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ITERATIVE ROOTS OF TWO-DIMENSIONAL MAPPINGS

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Abstract As a weak version of embedding flow, the problem of iterative roots is studied extensively in one dimension, especially in monotone case. There are few results in high dimensions because the constructive method dealing with monotone mappings is unavailable. In this paper, by introducing a kind of partial order, we define the monotonicity for two-dimensional mappings and then present some results on the existence of iterative roots for linear mappings, triangle-type mappings, and co-triangle-type mappings, respectively. Our theorems show that even the property of monotonicity for iterative roots of monotone mappings, which is a trivial result in one dimension, does not hold anymore in high dimensions. At the end of this paper, the problem of iterative roots for two well-known planar mappings, that is, Hénon mappings and coupled logistic mappings, are also discussed.

Keywords: iterative root; two-dimensional mapping; monotonicity; partial order; differentiability

2010 Mathematics subject classification: Primary 39B12; 37B02; Secondary 37E30

1. Introduction

Given a mapping $F : X \to X$, where X is a nonempty set, a self-mapping $f : X \to X$ is called an iterative root of F of order n if

$$f^n(x) = F(x), \qquad \forall\, x \in X,$$

where $n > 1$ is an integer, f^n is nth iterate of f, defined recursively by $f^n(x) := f(f^{n-1}(x))$ and $f^0(x) := x$ for all $x \in X$.

There are plentiful results about iterative roots in one dimension [4, 7, 11, 12, 19, 21–23, 25, 29, 33, 37, 41]. For example, the famous tent mapping $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} \frac{1}{4}x, & 0 \leq x \leq \frac{1}{2}, \\ -\frac{1}{4}(x-1), & \frac{1}{2} < x \leq 1, \end{cases}$$

has a continuous square iterative root $g : [0,1] \to [0,1]$, given as

$$g(x) = \begin{cases} \frac{1}{2}x, & 0 \leq x \leq \frac{1}{2}, \\ -\frac{1}{2}(x-1), & \frac{1}{2} < x \leq 1. \end{cases}$$

However, f has no continuous square iterative roots if $f(x)$ reaches 1 at $x = \frac{1}{2}$. In particular, a full description for strictly monotone self-mappings on an arbitrary compact interval was given by Kuczma and his colleagues [23, 25]. For the case of high dimensions greater than 2, this problem becomes difficult since the celebrated method called piece by piece in monotone case does not work anymore. Up to now, only planer Sperner homeomorphisms and Brouwer homeomorphisms were considered [27, 28], which are a kind of homeomorphisms without fixed points and keeping the monotonicity of area regions under iteration. Recently, by applying the theory of polynomial algebra, a topological classification for a class of two-dimensional quadratic mappings was presented in [39]. The topological relation was used to discuss square iterative roots of preserving-orientation Hénon mappings, which is a special two-dimensional quadratic polynomial mapping. Although some regularity conditions for the roots were given there, a general investigation for the iterative roots of two-dimensional mappings is still unknown. For more results on the nonexistence of iterative roots, see references [10, 13].

Our paper is organized as follows. In § 2, we first introduce a kind of partial order and then the monotonicity for two-dimensional mappings is defined accordingly. Some properties about the monotonicity for two-dimensional mappings are discussed there. Section 3 is devoted to a linear mapping; we present necessary and sufficient conditions for the existence of its square iterative roots. In particular, we show the fact that the monotonicity for each iterative root of a strictly monotone function, does not hold in high dimensions. The regularity of iterative roots for a triangle-type mapping and a co-triangle-type mapping is considered in § 4 and § 5, separately. In § 6, we study two well-known planar mappings, that is, Hénon mappings and coupled logistic mappings. Different from the case of preserving-orientation, as considered in [39], we give a completed answer to the problem of square iterative roots for reserving-orientation Hénon mappings, and a nonexistence result of roots for the coupled logistic mappings is also given.

2. Preliminary

Let a two-dimensional mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x, y) := (u(x, y), v(x, y)), \qquad x, y \in \mathbb{R},$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$. We say that F is strictly increasing (respectively, strictly decreasing) with respect to variable x (or y) if $u(x, y), v(x, y)$ are strictly increasing (respectively, strictly decreasing) for x (or y). Further, we say that F is *strictly increasing* (respectively, *strictly decreasing*) if F is strictly increasing (respectively, strictly decreasing) with respect to each variable. Thus, F is *strictly monotone* if it is strictly increasing or strictly decreasing.

Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we say that $(x_2, y_2) \succ (x_1, y_1)$ if and only if $x_2 > x_1$ and $y_2 > y_1$. Hence, we define a relation of partial order '$\succ$' for these $(x, y) \in \mathbb{R}^2$, and the symbol '$\succ$' means monotonicity for planar mappings.

Based on the theory of monotone iterative roots with a single variable, as shown in [23, 25], we have the following general properties for strictly monotone two-dimensional mappings.

Remark 2.1. If $(x_0, y_0) \in \mathbb{R}^2$ is a unique fixed point of F, so is its iterative roots.

Actually, assume that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an iterative root of F of order $n \in \mathbb{N}$. Let $f(x_0, y_0) = (x_1, y_1)$, then

$$(x_0, y_0) = F(x_0, y_0) = f^{n-1} \circ f(x_0, y_0) = f^{n-1}(x_1, y_1).$$

Hence, $F(x_1, y_1) = f \circ f^{n-1}(x_1, y_1) = f(x_0, y_0) = (x_1, y_1)$, implying $f(x_0, y_0) = (x_1, y_1) = (x_0, y_0)$ by the uniqueness of the fixed point.

Remark 2.2. If F is strictly monotone, then F^n is also strictly monotone. If F is strictly increasing, so is F^n, where $n \in \mathbb{N}$. If F is strictly decreasing, then F^{2n+1} is strictly decreasing and F^{2n} is strictly increasing.

3. Iterative roots of linear mappings

In this section, we first consider a linear two-dimensional mapping. We begin with the following.

Proposition 3.1. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$,*

$$F(x, y) := (ax + by, cx + dy),$$

where a, b, c, and d are arbitrary fixed real numbers such that $b \neq 0$, be a given linear map, and let $f : \mathbb{R}^2 \to \mathbb{R}^2$,

$$f(x, y) := (px + qy, rx + sy),$$

where p, q, r, and $s \in \mathbb{R}$. If $a = d$, then a linear square iterative root f of F exists if and only if $a^2 - bc \geq 0$ and

$$\alpha := \max\left(\frac{a - \sqrt{a^2 - bc}}{2}, \frac{a + \sqrt{a^2 - bc}}{2}\right) > 0.$$

Proof. Assume that $f^2 = F$. Then for all $x, y \in \mathbb{R}$,

$$p(px + qy) + q(rx + sy) = ax + by, \qquad r(px + qy) + s(rx + sy) = cx + dy,$$

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which holds true if and only if

$$p^2+qr=a,\quad q(p+s)=b,\quad r(p+s)=c,\quad rq+s^2=d. \tag{3.1}$$

The remaining results in this proposition are obvious. □

Proposition 3.1 deals with the case that $b\neq 0$ in the presentation of F. If $b=0$, we get the following result.

Proposition 3.2. *Let* $F:\mathbb{R}^2\to\mathbb{R}^2$,

$$F(x,y)=(ax+by,cx+dy),$$

where a,b,c,d *are arbitrary fixed real numbers such that* $b=0$, *be a given linear map, and let* $f:\mathbb{R}^2\to\mathbb{R}^2$ *be of the form*

$$f(x,y)=(px+qy,rx+sy),$$

where $p,q,r,s\in\mathbb{R}$.
If f *is a square iterative root of* F, *then*

$$q=0 \quad or \quad p+s=0.$$

Moreover,
if $q=0$ *and* $p+s\neq 0$, *then* $f^2=F$ *if and only if* $a\geq 0$, $d\geq 0$ *and either*

$$p=\sqrt{a},\ s=\sqrt{d},\quad r=\frac{c}{\sqrt{a}+\sqrt{d}}$$

or

$$p=-\sqrt{a},\ s=-\sqrt{d},\quad r=-\frac{c}{\sqrt{a}+\sqrt{d}};$$

if $p+s=0$, *then* f *is a square iterative root of* F *if and only if one of the following cases occurs*

(i) $q=0$; *then* $a\geq 0$ *and* $(p=\sqrt{a}$ *and* $s=-\sqrt{a}$ *and* r *is arbitrary) or*

$$\left(p=\sqrt{a} \text{ and } s=-\sqrt{a} \text{ and } r \text{ is arbitrary}\right);$$

(ii) $r=0$; *then* $a\geq 0$ *and* $(p=\sqrt{a}$ *and* $s=-\sqrt{a}$ *and* q *is arbitrary) or*

$$\left(p=\sqrt{a} \text{ and } s=-\sqrt{a} \text{ and } q \text{ is arbitrary}\right);$$

(iii) $qr\neq 0$; *then either* $s=-p$ *and* $r=\frac{a-p^2}{q}$, *where* p *and* $q\neq 0$ *are arbitrary or*

$$s=-p \quad \text{and} \quad q=\frac{a-p^2}{r},\ \text{where } p \text{ and } r\neq 0 \text{ are arbitrary.}$$

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Proof. Assume that $f^2 = F$. Now the system (3.1) takes the form

$$p^2 + qr = a, \quad q(p+s) = 0, \quad r(p+s) = c, \quad rq + s^2 = d, \tag{3.2}$$

so either $q = 0$ or $p + s = 0$.
Assume first that $q = 0$. Then (3.2) reduces to

$$p^2 = a, \quad r(p+s) = c, \quad s^2 = d.$$

It follows that $a \geq 0$, $d \geq 0$ and, consequently,

$$\left(p = \sqrt{a} \ \text{ or } \ p = -\sqrt{a}\right) \quad \text{and} \quad \left(s = \sqrt{d} \ \text{ or } \ s = -\sqrt{d}\right).$$

Moreover, if $p + s \neq 0$, then $r = \frac{c}{p+s}$ and, if $p + s = 0$, then r can be arbitrary.
Assume that $p + s = 0$. Then the system (3.1) reduces to

$$p^2 + qr = a, \qquad rq + s^2 = d,$$

and the remaining results are obtained accordingly. □

Remark 3.1. According to Proposition 3.2, if $b = 0$ in the representation of F, then there are one-parameter families of the square iterative roots of the mapping F.

Remark 3.2. By Propositions 3.1-3.2, if $b \neq 0$ and $a = d$ in the representation of F, then f is a unique linear increasing (respectively, decreasing) square iterative root of F if and only if $\beta \leq 0 < \alpha$; if $b = 0$ in the representation of F, then f is a unique linear increasing (respectively, decreasing) square iterative root of F if and only if $q = 0$ and $p + s \neq 0$.

Note that the linear mapping F considered in Propositions 3.1–3.2 does not require monotonicity. It is well known in single variable case that every iterative root of a strictly monotone function is also strictly monotone (see [24, 25]). However, it is not true in high dimensions.
To show that the answer is negative, we will take the linear mapping as a counter-example and prove the following.

Theorem 3.1. *Let $b, c, d > 0$ be fixed numbers. Then for every $n \in \mathbb{N}$, $n \geq 2$, there is a real $a < 0$ such that the map $F : \mathbb{R}^2 \to \mathbb{R}^2$*

$$F(x, y) = (ax + by,\ cx + dy),$$

is not increasing and its iterates $F^2, F^3, \ldots, F^n$ are increasing.

Proof. Consider the map F where $a \in \mathbb{R}$ is arbitrary. Then, for every $n \in \mathbb{N}$, we have

$$F^n(x, y) = (a_n x + b_n y,\ c_n x + d_n y),$$

where a_n, b_n, c_n, and d_n are uniquely determined.

It is easy to verify that for $n=2$, we have

$$F^2(x,y) = (a_2(a)\,x + b_2(a)\,y, c_2(a)\,x + d_2(a)\,y),$$

where

$$\lim_{a\to 0} a_2(a) = bc > 0, \qquad \lim_{a\to 0} b_2(a) = bd > 0,$$

$$\lim_{a\to 0} c_2(a) = cd > 0, \qquad \lim_{a\to 0} d_2(a) = bc + d^2 > 0.$$

The continuity of the functions $a_2(a)$, $b_2(a)$, $c_2(a)$, and $d_2(a)$ implies that there is a $\delta > 0$ such that these functions are positive for all $a \in \mathbb{R}$ such that $|a| < \delta$. In particular, if a is negative and $|a| < \delta$, the mapping F^2 is increasing and, obviously, F is not.

Assume that for some $n \in \mathbb{N}$, $n \geq 2$, we have

$$F^n(x,y) = (a_n(a)\,x + b_n(a)\,y,\ c_n(a)\,x + d_n(a)\,y),$$

where

$$\lim_{a\to 0} a_n(a) > 0, \quad \lim_{a\to 0} b_n(a) > 0, \quad \lim_{a\to 0} c_n(a) > 0, \quad \lim_{a\to 0} d_n(a) > 0. \tag{3.3}$$

Hence, a simple calculation, we get

$$F^{n+1}(x,y) = ((aa_n + cb_n)\,x + (ba_n + db_n)\,y,\ (ac_n + cd_n)\,x + (bc_n + dd_n)).$$

Using Equation (3.3), we obtain

$$\lim_{a\to 0} a_{n+1}(a) = c \lim_{a\to 0} b_n(a) > 0, \qquad \lim_{a\to 0} b_{n+1}(a) = b \lim_{a\to 0} a_n(a) + d \lim_{a\to 0} b_n(a) > 0,$$

$$\lim_{a\to 0} c_{n+1}(a) = c \lim_{a\to 0} d_n(a) > 0, \qquad \lim_{a\to 0} d_{n+1}(a) = b \lim_{a\to 0} c_n(a) + d \lim_{a\to 0} d_n(a) > 0.$$

Then, there exists a $\delta > 0$ such that for all real a with $|a| < \delta$, the numbers $a_{n+1}(a)$, $b_{n+1}(a)$, $c_{n+1}(a)$, and $d_{n+1}(a)$ are positive. In particular, for a negative a such that $|a| < \delta$, the mapping F^n is increasing and, of course, F is not. □

4. Iterative roots of a triangle-type mapping

Let $I \subset \mathbb{R}$ be an interval and $F : I^2 \to I^2$ be called a *triangle-type* mapping if $F(x,y) = (F_1(x), F_2(x,y))$, where $F_1 : I \to I$ is a single variable and $F_2 : I^2 \to I$. Clearly, a mapping $f : I^2 \to I^2$, $f = (f_1, f_2)$, satisfies the equality $f^2 = F$ if and only if $f_1 : I \to I$,

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$f_2 : I^2 \to I$ fulfill

$$f_1^2(x) = F_1(x), \qquad x \in I$$

and

$$f_2(f_1(x), f_2(x,y)) = F_2(x,y), \qquad x, y \in I. \tag{4.1}$$

We begin with the following

Remark 4.1. A continuous function $F_2 : (0,\infty)^2 \to (0,\infty)$ is *homogeneous* if and only if

$$F_2(x,y) = xh\left(\frac{y}{x}\right), \qquad x, y > 0, \tag{4.2}$$

where $h := F_2(1,.)$. Assume that f_2 is also homogeneous, that is,

$$f_2(x,y) = xg\left(\frac{y}{x}\right), \qquad x, y > 0, \tag{4.3}$$

here $g := f_2(1,.)$. Consequently, the mappings F_2 and f_2 satisfy Equation (4.1) with $I = (0,\infty)$ if and only if

$$f_1 g\left(\frac{f_2}{f_1}\right) = f_1 g\left(\frac{xg\left(\frac{y}{x}\right)}{f_1}\right) = xh\left(\frac{y}{x}\right), \qquad x, y \in I.$$

Taking $x = 1$ and setting $c := \frac{1}{f_1(1)}$, one gets

$$g(cg(y)) = ch(y), \qquad y \in I,$$

which implies that $(cg) \circ (cg) = ch$.

Clearly, the mapping g exists provided that $h : (0,\infty) \to (0,\infty)$ is strictly increasing. Thus, we have proved the following.

Proposition 4.1. *Let $F : (0,\infty)^2 \to (0,\infty)^2$ be a triangle-type mapping, that is, $F(x,y) = (F_1(x), F_2(x,y))$, where $F_1 : (0,\infty) \to (0,\infty)$ is strictly increasing and $F_2 : (0,\infty)^2 \to (0,\infty)$ is homogeneous, as defined in Equation (4.2). Let $f : (0,\infty)^2 \to (0,\infty)^2$ and $f(x,y) = (f_1(x), f_2(x,y))$, where f_1 is an arbitrary square iterative root of F_1 and $f_2 : (0,\infty)^2 \to (0,\infty)$ is also homogeneous, given in Equation (4.2). Then f is a square*

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iterative root of F if and only if the single variable mappings g and h satisfy

$$\left(\frac{1}{f_1(1)}g\right)^2(x) = \frac{1}{f_1(1)}h(x), \qquad x \in I.$$

In what follows, we turn to consider a special triangle-type mapping, that is, the coordinate function F_2 is also a single variable, that is,

$$F(x, y) := (F_1(x), F_2(y)), \tag{4.4}$$

where $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ are continuous functions.

Theorem 4.1. *Let F be in the form of Equation (4.4). If F is strictly increasing (respectively, strictly decreasing), then F has iterative roots of any order (respectively, odd order). If F_1 is strictly increasing (respectively, decreasing) and F_2 is strictly decreasing (respectively, increasing), then F has iterative roots of any odd order.*

Proof. The proof for the first part is followed from the theory of iterative roots for strictly increasing single variable functions, as shown in [23]. Actually, by [23, Theorem 15.7] (also see [25, Theorem 11.2.2]), each function $F_i (i = 1, 2)$ has a strictly increasing iterative root $f_i : \mathbb{R} \to \mathbb{R}$ satisfying $f_i^n = F_i$, where $n \geq 1$ is an arbitrary integer. Then, define $f(x, y) := (f_1(x), f_2(y))$, we get

$$f^n(x, y) = f^{n-1} \circ f(x, y) = f^{n-1}(f_1(x), f_2(y)) = (f_1^n(x), f_2^n(y)) = (F_1(x), F_2(y)).$$

Therefore, f is an iterative root of F of any order n. Similarly, by employing the theory of iterative roots for strictly decreasing functions [23], we obtain the second result directly. □

Concerning the decreasing functions, it is known that any continuous and strictly decreasing function defined on $\mathbb{R}$ has no continuous iterative roots of even order. For high dimensions, we also get a similar result.

Theorem 4.2. *Let F be in the form of Equation (4.4), where F_2 (respectively, F_1) is strictly decreasing. If F_1 (respectively, F_2) has a unique fixed point, then F has no continuous iterative roots of even order.*

Proof. It suffices to consider iterative roots of F of order 2. For the reduction to absurdity, assume that $f(x, y) = (f_1(x, y), f_2(x, y))$ is a continuous square iterative root of F, that is,

$$f^2(x, y) = (f_1(f_1(x, y), f_2(x, y)), f_2(f_1(x, y), f_2(x, y))) = (F_1(x), F_2(y)).$$

By the monotonicity of F_2 and the assumption that F_1 has a unique fixed point, there exists a unique fixed point of F, denoted by (x_0, y_0). Then $f(x_0, y_0) = (x_0, y_0)$ by

Remark 2.1, which implies that $f_1(x_0, y_0) = x_0$ and $f_2(x_0, y_0) = y_0$. Further, using the monotonicity of F_2 again, there exists a number $\delta > 0$ such that

$$F_2(y) > y_0, \quad \forall y \in (y_0 - \delta, y_0), \qquad F_2(y) < y_0, \quad \forall y \in (y_0, y_0 + \delta). \tag{4.5}$$

Choose small enough $\delta_1 \in (0, \delta)$, so that

$$|f_1(x, y) - x_0| < \delta \quad \text{and} \quad |f_2(x, y) - y_0| < \delta$$

for all $|x - x_0| \le \delta_1$ and $|y - y_0| \le \delta_1$. Let

$$\max f(x, y) = (\max f_1(x, y), \max f_2(x, y)) = (x_1, y_1)$$

for $x \in [x_0 - \delta_1, x_0 + \delta_1]$ and $y \in [y_0 - \delta_1, y_0]$. Then, the following two cases occur:

Case (i): If $y_1 \le y_0$, then there exists small enough $\delta_2 \in (0, \delta_1)$ such that $f_1(x, y) \in [x_0 - \delta_1, x_0 + \delta_1]$ and $f_2(x, y) \in [y_0 - \delta_1, y_0]$ for $x \in [x_0 - \delta_2, x_0 + \delta_2]$, $y \in [y_0 - \delta_2, y_0]$. We further obtain

$$F_2(y) = f_2(f_1(x, y), f_2(x, y)) \le y_1 \le y_0,$$

a contradiction to the first inequality of Equation (4.5).

Case (ii): If $y_1 > y_0$, then $f_2(x, y) > y_0$ for all $x \in [x_0 - \delta_1, x_0 + \delta_1]$ and $y \in (y_0, y_1]$. Otherwise, $F_2(y)$ cannot reach y_0 for all $x \in [x_0 - \delta_2, x_0 + \delta_2]$ and $y \in [y_0 - \delta_1, y_0)$. On the other hand, the fact $f_2(x, y) > y_0$ implies that there is a number $\delta_3 \in (0, \delta_2)$ such that $f_1(x, y) \in [x_0 - \delta_1, x_0 + \delta_1]$ and $f_2(x, y) \in (y_0, y_1]$ for all $x \in [x_0 - \delta_2, x_0 + \delta_2]$, $y \in (y_0, y_0 + \delta_3]$. Hence, $F_2(y) = f_2(f_1(x, y), f_2(x, y)) > y_0$, which is a contradiction to the second inequality of Equation (4.5). Therefore, F has no continuous square iterative roots. This completes the whole proof. □

Corollary 4.1. *Let F be in the form of Equation (4.4) and strictly decreasing, then F has no continuous iterative roots of even order.*

Example 4.1. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi(x, y) := (x, -y)$. Clearly, $(0, 0)$ is a unique fixed point of φ. According to Theorems 4.1–4.2, the mapping φ has iterative roots of any odd order but no continuous iterative roots of even order.

The above arguments are concerning iterative roots in the sense of monotonicity. In the end of this section, without monotonicity, we first present a necessary condition for the existence of differentiable iterative roots. Let $C^1(\mathbb{R}, \mathbb{R})$ be the set of all differentiable functions from $\mathbb{R}$ into itself.

Theorem 4.3. *Let F be in the form of* Equation (4.4)*, where $F_1, F_2 \in C^1(\mathbb{R}, \mathbb{R})$ and $(x_0, y_0) \in \mathbb{R}^2$ is a unique fixed point of F. Assume that $f(x, y) = (f_1(x, y), f_2(x, y))$,*

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where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ *is a square differentiable iterative root of* F. *Then either*

$$\left(\frac{\partial f_1}{\partial x}(x_0,y_0)\right)^2 = F_1'(x_0), \qquad \left(\frac{\partial f_2}{\partial y}(x_0,y_0)\right)^2 = F_2'(y_0)$$

or

$$F_1'(x_0) = F_2'(y_0).$$

Proof. Since f is a square iterative root of F, we have

$$f_1(f_1(x,y), f_2(x,y)) = F_1(x), \qquad f_2(f_1(x,y), f_2(x,y)) = F_2(y). \tag{4.6}$$

Differentiating the above equations of (4.6) with respect to y and x, respectively, we obtain

$$\frac{\partial f_1}{\partial x}(f_1(x,y), f_2(x,y))\frac{\partial f_1}{\partial y}(x,y) + \frac{\partial f_1}{\partial y}(f_1(x,y), f_2(x,y))\frac{\partial f_2}{\partial y}(x,y) = 0. \tag{4.7}$$

$$\frac{\partial f_2}{\partial x}(f_1(x,y), f_2(x,y))\frac{\partial f_1}{\partial x}(x,y) + \frac{\partial f_2}{\partial y}(f_1(x,y), f_2(x,y))\frac{\partial f_2}{\partial x}(x,y) = 0. \tag{4.8}$$

Note that (x_0, y_0) is a unique fixed point of F. According to Remark 2.1, the point (x_0, y_0) is also a fixed point of f, that is, $f(x_0, y_0) = (f_1(x_0, y_0),\ f_2(x_0, y_0)) = (x_0, y_0)$. Substituting (x_0, y_0) into Equations (4.7)–(4.8), we get

$$\frac{\partial f_1}{\partial y}(x_0,y_0)\left(\frac{\partial f_1}{\partial x}(x_0,y_0) + \frac{\partial f_2}{\partial y}(x_0,y_0)\right) = 0,$$
$$\frac{\partial f_2}{\partial x}(x_0,y_0)\left(\frac{\partial f_1}{\partial x}(x_0,y_0) + \frac{\partial f_2}{\partial y}(x_0,y_0)\right) = 0.$$

Therefore, we have either

$$\frac{\partial f_1}{\partial y}(x_0,y_0) = 0, \qquad \frac{\partial f_2}{\partial x}(x_0,y_0) = 0 \tag{4.9}$$

or

$$\frac{\partial f_1}{\partial x}(x_0,y_0) = -\frac{\partial f_2}{\partial y}(x_0,y_0) \tag{4.10}$$

On the other hand, differentiating equations of (4.6) with respect to x and y, respectively, then

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$$\frac{\partial f_1}{\partial x}(f_1(x,y),f_2(x,y))\frac{\partial f_1}{\partial x}(x,y)+\frac{\partial f_1}{\partial y}(f_1(x,y),f_2(x,y))\frac{\partial f_2}{\partial x}(x,y)=F_1'(x).$$

$$\frac{\partial f_2}{\partial x}(f_1(x,y),f_2(x,y))\frac{\partial f_1}{\partial y}(x,y)+\frac{\partial f_2}{\partial y}(f_1(x,y),f_2(x,y))\frac{\partial f_2}{\partial y}(x,y)=F_2'(y).$$

With a similar discussion, we get

$$\left(\frac{\partial f_1}{\partial x}(x_0,y_0)\right)^2+\frac{\partial f_1}{\partial y}(x_0,y_0)\frac{\partial f_2}{\partial x}(x_0,y_0)=F_1'(x_0). \tag{4.11}$$

$$\frac{\partial f_2}{\partial x}(x_0,y_0)\frac{\partial f_1}{\partial y}(x_0,y_0)+\left(\frac{\partial f_2}{\partial y}(x_0,y_0)\right)^2=F_2'(y_0).$$

It follows that

$$\left(\frac{\partial f_1}{\partial x}(x_0,y_0)\right)^2-\left(\frac{\partial f_2}{\partial y}(x_0,y_0)\right)^2=F_1'(x_0)-F_2'(y_0). \tag{4.12}$$

If f satisfies Equation (4.9), then Equation (4.11) becomes

$$\left(\frac{\partial f_1}{\partial x}(x_0,y_0)\right)^2=F_1'(x_0), \qquad \left(\frac{\partial f_2}{\partial y}(x_0,y_0)\right)^2=F_2'(y_0).$$

If f satisfies Equation (4.10), then Equation (4.12) reduces to $F_1'(x_0)=F_2'(y_0)$. This completes the whole proof. □

Reconsider the mapping $\varphi(x,y)=(x,-y)$ for $(x,y)\in\mathbb{R}^2$, as discussed in Example 4.1, which is a simple but important two-dimensional mapping. Since conjugate relation preserves the dynamical properties for mappings in topological sense, we may study complicated systems via a simple one. Hence, by using the theory of polynomial algebra [34, 39], we obtain a topological classification of the mapping φ.

Theorem 4.4. *Let $F:\mathbb{R}^2\to\mathbb{R}^2$, $F(x,y):=(u_0+u_1x+u_2y, v_0+v_1x+v_2y)$, where $u_i,\ v_i\in\mathbb{R}$ for $i=0,1,2$ is a linear polynomial. Then F is topologically conjugate to $\varphi(x,y)=(x,-y)$ by a linear transformation $\Lambda(x,y):=(a_1x+a_2y, b_1x+b_2y)$, where $a_1,a_2,b_1,b_2\in\mathbb{R}$ and $a_1b_2-a_2b_1\neq 0$ if and only if $u_0=v_0=0$ and one of the following conditions holds:*

$$\text{(i) } u_1=-v_2,\ v_1=\frac{-(v_2^2-1)}{u_2}, \quad u_2\neq 0;\ \text{(ii) } u_1=-1,\ u_2=0,\ v_2=1;$$
$$\text{(iii) } u_1=1,\ u_2=0,\ v_2=-1.$$

Moreover, in case (i) $b_1=\frac{a_1(v_2+1)}{u_2}$, $b_2=\frac{a_2(v_2-1)}{u_2}$; *in case* (ii) $a_1=0$, $b_2=-\frac{a_2v_1}{2}$; *in case* (iii) $a_2=0$, $b_1=\frac{a_1v_1}{2}$.

Proof. Clearly, we have

$$\Lambda^{-1} : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \frac{b_2}{a_1b_2-a_2b_1}x + \left(-\frac{a_2}{a_1b_2-a_2b_1}\right)y \\ -\frac{b_1}{a_1b_2-a_2b_1}x + \frac{a_1}{a_1b_2-a_2b_1}y \end{bmatrix}.$$

Compute $\Lambda^{-1} \circ F \circ \Lambda$ and equate the corresponding coefficients on both sides of

$$\Lambda^{-1} \circ F \circ \Lambda = \begin{bmatrix} x \\ -y \end{bmatrix},$$

we get the following semi-algebraic system **SPS**

$$\begin{aligned}
&P_1 := a_1a_2v_1 - a_1b_2u_1 + a_2b_1v_2 - b_1b_2u_2 + a_1b_2 - a_2b_1 = 0,\\
&P_2 := a_2^2v_1 - a_2b_2u_1 + a_2b_2v_2 - b_2^2u_2 = 0,\\
&P_3 := a_1^2v_1 - a_1b_1u_1 + a_1b_1v_2 - b_1^2u_2 = 0,\\
&P_4 := a_1a_2v_1 + a_1b_2v_2 - a_2b_1u_1 - b_1b_2u_2 + a_1b_2 - a_2b_1 = 0,\\
&P_5 := a_2v_0 - b_2u_0 = 0,\\
&P_6 := a_1v_0 - b_1u_0 = 0,\\
&P_7 := a_1b_2 - a_2b_1 \neq 0.
\end{aligned}$$

In order to 'solve' semi-algebraic system **SPS**, as done in [34, 40], we only need to consider the algebraic system

$$\widetilde{\mathbf{SPS}} := \{P_1 = 0, P_2 = 0, P_3 = 0, P_4 = 0, P_5 = 0, P_6 = 0, 1 - \kappa P_7 = 0\}$$

instead, where $\kappa \in \mathbb{R}$. Based on a computation of the reduced Gröbner basis (see [6, 14]) G for ideal $\mathbf{J} := \langle P_1, \ldots, P_5, P_6, 1 - \kappa P_7\rangle$ with respect to any term order on $\mathbb{C}[a_i\,\mathrm{s}, b_i\,\mathrm{s}, u_i\,\mathrm{s}, v_i\,\mathrm{s}, \kappa]$, we conclude that G is different from $\{1\}$. Furthermore, using the Elimination Theorem (see [34]) to eliminate κ from G, we obtain the first elimination ideal $\mathbf{J}_1 := \langle Q_1, \ldots, Q_8, Q_9\rangle$, where

$$\begin{aligned}
&Q_1 := u_0;\ \ Q_2 := v_0;\ \ Q_3 := u_1 + v_2;\ \ Q_4 := u_2v_1 + v_2^2 - 1;\ \ Q_5 := a_1v_1 + b_1v_2 - b_1;\\
&Q_6 := a_2v_1 + b_2v_2 + b_2;\quad Q_7 := b_1u_2 - a_1v_2 - a_1;\quad Q_8 := b_2u_2 - a_2v_2 + a_2;\\
&Q_9 := b_1a_2v_2 - b_2a_1v_2 - b_1a_2 - b_2a_1.
\end{aligned}$$

Then, computing the minimal irreducible decomposition of the corresponding variety for $\mathbf{J}_1$, we get the conditions of conjugation between F and φ via Λ as shown in this theorem. This completes the proof. □

Therefore, combining with the discussions in Example 4.1 and Theorem 4.4, we get results of iterative roots for the following linear mappings.

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Corollary 4.2. *Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$,

$$F(x,y) := \left(ax + by, \frac{-(a^2-1)}{b}x - ay\right)$$

for $a, b \in \mathbb{R}$ *and* $b \neq 0$ *be a linear mapping. Then* F *has iterative roots of any odd order but no continuous roots of even order.*

Remark 4.2. Note that the linear mapping presented in Corollary 4.2 is different from that in Proposition 3.1. Indeed, the linear mapping considered in Proposition 3.1 requires the condition that $a = d$, while $a = -d$ in Corollary 4.2.

5. Iterative roots of a co-triangle-type mapping

Let $I \subset \mathbb{R}$ be an interval and $F : I^2 \to I^2$ be called a *co-triangle-type* mapping if $F(x,y) = (F_1(x,y), F_2(y))$, where $F_2 : I \to I$ is a single variable and $F_1 : I^2 \to I$. Clearly, mapping $f : I^2 \to I^2$, $f = (f_1, f_2)$, satisfies the equality $f^2 = F$ if and only if $f_1 : I^2 \to I$, $f_2 : I \to I$ fulfill

$$f_1(f_1(x,y), f_2(y)) = F_1(x,y), \qquad x, y \in I$$

and

$$f_2^2(y) = F_2(y), \qquad y \in I. \tag{5.1}$$

In a similar way as in Proposition 4.1, we can prove the following.

Proposition 5.1. *Let* $F : (0,\infty)^2 \to (0,\infty)^2$ *be a co-triangle-type mapping, that is,* $F(x,y) = (F_1(x,y), F_2(y))$*, where* $F_2 : (0,\infty) \to (0,\infty)$ *is strictly increasing and* $F_1 : (0,\infty)^2 \to (0,\infty)$ *is homogeneous, given by*

$$F_1(x,y) = y\varphi(\frac{x}{y}), \qquad x, y > 0.$$

Let $f : (0,\infty)^2 \to (0,\infty)^2$, $f(x,y) = (f_1(x,y), f_2(y))$*, where* f_2 *is an arbitrary square iterative root of* F_2 *and* $f_1 : (0,\infty)^2 \to (0,\infty)$ *is also homogeneous, given by*

$$f_1(x,y) = y\psi(\frac{x}{y}), \quad x, y > 0.$$

Then f *is a square iterative root of* F *if and only if the single variable mappings* φ *and* ψ *satisfy*

$$\left(\frac{1}{f_2(1)}\psi\right)^2(x) = \frac{1}{f_2(1)}\varphi(x), \qquad x \in I.$$

In what follows, we turn to consider another special mapping, that is, $F(x,y) := (F_1(y), F_2(x))$, where $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ are continuous functions with a single variable.

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Different from the mentioned mappings in § 4, the problem of iterative roots for this kind of mappings becomes very difficult since the two variables exchanged their locations.

Theorem 5.1. *Let $F:\mathbb{R}^2\to\mathbb{R}^2$, $F(x,y)=(F_1(y),F_2(x))$, where $F_1,F_2:\mathbb{R}\to\mathbb{R}$ are continuous. Assume that $f(x,y)=(f_1(y),f_2(x))$ is an iterative root of F of order $n>1$. Then n is odd and $f_1\circ F_2=F_1\circ f_2$. In particular, if $F_1,F_2:\mathbb{R}\to\mathbb{R}$ are strictly monotone, then F is an iterative root of itself of order n if and only if n is odd, and $(F_1\circ F_2)^2=id$ or $F_1\circ F_2=id$.*

Proof. Suppose that $f^n=F$. It is easy to verify that $f^{2m}(x,y)=((f_1\circ f_2)^m(x),(f_2\circ f_1)^m(y))$ and $f^{2m+1}(x,y)=((f_1\circ f_2)^m\circ f_1(y),f_2\circ(f_1\circ f_2)^m(x))$ for any integer $m>1$. Clearly, n is odd. Furthermore, we have

$$(f_1\circ f_2)^m\circ f_1=F_1\quad\text{and}\quad f_2\circ(f_1\circ f_2)^m=F_2,$$

which implies that $F_1\circ f_2=f_1\circ F_2$. For the second result, suppose that $F^n=F$, where $n=2m+1$ for some integer $m\geq 1$, we have

$$F^{2m+1}(x,y)=((F_1\circ F_2)^m\circ F_1(y),F_2\circ(F_1\circ F_2)^m(x))=(F_1(y),F_2(x)),$$

that is, $(F_1\circ F_2)^m=id$ according to the monotonicity of F_1 and F_2. Therefore, the Babbage equation indicates $(F_1\circ F_2)^2=id$ or $F_1\circ F_2=id$. □

Remark 5.1. Theorem 5.1 shows that the mapping $F(x,y)=(F_1(y),F_2(x))$ has no iterative root $f(x,y)=(f_1(y),f_2(x))$ of even order.

From the idea in the proof of Theorem 4.2, we get the following general nonexistence result of iterative roots for even order.

Corollary 5.1. *Let $F:\mathbb{R}^2\to\mathbb{R}^2$, $F(x,y)=(F_1(y),F_2(x))$, where $F_1,F_2:\mathbb{R}\to\mathbb{R}$ are continuous and F_1 (respectively, F_2) is strictly decreasing. If $(x_0,y_0)\in\mathbb{R}^2$ is a unique fixed point of F, then F has no continuous iterative roots of even order.*

Finally, by a similar discussion as Theorem 4.3, we obtain the following necessary condition for the existence of differentiable iterative roots.

Corollary 5.2. *Let $F:\mathbb{R}^2\to\mathbb{R}^2$, $F(x,y)=(F_1(y),F_2(x))$, where $F_1,F_2\in C^1(\mathbb{R},\mathbb{R})$ and $(x_0,y_0)\in\mathbb{R}^2$ is a unique fixed point of F. Assume that $f(x,y)=(f_1(x,y),f_2(x,y))$, where $f_1,f_2:\mathbb{R}^2\to\mathbb{R}$ is a square differentiable iterative root of F. Then either*

$$\frac{\partial f_1}{\partial x}(x_0,y_0)\frac{\partial f_1}{\partial y}(x_0,y_0)=\frac{1}{2}F_1'(y_0),\quad \frac{\partial f_1}{\partial x}(x_0,y_0)\frac{\partial f_2}{\partial x}(x_0,y_0)=\frac{1}{2}F_2'(x_0)$$

or

$$F_1'(y_0)=F_2'(x_0)=0.$$

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6. Two important two-dimensional mappings in natural science

In this section, we will investigate square iterative roots of two well-known planar mappings, that is Hénon mappings and coupled logistic mappings, defined on the plane.

6.1. Hénon mappings

Consider the maps $H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ with parameters $a, b \in \mathbb{R}$, defined by

$$H_{a,b} : (x, y) \mapsto (a - x^2 - y, bx),$$

which was first introduced by Hénon [20] in order to investigate the properties of strange attractors. Later, Friedland and Milnor [18] proved that every polynomial diffeomorphism is conjugate to a composition of Hénon maps or an elementary transformation. The fact shows the importance of Hénon maps for the dynamics of polynomial diffeomorphisms that are of special interests [1–3, 8, 31, 32, 36, 38]. Since the Jacobian determinant of $H_{a,b}$ is equal to b, the map is area preserving if $|b| = 1$. Further, it is called *preserving-orientation* if $b = 1$ and *reversing-orientation* if $b = -1$ [9]. Although some results about square iterative roots for preserving-orientation Hénon maps were given in [39], the description for reversing-orientation case is still open.

In this subsection, we continue to study the square iterative roots of reserving-orientation Hénon maps and solve the problem completely.

We first present an useful lemma, which was given in [24].

Lemma 6.1. *Let E be an arbitrary set and g arbitrary function on E taking values in E. Further suppose that there exist in E points $a \neq b$ such that $g(a) = b$, $g(b) = a$, and $g^2(x) = x$, which implies that either $x = a$ or $x = b$ or $g(x) = x$. Then equation $\varphi^2(x) = g(x)$, $\forall x \in E$ has no solution in E.*

According to Lemma 6.1, we have the following nonexistence results of iterative roots for reserving-orientation Hénon mappings.

Theorem 6.1. *For $b = -1$ and $a > 0$, the Hénon mappings have no square iterative roots at all.*

Proof. Since $a > 0$, choose two points $(x_1, y_1) := (\sqrt{a}, \sqrt{a})$ and $(x_2, y_2) := (-\sqrt{a}, -\sqrt{a})$. Under the condition of $b = -1$, it is easy to verify that $H_{a,-1}(x_1, y_1) = (x_2, y_2)$ and $H_{a,-1}(x_2, y_2) = (x_1, y_1)$. Moreover, the equality $H^2_{a,-1}(x, y) = (x, y)$ implies that $(x, y) \in \{(x_1, y_1), (x_2, y_2)\}$ or $H_{a,-1}(x, y) = (x, y)$. Therefore, we infer from Lemma 6.1 that the Hénon mappings $H_{a,-1}$ have no square iterative roots for $a > 0$. □

On the other hand, if $a < 0$, as indicated in [16], there are no periodic points for the reserving-orientation Hénon maps. Bera and Rocha [9] further showed that there is a fundamental domain and an area-preserving topological conjugacy between $H_{a,-1}$ and the translative mapping $T(x, y) := (x+1, -y)$. Therefore, by Theorems 4.1–4.2, we obtain the following result directly.

Corollary 6.1. *For $b = -1$ and $a < 0$, the Hénon mappings $H_{a,-1}$ have iterative roots of any odd order but no continuous iterative roots of even order.*

6.2. Coupled logistic mappings

Coupled logistic maps, one of simple nonlinear dynamical systems, have had special interest for scientists working in pure and applied mathematics, which allow to describe a huge variety of important problems in natural science [5, 17, 26, 30, 35]. The most famous logistic map in one dimension is of the form $f_a : [0,1] \to \mathbb{R}$ such that $x \mapsto ax(1-x)$ with the parameter $a > 0$. By introducing the lattices version, the dynamics of coupled logistic mappings can be described by a planar mapping $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\mathcal{L}(x, y) = ((\lambda - \beta y)x + \sigma y, \gamma y - \mu y^2) \tag{6.1}$$

for real numbers $\lambda, \beta, \sigma, \gamma, \mu$ and $\mu \neq 0$, which is a more general version considered in [15]. Before presenting the main result in this subsection, we need the following auxiliary lemma that was given in [33].

Lemma 6.2. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$. Then a necessary condition for F to have a square iterative root is that for any positive even integer $2m$, the number (if finite) of $2m$-cyclic F-orbit is even.*

According to Lemma 6.2, we get the following non-existence result of iterative roots.

Theorem 6.2. *For $\gamma > 3$ or $\gamma < -1$, the planar mapping $\mathcal{L}$ defined in Equation (6.1) has no square iterative roots at all.*

Proof. Obviously, the fixed points of mapping $\mathcal{L}$ are determined by the following equations

$$\begin{cases} (\lambda - \beta y)x + \sigma y = x, \\ \gamma y + \mu y^2 = y. \end{cases}$$

It is easy to compute that the above equation has two solutions in $\mathbb{R}^2$, that is, $(0, 0)$ and $(\sigma(\gamma - 1)/(\beta\gamma - \lambda\mu - \beta + \mu), (\gamma - 1)/\mu)$. Further, we calculate that $(-(\sigma x^* \mu\lambda - x^*\mu - \beta\gamma - \gamma\mu - \beta - \mu)/(\beta\gamma\lambda\mu + \lambda^2\mu^2 + \beta^2\gamma + \beta\lambda\mu + \beta^2 - \mu^2), \frac{x^*}{\mu})$ are 2-periodic points of $\mathcal{L}$, where x^* is a solution of equation $x^2 - (\gamma + 1)x + \gamma + 1 = 0$. Consequently, $\mathcal{L}$ has two 2-periodic points if and only if $\gamma > 3$ or $\gamma < -1$, which further implies that $\mathcal{L}$ has only one 2-cyclic orbit. Therefore, by Lemma 6.2, the mapping $\mathcal{L}$ defined in (6.1) has no square iterative roots. □

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