Session A

Modeling convection and radiative transfer

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Theoretical modeling of convection I. Key physical processes

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Abstract. If one strives for a reliable description of "Turbulent Mixing in Stars" one must account for a large variety of physical processes. These include non-locality, that is needed in unstably stratified regimes, overshooting, which occurs in a stably stratified regime, double-diffusion processes (semi-convection and salt-fingers), transport of angular momentum, the Li^7 problem, compressibility, and magnetic fields. While phenomenological models are manifestly inadequate, LES are too computer intensive to tackle this large variety of processes. Since the requirement of completeness of the description of these processes must also result in models that are usable in stellar structure-evolution codes, we conclude that only the RSM (Reynolds Stress Model) can do so and a description of the state of the art in that field is presented in Part 2.

Keywords. Convection, turbulence, stars: interiors

1. General considerations

In the description of stellar convection one faces a well known dichotomy: on the one hand, LES (large eddy simulations) provide a wealth of data that are rich in content, information, details etc., while in stellar structure-evolution codes the models used to describe turbulent processes are still quite rudimentary, as the MLT-type models clearly show. Due to the large computational requirements that LES entail, there is little hope that astrophysical LES codes (e.g., Stein and Nordlund 1998; Rosenthal *et al.* 1999; Asplund *et al.* 2000; Miesch *et al.* 2000; Brun and Toomre 2002; Robinson *et al.* 2003; Brun *et al.* 2004; Wedemeyer *et al.* 2004) will be hooked-up to a stellar structure and evolution code any time soon. What is the solution? Can one find a middle of the road approach that uses the LES data to test a model of turbulent convection that is superior to the MLT while retaining the manageability that stellar structure codes require? The only *viable route is to employ the RSM, Reynolds Stress Model*. The RSM solves the basic NSE (Navier-Stokes Equations) and the equations for temperature, mean molecular weight, etc., can accommodate diverse physical processes without having to change the rules of the game every time a new process is included, gives very frequently algebraic results, can be local and non-local, can be hooked-up to a general code, has an extensive and rather successful pedigree for it has been used for several decades to describe turbulent phenomena that exhibit regimes often similar to those occurring in stars. It must also be stressed that in both atmospheric and oceanic studies, the RSM has been in use for several decades while in astrophysics it is relatively new. This means that one can benefit from the progress made in other fields.

Before we describe the RSM, its accomplishments and limitations, we must stress that even the LES have their own limitations. The reason is simple. The key difficulty in the description of any turbulent flow are the non-linear interactions (NLI) which are the origin of turbulence, the treatment of which has challenged all those who dealt with them.

Difficult as it is to account for them, the NLI have a remarkable property: they appear under the divergence operator and thus when one integrates over the volume of the system, the NLI yield zero. Turbulence is a somewhat secretive process, an insider that wields much power but that globally acts only as the perfect "transferer" that actually works for free (the zero integral just discussed). What does turbulent transfer and is this its main job? The answer can be seen by a simple argument. Consider the non-linear term $\nabla\cdot\mathbf{uu}$ or $\mathbf{u}\cdot\nabla\mathbf{u}$ (since we are considering incompressible flows, $\nabla\cdot\mathbf{u}=0$). If one Fourier transforms the velocity field $\mathbf{u}(\mathbf{r})=\Sigma_{(\mathbf{k})}\mathbf{u}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})$, it is clear that the velocity $u_i(\mathbf{k})$ corresponding to the mode $\mathbf{k}$ will entail the non-linear term $\alpha_{ijm}\Sigma_{(\mathbf{k}')}u_j(\mathbf{k}')u_m(\mathbf{k}-\mathbf{k}')$ where $\mathbf{k}'$ are all the modes different than $\mathbf{k}$ (McComb 1990). If we view the mode $k\sim l^{-1}$ as a length scale l, it is clear that the dynamics of l will be governed by its interactions with all the other length scales. The non-linear interactions are the ones responsible for the transfer process we referred to before. The extent of these "sizes" l is governed on the large size by the geometrical extent of the flow and at the opposite end by the molecular sizes which have a dynamics of their own. Contrary to the case of laminar flows, in which the molecular sizes are dictated by molecular forces and do not change, the "sizes" that characterize a turbulent flow span a wide range of values. It is known that "viscosity" is at the top of the list of anti-turbulence agents, a fact usually represented by the Reynolds number Re that gauges the importance of the non-linear interactions to the viscous effects:

$$\mathrm{Re}=\frac{\overline{u}\nabla\overline{u}}{\nu\nabla^2\overline{u}}=UL\nu^{-1} \tag{1.1}$$

where U and L represent typical velocity and scale of the flow under consideration and ν is the kinematic viscosity. Flows with $\mathrm{Re}>10^3$ are considered turbulent, a rather modest value (a street car can reach $\mathrm{Re}\approx 10^4$). With the two variables ε and ν (characterizing the rate of energy input and the type of fluid), one constructs (using dimensional arguments) the dissipation length scale l_d where the eroding action of viscosity begins to be felt. The variables of interest are therefore:

$$l_d=\left(\nu^3\varepsilon^{-1}\right)^{1/4} \tag{1.2}$$

$$L/l_d\sim\mathrm{Re}^{3/4} \tag{1.3}$$

$$N\sim(L/l_d)^3\sim\mathrm{Re}^{9/4} \tag{1.4}$$

In the first relation, the larger the viscosity, the larger l_d, so that for large ν, l_d can be of the order of the largest eddy size in which case turbulence has no room to develop. In the Sun's interior, the viscosity of a fully ionized gas is given by (Chapman 1954):

$$\nu[\mathrm{cm}^2\mathrm{s}^{-1}]=1.2^{\star}10^{-16}T^{5/2}\rho^{-1} \tag{1.5}$$

If $L_\odot$ and $M_\odot$ denote the luminosity and the mass of the Sun, we have $\varepsilon\sim L_\odot/M_\odot\sim$ O(1) cm$^2s^{-3}$. Furthermore, at the bottom of the convective zone we can take the representative values $T\approx 10^6$ K, $\rho\approx 0.1$ g cm^{-3} and $\nu\approx 1$ cm^2 s^{-1}. From (1.2), we obtain $l_d\approx 1$ cm which gives an idea of how small are the scales at which dissipation occurs. Next, consider (1.3). This simple looking relation has several consequences the most important of which is that even LES cannot resolve all the scales of a turbulent flow. To do so, (1.4) tells us the number of points N that an LES needs to simulate. If we use values of Re corresponding to a street car, $\mathrm{Re}\approx 10^4$, $L/l_d\approx 10^3$, that is, we have 10^3 "sizes" to account for which means $N\sim 10^9$ degrees of freedom, which is feasible with today's computers. However, in stars we deal with $N\geqslant 10^{20}$ which is orders of magnitude larger than what modern computers can handle. Thus, LES resolve numerically only the largest scales and must model a huge number of numerically unresolved scales using a SGS (sub grid scale) model. A discussion of the SGS models for use in LES can be found in Canuto (1994, 1997).

2. Heat fluxes: local and non-local models

Stellar structure codes solve the equations for the mean variables, mean velocity, mean temperature, mean molecular weight etc. Toward the end of the XIX century, Osborne Reynolds suggested a procedure to treat the effect of the non-linear interactions on the mean flow variables. He suggested that every field should be the sum of a mean and a fluctuating component:

$$\varphi = \overline{\varphi} + \varphi', \qquad \overline{\varphi'} = 0 \tag{2.1}$$

Substituting the first of (2.1) into the equation for the complete field, taking the average and using the second of (2.1), one obtains the equation for the mean fields. For example, for temperature and velocity, one has:

$$\frac{\partial T}{\partial t} + \cdots = -\frac{\partial}{\partial z}\overline{w\theta} \tag{2.2}$$

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \cdots = -\nabla \cdot \overline{\mathbf{u}\mathbf{u}} \tag{2.3}$$

The rhs represent correlations between two fluctuating variables: in (2.2) it is the turbulent heat flux (in units of $c_{\mathrm{p}}\rho$) while in (2.3) we have the *Reynolds Stresses*:

$$R_{ij} = \overline{u_i u_j} \tag{2.4}$$

In early works, turbulent fluxes were treated with heuristic models such as:

$$\overline{uw} = -K_m \partial_z \overline{u}, \qquad \overline{w\theta} = -K_h \partial_z T, \qquad K_{m,h} = wl, \tag{2.5}$$

where the *turbulent momentum and heat diffusivities* $K_{m,h}$ were written on dimensional grounds in terms of a velocity and a mixing length l. In the stellar case, one needs to model *Turbulent Convection* and in this respect the MLT, Mixing Length Theory, constructed the heat diffusivity K_h on a phenomenological basis rather than on a Reynolds stress model (Cox and Giuli 1968; Gough and Weiss 1976). In that sense, it was less well grounded than the turbulence models used in the engineering context. However, the MLT turned out to be successful for reasons we now discuss. It so happens that in most stellar cases convection is actually governed not by heating from below but by cooling from above much as it occurs in geophysical regimes such as oceanic convection (in the Labrador Sea, Gulf of Lyon and Weddell Sea the loss of buoyancy by surface waters due to both evaporation and winds makes them heavy and thus prone to fall) and in the Earth's atmosphere when the latter is cloud-capped. It turns out that in these regimes, the flow is a combination of well organized, narrow, vigorous, descending "plumes" accompanied by disordered, broad plumes which are the ones that the MLT modeled. Pioneering studies by Cattaneo *et al.* (1991) showed that:

$$\text{Downflows:} \quad F(\text{convected}) = F_{\mathrm{KE}} + c_{\mathrm{p}}\overline{w\theta} \approx 0 \tag{2.6}$$

$$\text{Upflows:} \quad F(\text{convected}) = F_{\mathrm{KE}} + c_{\mathrm{p}}\overline{w\theta} \neq 0 \tag{2.7}$$

which tell us that the convected flux of the descending plumes is almost entirely canceled by the flux of turbulent kinetic energy $F_{\mathrm{KE}} = \overline{wK}$ leaving only the upflows. Later studies showed that the situation is more complicated in the sense that the convective layer should have stable layers on both sides whereas the one studied by Cattaneo *et al.*

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(1991) had fixed plates as boundary conditions. In the simulation of Chan and Gigas (1992), there was an extended stable layer at the bottom of the convective zone and a tiny stable layer at the top. In this case, the cancellation (2.6) was not as strong, in effect only 30%. The conclusion seems to be that the net flux contribution from the downflows is not zero and is directed upward and since the enthalpy flux (up) is about 50% of the total flux, and the convected flux (up) is about 2/3 of the total flux, one could argue that within a factor of about 2, a local convective model may be a good estimate for the total convective flux. To estimate the latter, one may employ the second of (2.5) and express K_h in terms of the mean stellar parameters, one of which is the rate of radiative cooling by raising blobs with a time scale t_χ and the other is the time scale t_b on which buoyancy operates:

$$t_\chi = l^2/\chi, \qquad t_b = (g\alpha\beta)^{-1/2}, \qquad S = (t_\chi/t_b)^2 = g\alpha\beta l^4 \chi^{-2} \tag{2.8}$$

where $\beta = -\partial T/\partial z + (\partial T/\partial z)_{\rm ad}$. When $S < 1$, convection is inefficient since the buoyancy time is long enough to allow radiative processes to cool off the blob of gas while it raises; the case of efficient convection is when the opposite is true, the buoyancy time is so short that the blob raises before it loses heat via radiative processes. The MLT (Cox and Giuli 1968; Gough and Weiss 1976; Canuto 1996) yields the following expression for K_h/χ:

$$K_h/\chi \sim S^{-1}[(1+S)^{1/2} - 1]^3 \tag{2.9}$$

that embraces efficient $S >> 1$, $K_h/\chi \sim S^{1/2}$ and inefficient $S << 1$, $K_h/\chi \sim S^2$ convection.

We must note that (2.9) was derived from a turbulence model (Canuto and Goldman 1985; Canuto *et al.* 1987; Canuto 1996) and that improvements of it were also proposed (Canuto and Mazzitelli 1991) and tested (Althaus and Benvenuto 1996; D'Antona and Mazzitelli 1996; Samadi *et al.* 2006). The cancellation described by the first of (2.6) does not mean however that in the second of (2.7) we can neglect $F_{\rm KE}$. To understand its physical meaning and implications, consider the figure below in which we have sketched an eddy as a spherical blob of the same size H of the region in which it is formed.

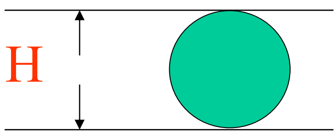

Figure 1. An eddy as a spherical blob.

Can such a large eddy exist? In an unstably stratified regime, buoyancy overpowers gravity (so to speak) and thus it is possible to have eddies of the size of the "container". On these grounds, one would expect H to appear in the expression for the heat flux, but if one looks at the second of (2.5), there is no H and therefore the formula is incomplete. In fact, it represents the flux only with local variables since $J(z)$ is given by the temperature gradient and the diffusivity evaluated at the same z. *It is a local flux* that does not account for the fact that there are large eddies of the size of the "container". One way to construct a non-local flux is to use an expression of the type:

$$J(z) = -\int_0^z \tilde{K}_h\,(z,z')\frac{\partial T(z')}{\partial z'}dz' \tag{2.10}$$

which implies that at every height z, the flux is contributed by all the fluxes below it,

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and at $z = H$ the upper limit brings in the height H of the convective region. It is not difficult to find out what a non-local term looks like, a result that the RSM discussed in Canuto (2007) justifies. The dynamic equation for any field φ has the general form:

$$\frac{\partial \varphi}{\partial t} + \nabla \cdot \mathbf{u}\,\varphi = \text{sources} - \text{sinks} \tag{2.11}$$

Suppose $\varphi = \overline{w\theta}$ and a 1D case. In the stationary limit, we have:

$$\frac{\partial \overline{w^2\theta}}{\partial z} = \text{sources} - \text{sinks} \tag{2.12}$$

Though the steps to arrive at (2.12) are more intuitive than rigorous, the message is clear: the lhs of (2.12), the d/dz of the flux of the heat flux, has a clear physical interpretation: turbulence not only creates non-zero correlations such as $\overline{w\theta}$, but it also transports them via the term on the lhs which represents *non-locality*. In fact, even if the "sources" on the rhs where zero, the non-local term on the lhs could act as a source that balances the sink. That is to say, in places where there is no local source of turbulence, there may be mixing which, originated somewhere else in the flow, is transported there by the term on the lhs. For the turbulent kinetic energy K the equivalent of (2.12) reads:

$$\frac{\partial F_{\mathrm{KE}}}{\partial z} = g\alpha J - \varepsilon, \qquad F_{\mathrm{KE}} \equiv \overline{wK} \tag{2.13}$$

where F_{KE} is the flux of kinetic energy, which we already introduced in Eqs. (2.6), (2.7). We have also specified the source, buoyancy, and re-introduced the rate of dissipation ε. $\mathrm{F_{KE}}$ is a third-order moment the closure of which will be discussed in detail in Part 2 (Canuto 2007). At this point, it suffices to say that the *local limit* of (2.13) corresponds to $g\alpha J = \varepsilon$ and gives rise to the MLT model (Canuto and Dubovikov 1998). It is simple to reproduce the $S \gg 1$ limit of (2.9). In fact, the dissipation ε must equal the rate of injection, that is:

$$\varepsilon \sim \int n(k)E(k)dk \tag{2.14}$$

where $n(k)$ is the growth rate of the convective instability (Canuto and Mazzitelli 1991). Since in the $S > 1$ case, $n(k) \sim (g\alpha\beta)^{1/2}$, use of the Kolmogorov spectrum $E(k)$=Ko $\varepsilon^{2/3}k^{-5/3}$ in (2.14) and then in $g\alpha J = \varepsilon$ leads to the $S > 1$ limit of (2.9).

Simple non-local model. In the case of the PBL (planetary boundary layer), Holtslag and Moeng (1991) and Holtslag and Boville (1993) suggested the simple parameterization:

$$\frac{\partial \overline{w^2\theta}}{\partial z} \sim H^{-1} w_{\mathrm{s}} J_{\mathrm{s}}, \qquad w_{\mathrm{s}} = [g\alpha H J_{\mathrm{s}}]^{1/3} \tag{2.15}$$

where the second relation is due to Deardorff (1972); the subscript "s" represents the surface value. What is relevant is the presence of H which is the hallmark of non-locality. Therefore, one now has that the total heat flux is given by:

$$J(z) = J_{\mathrm{L}}(z) + J_{\mathrm{NL}} = -K_h \frac{\partial T}{\partial z} + \frac{c}{H}\tau w_{\mathrm{s}} J_{\mathrm{s}} \tag{2.16}$$

where c is a numerical constant. While use of (2.15, 2.16) has considerably improved the description of the PBL, no stellar test has yet been made of them. In conclusion, the LES data (Cattaneo *et al.* 1991; Chan and Gigas 1992) suggested an interesting cancellation in the downflows leaving behind the disordered upflows described by the MLT. That may explain the success the MLT has enjoyed in stellar structure/evolution studies over the years. The MLT is however a local theory and thus incomplete since in unstably stratified regimes non-locality plays a major role. The challenge is to account for non-locality with the Reynolds Stress Model (see Part 2, Canuto 2007).

3. Overshooting regions (OV)

Outside the *unstably stratified* CZ (Convective Zone) there are dynamically important *stably stratified* regions referred to as OV (overshooting) which have attracted a great deal of attention over the years (Roxburgh 1978, 1989; Anderson *et al.* 1990; Spiegel and Zahn 1992). In Fig. 2 we present a simple sketch of the stellar CZ together with the ocean case (stable stratification is the norm) and the Earth's atmosphere where in the daytime one has unstable stratification (heated from below) and the opposite at night.

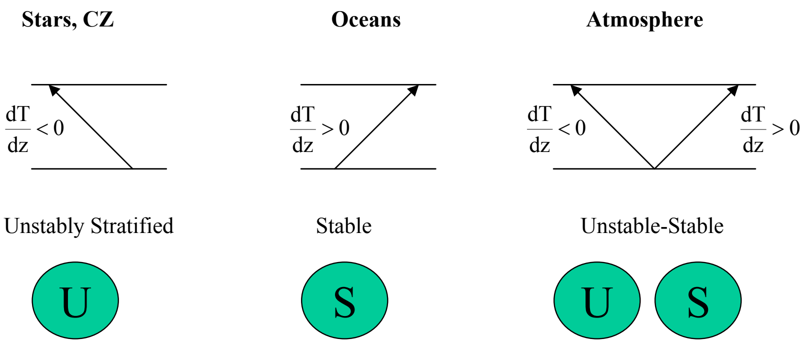

Figure 2. Comparison of stratifications.

A question arises: what is the source of turbulent mixing in a stably stratified regime? Consider the earth's oceans which are a large body of a stably stratified fluid since cold, dense waters are at the bottom and warm, light waters are on top. Without external disturbances, such a fluid would not become turbulent which, on the other hand, is the true state of the ocean where the main source of the strong mixing is the shear produced by the external wind. Without mixing, there would be no upwelling of nutrients, of which deep waters are rich, leading to a desert sea.

In the stellar OV, the two most obvious sources are: 1) differential rotation (which we call shear) and 2) non-local flow from the CZ, that is, the transport of turbulent kinetic energy represented by the term F_{KE} in (2.13) which we now generalize to:

$$\frac{\partial K}{\partial t} + \frac{\partial F_{\mathrm{KE}}}{\partial z} = -R_{ij}\overline{u}_{i,j} - gJ_\rho - \varepsilon \tag{3.1}$$

where we have added a time dependence of K, the shear contribution (first term on the right, where $a_{i,j} = \partial a_i/\partial x_j$) and generalized the heat flux to the mass flux:

$$J_\rho = \rho_0^{-1}\overline{\rho w} = -\alpha_T J_h + \alpha_\mu J_\mu = -\rho_0^{-1} K_\rho \frac{\partial \overline{\rho}}{\partial z} = g^{-1} K_\rho N^2 \tag{3.2}$$

where $\alpha_T = -\partial \ln \rho/\partial T, \alpha_\mu = +\partial \ln \rho/\partial \mu$, $N^2 = -g\rho_0^{-1}\partial\overline{\rho}/\partial z$, and μ is the mean molecular weight. Since in a stably stratified regime $N^2 > 0$, $J_\rho > 0$, in the absence of shear, the rhs of (3.1) is negative. It follows that the only way to satisfy (3.1) is by having a negative gradient of the non-local flux of turbulent kinetic energy. In the presence of differential rotation, the first term on the rhs is positive (source) and, together with a negative lhs, constitutes a second source of mixing. The often discussed gravity waves (whose power $\Pi_{\rm gw}$ was computed by Kumar *et al.* 1999) are in effect an energy flux that originates from the CZ and thus they may be interpreted as part of the lhs of (3.1). Using (2.5) and (3.2), Eq. (3.1) becomes:

$$\frac{\partial K}{\partial t}+\frac{\partial F_{\rm KE}}{\partial z}=K_m\Sigma^2-K_\rho N^2-\varepsilon=K_m\Sigma^2(1-\sigma_t^{-1}{\rm Ri})-\varepsilon \tag{3.3}$$

where Ri is the Richardson number and σ_t is the turbulent Prandtl number:

$${\rm Ri}=\frac{\rm Sink}{\rm Source}=\frac{N^2}{\Sigma^2},\qquad \Sigma^2=\overline{u}^2_{,z}+\overline{v}^2_{,z},\qquad \sigma_t=\frac{K_m}{K_\rho} \tag{3.4}$$

The physical reason why in stably stratified flows a local model may not be as poor an approximation as in the unstable case is because eddies are generally small in a stably stratified regime. In that case, Eq. (3.3) becomes:

$$K_m\Sigma^2=\varepsilon+K_\rho N^2 \tag{3.5}$$

which we interpret by saying that the mixing caused by shear has to "work" against dissipation *and* the natural stability of the fluid. From (3.5, 3.4) we see that the dominance of the source over the sink (the stable stratification) is largely dictated by the value of the Richardson number Ri. This has given rise to confusing statements over the years about the "critical Ri" above which the source can no longer sustain the eroding action of the sink. Using linear stability analysis, Miles (1961) and Howard (1961) showed that $\rm Ri_L=1/4$ is the value at which *laminarity* ceases to exist. After that, the system first enters a weakly non-linear regime and then finally a turbulent state where non-linearities dominate. Woods (1969) was the first to give a physical picture of the different regimes leading to turbulence. Given a stable laminar sheet of thickness h, Kelvin-Helmholtz instabilities gradually erode and entrain fluid parcels above and below h. The process leads to an increase of h which ceases when the thickness has become four times the original value h. Woods concluded that "since the final thickness is nearly four times the original value, the final Richardson number is also four times the value prior to the instability", that is, the inception of turbulence occurs approximately at:

$${\rm Ri_t}=4\,{\rm Ri_L}\approx {\rm O}(1) \tag{3.6}$$

Abarbanel *et al.* (1984) carried out a stability analysis with the inclusion of non-linearities, and concluded that the instability occurs at the value given by (3.6).

In spite of this collective evidence, most authors who studied stable stratification in stars used $\rm Ri_{cr}=1/4$ which underestimates turbulent mixing (Zahn 1974; Maeder 1995; Maeder and Meynet 1996; Talon and Zahn 1997; Schatzman *et al.* 2000). Due to the physical role of Ri, it is clear that the intensity of turbulence must decrease with increasing Ri and several heuristic expressions were proposed over the years primarily because no reliable theory was available. In 2001-2002, Canuto *et al.* developed a model for turbulence under stable stratification and shear that quite naturally reproduced (3.6) as the point at which turbulence has weakened as to become practically inefficient. The model also includes DD (Double Diffusion) processes which can be quite relevant (see below). In conclusion, in the OV the situation is rather complex since one can think of at least three possible sources and one possible sink.

Sources: Differential rotation (shear), gravity waves, non-local K-fluxes.

Sinks: a μ-gradient $\nabla_\mu>0$ (a positive $\nabla_\mu\sim\partial\mu/\partial P$ corresponds to a mean molecular weight large at the center and low at the surface) acts to increase the local dissipation (as we show below) thus reducing the penetration of turbulence into the OV region.

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4. Semi-convection and salt-fingers: double diffusion processes

Double diffusion processes occur when two different fields exist which have very different kinematic diffusivities. In stars we have the fields (T, μ) where the latter is the mean molecular weight. In oceanography, one has (T, S) where the salinity field has a kinematic diffusivity that is two order of magnitude smaller than heat. Such processes are also referred to as thermohaline and/or thermosolutal convection. When both T and S increase from the ocean surface toward the bottom, the result is cold fresh water over warm salty water. The S field is stable, the T field is unstable (heavy at the top), and one has *diffusive convection* (Schmitt 1994). Examples are lakes, water underneath an ice island, and the Red Sea. In stars, *diffusive convection* is called *semi-convection* and was studied by several authors (Stothers 1970; Spiegel 1972; Stothers and Chin 1975, 1976; Stevenson 1979; Langer *et al.* 1983, 1985, 1989; Spruit 1992; Grossman and Taam 1996; Umezu 1998). Yet, there does not seem to be a generally accepted procedure to treat the phenomenon. Stothers (1970) critically analyzed 11 different prescriptions and concluded that only two were physically acceptable: one used by Schwarzschild and Harm (1958), who adopted the Schwarzschild criterion, and the other by Sakashita and Hayashi (1959) who adopted the Ledoux criterion (Ledoux 1947). In the absence of a turbulence model, Langer *et al.* (1983, 1985, 1989) suggested a phenomenological model that we shall discuss below. Merryfield (1995) found that none of his two-dimensional numerical simulations exhibited any close resemblance to the models by Stevenson and/or Spruit and that the closest similarity is with a Langer *et al.* model. Xiong (1985a,b; 1986) and Grossman and Taam (1996) carried out nonlinear studies of semi-convection which is characterized by the conditions:

$$\nabla - \nabla_{\rm ad} > 0, \quad \nabla_\mu > 0, \quad \nabla_{\rm r} > \nabla, \tag{4.1}$$

with $\nabla = \partial \ln T/\partial \ln P$, $\nabla_{\rm ad} = (\partial \ln T/\partial \ln P)_{\rm ad}$, $\nabla_{\rm r} = (\partial \ln T/\partial \ln P)_{\rm rad}$, and $\nabla_\mu \equiv \partial \ln \mu/\partial \ln P$ and thus $\nabla_{\rm r} > \nabla > \nabla_{\rm ad}$.When both the T and S fields increase from the bottom to the top of the ocean, the result is warm salty water over cold fresh water. Since the T field is stable while the S field is unstable (heavy at the top), the latter causes an instability called *salt fingers.* An example is the Atlantic Ocean underneath the Mediterranean outflow of very salty water. In astrophysics, this instability occurs when a layer with a higher μ lies above a region of lower μ, for example, when the He flash does not occur at the center of a star, as first discussed by Thomas (1967). Salt fingers were first suggested by Stothers and Simon (1969) and later studied by Ulrich (1972) and Kippenhahn *et al.* (1980). The μ field causes the instability, while $\nabla - \nabla_{\rm ad}$ plays the role of a stabilizing gradient. The *salt fingers* phenomenon is characterized by the following conditions:

$$\nabla - \nabla_{\rm ad} < 0, \quad \nabla_\mu < 0, \quad \nabla_{\rm r} < \nabla, \quad \nabla_{\rm ad} > \nabla > \nabla_{\rm r}. \tag{4.2}$$

Since $N^2 = gH_{\rm p}^{-1}[\nabla_\mu - (\nabla - \nabla_{\rm ad})]$, where $H_{\rm p} = p/g\rho$ is the pressure scale height, we must distinguish between the following cases. With $R_\mu = \nabla_\mu(\nabla - \nabla_{\rm ad})^{-1}$, we have:

$$\begin{array}{rlll} \text{Semi-convection:} & \nabla - \nabla_{\rm ad} > 0, & \nabla_\mu > 0, & R_\mu > 0 \\ \text{Ledoux stable:} & N^2 > 0, & \nabla_\mu > \nabla - \nabla_{\rm ad}, & R_\mu > 1 \\ \text{Ledoux unstable:} & N^2 < 0, & \nabla - \nabla_{\rm ad} > \nabla_\mu, & R_\mu < 1 \end{array} \tag{4.3}$$

$$\begin{array}{rlll} \text{Salt Fingers:} & \nabla_\mu < 0, & \nabla_{\rm ad} - \nabla > 0, & R_\mu > 0 \\ \text{Ledoux stable:} & N^2 > 0, & \nabla_{\rm ad} - \nabla > |\nabla_\mu|, & R_\mu < 1 \\ \text{Ledoux unstable:} & N^2 < 0, & |\nabla_\mu| > \nabla_{\rm ad} - \nabla, & R_\mu > 1 \end{array} \tag{4.4}$$

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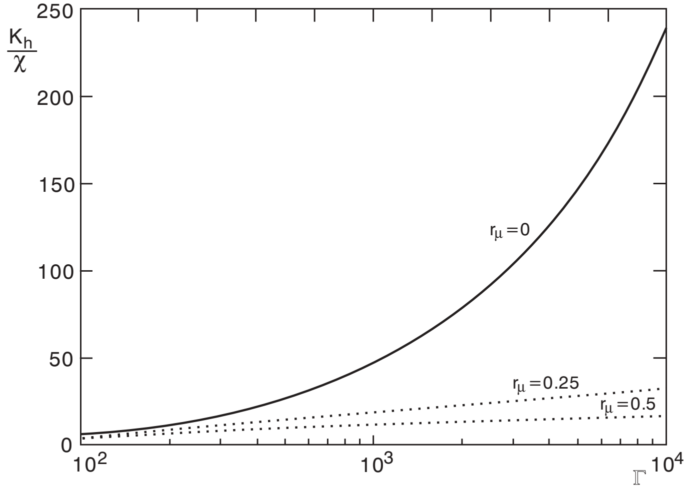

Figure 3. Semiconvection. The ratio K_h/χ vs. Γ for different values of r_μ. The $r_\mu = 0$ case corresponds to the standard local model of convection.

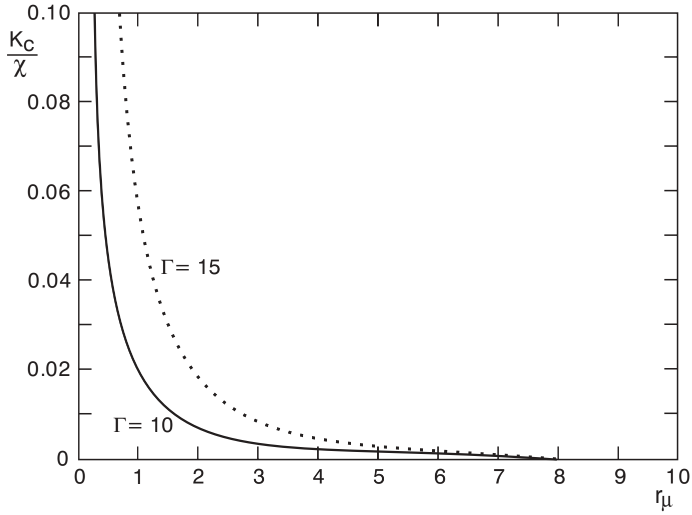

Figure 4. Semiconvection. Turbulent concentration diffusivity ($K_{\mathrm{c}} \equiv K_\mu$).

The difference between Ledoux and Schwarzschild criteria was discussed in Canuto (1999) who developed and solved an RSM that includes a) salt-fingers, b) semi-convection, and c) solid and differential rotation. Some interesting results will be discussed here. The stationary, local limit of (3.1, 3.2) takes the form:

$$gH_{\mathrm{p}}^{-1}[K_h(\nabla - \nabla_{\mathrm{ad}}) - K_\mu \nabla_\mu] = \varepsilon \tag{4.5}$$

which is the generalization of $g\alpha J = \varepsilon$ to include a μ-gradient. The algebraic expressions for the two diffusivities $K_{h,\mu}$ were derived in Canuto (1999). From (4.1, 4.2) wee see that in the presence of a μ-gradient, semi-convection acts like a sink while in the case of salt-fingers, it acts like a source. In Fig. 3 we plot the ratio K_h/χ which is now a function

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of the two parameters r_μ, Γ defined as follows:

$$r_\mu = \nabla_\mu(\nabla_{\rm r} - \nabla_{\rm ad})^{-1}, \quad \Gamma = \frac{8\pi^2}{125}[A_{\rm p}(\nabla_{\rm r} - \nabla_{\rm ad})]^{1/2}, \quad A_{\rm p} = g\Lambda^4 H_{\rm p}^{-1}\chi^{-2} \tag{4.6}$$

where Γ can be viewed as a convective efficiency (within a numerical coefficient, Λ is the same as l in (2.8)). As expected on physical grounds, in the $r_\mu = 0$ case (no semi-convection), the heat diffusivity grows quite rapidly with Γ. On the other hand, in the case of semi-convection, the growth with Γ is considerably reduced.

In Fig. 4 we plot K_μ vs. the two previous parameters. One can compare the results of Fig. 4 with the heuristic relations of Langer *et al.* (1983) and Woosley *et al.* (1999):

$$K_\mu/\chi = \frac{1}{6}\alpha_{\rm sc}(R_\mu - 1)^{-1} \tag{4.7}$$

where the efficiency factor $\alpha_{\rm sc}$ was determined to be $0.008 < \alpha_{\rm sc} < 0.05$. Salasnich *et al.* (1998) and Eggleton (1971, 1972) suggested the expressions:

$$K_\mu/\chi = \alpha_2^{-1} = (50 - 100)^{-1}, \qquad K_\mu \sim r_\mu^{-n}, \quad n > 1 \tag{4.8}$$

From Fig. 4 it is clear that the RSM can reproduce these empirical laws and actually give more information. For example, to reproduce the first of (4.8), we see from Fig. 4 that $r_\mu \sim 2 - 3$ which imposes a further constraint on the model.

5. OV and double diffusion

Quantifying the effect of DD on the extent of OV is an interesting and thus far quantitatively unexplored problem that was analyzed in Canuto (1999). The gist of the qualitative argument can be seen by considering (3.3) written as:

$$\frac{\partial K}{\partial t} + \frac{\partial F_{\rm KE}}{\partial z} = K_m\Sigma^2 + gH_{\rm p}^{-1}K_h(\nabla - \nabla_{\rm ad}) - \varepsilon_{\rm eff} \tag{5.1}$$

where:

$$\varepsilon_{\rm eff} = \varepsilon + gH_{\rm p}^{-1}K_\mu\nabla_\mu \equiv Q\varepsilon. \tag{5.2}$$

Since $\varepsilon_{\rm eff} > \varepsilon$, semi-convection enhances dissipation and causes a smaller OV extent.

6. Effect of rotation on double diffusion and the OV

Though rotation is an important factor in stellar structure and evolution, its effect on mixing is still not fully understood. A turbulent convection model must be able to incorporate rotational effects and in particular, a key feature such as the Golystin's length scale defined as (Golystin 1980):

$$l_{\rm rot} \sim (B_{\rm s} f^{-3})^{1/2} \tag{6.1}$$

where $B_{\rm s}$ is some fiducial value of buoyancy and f is the Coriolis parameter. To understand the implication of (6.1), consider the ratio of (6.1) with the pressure scale height $H_{\rm p}$. For efficient convection we have:

$$\frac{l_{\rm rot}}{H_{\rm p}} = \left(\frac{N}{f}\right)^{3/2}. \tag{6.2}$$

If $N < f$, rotational effects are important (the ratio can be viewed as a Rossby number whose small size is an indication of the importance of rotation). How does rigid rotation affect turbulence? Rotation enters the turbulence equations not only through the Coriolis term in the dynamic equations for the velocity but, more importantly, it affects the structure of the non-linear interactions that, as we have discussed, are at the heart of turbulence. In the presence of rotation, velocity components with different vectors are rotated by the Coriolis force around different axes that coincide with the directions of the corresponding wave-vectors. Thus, the energy cascade from large to small eddies is inhibited. The Kolmogorov-inertial range is still present but only for wavenumbers larger than k(rot) where the latter is the inverse of Eq. (6.1). For wavenumbers $k < k$(rot), the spectrum is no longer Kolmogorov, that is, one has two regimes:

$$l < l_{\rm rot} : E(k) \sim \varepsilon^{3/2} k^{-5/3}, \qquad l > l_{\rm rot} : E(k) \sim (\varepsilon\Omega)^{1/2} k^{-2}. \tag{6.3}$$

Integrating the two spectra, the velocities with and without rotation are given by:

$$w_{\rm rot} \sim [l/l_{\rm rot}]^{1/2} (B_{\rm s} f^{-1})^{1/2}, \qquad w_0 \sim (B_{\rm s} H)^{1/3} \tag{6.4}$$

where we have taken the dissipation rate ε equal to the surface buoyancy flux $B_{\rm s}$. We notice that the second of (6.4) is just the second of (2.15). The presence of differential rotation has different consequences. In unstably stratified regions $\nabla - \nabla_{\rm ad} > 0$, the presence of shear in the rhs of (5.1) helps enhance turbulent mixing while the presence of a positive μ-gradient increases the dissipation and thus lowers the mixing. In a stably stratified regime $\nabla - \nabla_{\rm ad} < 0$ such as the OV, all the terms in the rhs of (5.1) except the first, are negative and act like sinks. It is therefore important to have both shear (differential rotation) and the kinetic energy flux.

7. The extent of the OV region

A question that is still unanswered is the extent of the OV since, among other things, it depends on how it is defined. Though several suggestions were made, none is particularly attractive since they do not take into account the physical fact that the OV is primarily a region of *extended mass flux*. We shall therefore suggest the following picture. Consider the mean density equation:

$$\frac{\partial \overline{\rho}}{\partial t} + \overline{\mathbf{u}}_h \nabla_h \overline{\rho} + \overline{w} \frac{\partial \overline{\rho}}{\partial z} = -\rho_0 \frac{\partial J_\rho}{\partial z} \tag{7.1}$$

where $\overline{\mathbf{u}}_h, \overline{w}$ are the mean flow velocities. In the stationary case and considering only the vertical velocity, we have:

$$\overline{w} = -\frac{\rho_0}{\overline{\rho}_z} \frac{\partial J_\rho}{\partial z} = \frac{\partial K_\rho}{\partial z} + K_\rho \frac{\overline{\rho}_{zz}}{\overline{\rho}_z} \tag{7.2}$$

where we have used (3.2). We suggest to view the OV as a region of additional mass transport which ceases when the vertical mass flux velocity $\overline{w}$ vanishes, which occurs at a $z_\star$ where:

$$\frac{\partial}{\partial z} (\ln K_\rho + \ln \overline{\rho}_z) = 0, \tag{7.3}$$

a relation that has not yet been tested in a stellar code.

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8. Transport of angular momentum: the Li^7 problem

Thus far, this problem has been treated in a way that is not fully satisfactory for the following reasons. Let us consider the angular momentum equation (Elliott *et al.* 2000):

$$\frac{\partial}{\partial t}(\rho L) + \nabla \cdot (\rho \mathbf{F}_L) = 0 \tag{8.1}$$

Here, $\mathbf{F}_L$ is the vector flux of the angular momentum L and Ω_0 is the solid-body rotation:

$$F_L^r = L\overline{u}_r + r\Gamma R_{r\varphi}, \quad F_L^\theta = L\overline{u}_\theta + r\Gamma R_{\theta\varphi}, \quad L = \Gamma r^2\Omega_0 + \Gamma r\overline{u}_\varphi \tag{8.2}$$

where $\Gamma \equiv \sin\theta$. As one can see, one needs two Reynolds stresses $R_{r\varphi} = \overline{u_r u_\varphi}, R_{\theta\varphi} = \overline{u_\theta u_\varphi}$. After integration/averaging over θ, the equation that is usually considered is the following (Chaboyer and Zahn 1992; Zahn 1992; Pinsonneault *et al.* 1989):

$$\frac{\partial}{\partial t}\left(r^2\Omega\right) = r^{-2}\frac{\partial}{\partial r}\left(r^4 K_m \frac{\partial \Omega}{\partial r}\right) + \cdots \tag{8.3}$$

where the momentum diffusivity was introduced in (2.5). Though (8.3) is generally referred to as a "*diffusion equation*", it is not since the latter has the form:

$$r^{-2}\frac{\partial}{\partial r}\left[r^2 K_m \frac{\partial}{\partial r}(r^2\Omega)\right] \tag{8.4}$$

Eqs. (8.3), (8.4) give similar results only for an $\Omega(r)$ that varies with r like a power law. From helio-seismological data we have however learned that $\Omega(r)$ is truly differential in the CZ but below it, it becomes Ω = constant (Tomczyk *et al.* 1995; Canuto and Christensen-Dalsgaard 1998). In that region, also known as the *tachocline* (Spiegel and Zahn 1992), the rhs of (8.3) vanishes but (8.4) does not. To understand the origin of such different equations, let us go back to equations (8.1) and (8.2). If one employs the first of (2.5) which in this case becomes $R_{r\varphi} = -K_m\Gamma r\partial\Omega/\partial r$ one recovers (8.3). Is there anything wrong with the above form of $R_{r\varphi}$? Since the mean velocity $\overline{\mathbf{u}}$ is a vector, there are two independent tensors that represent *shear and vorticity:*

$$\Sigma_{ij} = \frac{1}{2}\left(\overline{u}_{i,j} + \overline{u}_{j,i}\right) = \text{Shear}, \qquad V_{ij} = \frac{1}{2}\left(\overline{u}_{i,j} - \overline{u}_{j,i}\right) = \text{Vorticity} \tag{8.5}$$

and therefore the Reynolds stresses must be of the form:

$$R_{ij} = f(\Sigma_{ij}, V_{ij}). \tag{8.6}$$

The form $R_{r\varphi} = -K_m\Gamma r\partial\Omega/\partial r$ corresponds to having used only shear and not vorticity, and the inclusion of both gives rise to the following angular momentum equation:

$$\frac{\partial}{\partial t}\left(r^2\Omega\right) = Ar^{-2}\frac{\partial}{\partial r}\left(r^4 K_m \frac{\partial \Omega}{\partial r}\right) + Br^{-2}\frac{\partial}{\partial r}\left(r^2 K_m \frac{\partial}{\partial r}(r^2\Omega)\right) + \cdots \tag{8.7}$$

where the coefficients A, B can only be provided by a complete model of the Reynolds stresses. The last term in Eq. (8.7) is of the form (8.4), a true diffusion and represents the first modification of Eq. (8.3). There are other modifications, however, and thus we extend (8.6) to the more general form:

$$R_{ij} = f(\Sigma_{ij}, V_{ij}; \text{buoyancy, gravity waves}; \ \underbrace{\overline{u}_r, \overline{u}_\theta}_{\text{mer.curr.}}\ ) \tag{8.8}$$

where the buoyancy flux (or mass flux) is defined as $B_i = -g\overline{\rho u_i}$ and entails the gradients

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of temperature and mean molecular weight. Finally, the presence of gravity waves can be accounted for by adding the flux $\Pi_{\rm gw}$ (Kumar *et al.* 1999) to the source in (5.1).

The complete expression (8.8) was derived in algebraic form by Canuto and Minotti (2001). For example, the new Reynolds stress $R_{r\varphi} = \overline{u_r u_\varphi}$ now reads:

$$R_{r\varphi} = A_1[\Omega_0 + \Omega(r,\theta)] + A_2\Gamma\frac{\partial\Omega}{\partial\theta} + A_3\Gamma r\frac{\partial\Omega}{\partial r} + A_4 B_{r\varphi} + E_{\rm mer} \tag{8.9}$$

This expression contrasts quite significantly with the previous expression that has only the A_3-term. It is important to stress that the second Reynolds stress $R_{\theta\varphi} = \overline{u_\theta u_\varphi}$ also exhibits a structure of the type (8.9), salient features of which are the presence of rigid rotation (first term), of the meridional currents (last term) and the buoyancy flux (last but one term) which depends on both the T and μ–fluxes thus including the effect of double-diffusion processes on the transport of angular momentum. The first term in (8.9) was accounted for in heuristic models (Rudiger 1989).

These considerations may be relevant to the important problem of Li^7 (Charbonnel 2006; Korn *et al.* 2006). The basic facts are well known: big bang nucleosynthesis predicts a Li^7 abundance that is too high compared to what is observed in the oldest stars in the galaxy. Recent measurements by Korn *et al.* (2006) suggest a solution: these old stars have destroyed part of their pristine Li^7 (Charbonnel 2006) since diffusive processes have brought Li^7 to regions hot enough to have caused its burning. In other words, turbulent mixing is deemed responsible for the discrepancy between big bang predictions and stellar observations. Ad hoc mixing models have been suggested that can explain the Li^7 data but the problem remains since the physical processes underlying the mixing (one or many) are still unclear. The inclusion of several physical processes, as formally written in (8.8), may be a good starting point to sort out which of these processes or a combination of them, is capable of explaining the new data.

9. Compressibility and magnetic fields

In the Earth's atmosphere, the height of the PBL is about 1 km while the pressure scale height is about 8 km, yielding a ratio less than unity that ensures the validity of an incompressible treatment. Quite different is the situation in stars where the convective zone may be several pressure scale heights, just the opposite of the PBL situation. This implies that compressibility effects are important (Brummel *et al.* 1996). An RSM for compressible flows has been developed (Canuto 1997). It would be quite instructive to consider the stationary and local limits of the new compressible equations and derive the compressible equivalent of the standard MLT.

As for the effect of magnetic fields on the heat transport and the possible combination of magnetic fields and rotation, the study by Canuto and Hartke (1986) leads to analytic results. Depending on the angle between the vector **H** and the z-axis, as well as on the magnetic Rayleigh number and the ratio of magnetic energy (density) to kinetic energy (density), the heat flux exhibits different dependence on the convective efficiency S defined in (2.8). In other words, the heat flux can be either enhanced or reduced depending on those parameters. In Figs. 4-17 of the reference just cited one can find a set of heat flux vs. convective efficiency S results for different cases.

10. Conclusions

We have discussed several very interesting physical problems of astrophysical nature that are still unsolved. Since LES have several advantages but not that of being a flexible tool that is easily adaptable to describe scenarios with different physical characteristics, the alternative has been the use of heuristic models that are perennialy unpredictive and almost always built case by case.

For example, the Li^7 problem beautifully stitches together cosmology with stellar structure–evolution and the depletion process can be influenced by shear, vorticity, double-diffusion, rigid rotation, differential rotation and meridional currents. It is not even thinkable to approach such a problem with a heuristic model and is at the very least quite hard to deal with that many ingredients with an LES framework. On the other hand, the RSM model offers an analytic expression for the Reynolds stresses that encompasses all these processes and can therefore be easily used in the angular momentum transport equation. The relevance of each of the above ingredients can therefore be assessed.

We also note that the advent of helio-seismology and the wealth of information that it has brought to the fore can and has been used to assess the validity of mixing models. For an assessment, the reader can consult the review article by Canuto and Christensen-Dalsgaard (1998) and several contributions to this volume.

While atmospheric and oceanic mixing problems have been treated for years with RSM, astrophysical problems by and large have not. It is the goal and the hope of this review to suggest that it may be time to forgo heuristic models since the RSM is capable of including great many physical processes in a unified and manageable way.

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