1995), 67–86.

[CY] K. Cho and M. Yoshida, *Comparison of (co)homologies of branched covering spaces and twisted ones of basespaces*, Kyushu J. Math., **45** (1994), 111–122.

[G] Ph. Griffiths, ed., Topics in Transcendental Algebraic Geometry, Princeton Univ. Press, 1984.

[KY] M. Kita and M. Yoshida, *Intersection theory for twisted cycles*, Math. Nachr., **166** (1994), 287–304.

[M1] K. Matsumoto, *Quadratic identities for hypergeometric series of type* (k, l), Kyushu J. of Math., **48** (1994), 335–345.

[M2] K. Matsumoto, *Intersection numbers for logarithmic k-forms*, Preprint (1996).

[Y] M. Yoshida, Hypergeometric Functions, My Love, Vieweg, Wiesbaden, 1997.

[Zuc] S. Zucker, *Hodge theory with degenerating coefficients*: L_2 *cohomology in the Poincaré metric*, Ann. of Math., **109** (1979), 415–176.

Masaki Hanamura
Graduate School of Mathematics
Kyushu University 33
Fukuoka 812-8581
Japan

Masaaki Yoshida
Graduate School of Mathematics
Kyushu University 33
Fukuoka 812-8581
Japan