A BOUNDARY THEOREM FOR TSUJI FUNCTIONS

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To Professor K. NOSHIRO on his sixtieth birthday

1. Introduction. Let D denote the unit disc $|z|<1$, C the unit circle $|z|=1$ and C_r the circle $|z|=r$. Corresponding to any function $w(z)$ meromorphic in D we denote by $w^*(z)$ the spherical derivative

$$w^*(z) = \frac{|w'(z)|}{1+|w(z)|^2}.$$

We say that $w(z)$ is a Tsuji function if the spherical length of the curve $w(C_r)$ is a bounded function in $0<r<1$, in other words, provided

$$\sup_{r\to 1}\int_0^{2\pi} w^*(re^{i\theta})\, rd\theta < \infty. \tag{1}$$

A rectilinear segment S lying in D except for one endpoint $e^{i\theta}$ on C is called a segment of Julia for w, provided in each open triangle in D having one vertex at $e^{i\theta}$ and meeting S, the function w assumes all values on the Riemann sphere except possibly two. A point $e^{i\theta}$ is a Julia point for w provided each rectilinear segment lying in D except for one endpoint at $e^{i\theta}$ is a segment of Julia for w.

Corresponding to each θ and each α ($|\alpha|<\pi/2$), let $S(\theta, \alpha)$ be the segment that joins the points $e^{i\theta}$ and $(1-e^{i\alpha}\cos\alpha)e^{i\theta}$; in other words, let $S(\theta, \alpha)$ denote the chord of the circle with diameter $[0, e^{i\theta}]$ that forms a directed angle α with $[0, e^{i\theta}]$ at $e^{i\theta}$. In case $w(z)$ approaches a limit as $z\to e^{i\theta}$ on $S(\theta, \alpha)$, we denote this limit by $w(\theta, \alpha)$.

Further, let $L(\theta, \alpha)$ denote the spherical length of the image under the mapping w of $S(\theta, \alpha)$.

Tsuji [1] proved the following theorem.

For almost all θ, the radius $S(\theta, 0)$ is mapped on a rectifiable curve on the w-sphere so that a finite radial limit exists.

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More generally, there exists a set E of measure 2π on C which satisfies the following condition:

If $e^{i\theta} \in E$, then $L(\theta, \alpha)$ is an integrable function of α so that, for almost all α, $S(\theta, \alpha)$ is mapped on a rectifiable curve and the segmental limit $w(\theta, \alpha)$ exists and $w(\theta, \alpha) = w(\theta, \beta)$, $\alpha \neq \beta$, wherever both limits exist.

If α is an exceptional value for which $L(\theta, \alpha) = \infty$, then $S(\theta, \alpha)$ is a segment of Julia.

We may note that it follows from Bagemihl's ambiguous point theorem [2] that where both limits exist $w(\theta, \alpha) = w(\theta, \beta)$ for all values of θ except perhaps for a countable set.

Plainly, the set E of Tsuji's theorem contains the set $F(w)$ of the Fatou points of $w(z)$, if any such exist, although the Tsuji condition that $w(C_r)$ be of bounded spherical length does not imply that $F(w)$ is not empty.

In a previous paper [3] with G. Piranian an existence theorem was proved (Theorem 3) which showed that, given a set E of measure zero on C, a Tsuji function $w(z)$ of bounded characteristic exists for which every point of E is a Julia point. For this function, $F(w)$ is of measure 2π. Another theorem in the same paper showed the existence of Tsuji functions for which *every* point of C is a Julia point and $F(w)$ is consequently empty.

2. A number of theorems are known which show the relationship between the sets $F(w)$, or the corresponding set $\Gamma_F(w)$ of Fatou values of an analytic function and various classes of singularities of the function $w(z)$ on C defined in terms of cluster sets or omitted values. The purpose of this note is to give a theorem of this type for Tsuji functions similar to Plessner's theorem for functions meromorphic in the disc except that the set $I(w)$ of the Plessner points at which the cluster set $C_\Delta(w, e^{i\theta})$ is total, i.e. covers the Riemann sphere, for every Stolz angle Δ at $z = e^{i\theta}$, is replaced by the set $J(w)$ of the Julia points of $w(z)$. The theorem in question, which is thus stronger than Plessner's theorem, is

THEOREM 1. *If $w(z)$ is a Tsuji function, then almost all points of C are either Fatou points or Julia points.*

The key to the proof is a theorem of Kurt Meier [6] from which, in the light of Tsuji's theorem quoted above, it is an almost immediate deduction.

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To state Meier's theorem we need some notation and definitions. We denote by Π_θ the intersection $\bigcap_\alpha C_{S(\theta,\alpha)}(w, e^{i\theta})$ of all the chordal cluster sets of $w(z)$ at $e^{i\theta}$ and by Λ_θ the set of values of $w(z)$ which are taken an infinity of times in every Stolz angle at $e^{i\theta}$. Meier's theorem (Satz 1 of [6]) then states:

If $w(z)$ is meromorphic in $|z|<1$, then almost all points of C belong to at least one of the classes (a) of Fatou points, (b) of Julia points, or (c) of points such that $\Pi_\theta \cup \Lambda_\theta$ is total.

By Tsuji's theorem the Tsuji function $w(z)$ has a radial limit at almost all points of C so that Π_θ is almost everywhere either empty or consists of a single point. Hence at almost all the points in the class (c) of Meier's theorem the complement of Λ_θ with respect to the Riemann sphere contains at most one point. Therefore, almost all points of the class (c) fall into the class (b) of Julia points at which the complement of Λ_θ contains at most two points This proves Theorem 1.

3. An immediate deduction from Theorem 1 is

THEOREM 2. *If $w(z)$ is a non-constant Tsuji function such that the set $\Gamma_F(w)$ of its Fatou values is of capacity zero, then the set $J(w)$ of Julia points is of measure 2π and $F(w)$ is of first category on C.*

First, by a theorem of Privalov ([5], p. 210), cap. $\Gamma_F(w)=0$ implies that $mF(w)=0$ so that $mJ(w)=2\pi$.

Secondly, we recall that for a function meromorphic in D (and not only for a Tsuji function) the condition cap. $\Gamma_F(w)=0$ implies that every point of C is a Frostman point (i.e. a point for which $CR(w, e^{i\theta})$ is of capacity zero) and thus belongs to the set of Weierstrass points for which $C(w, e^{i\theta})$ is total [4]. From this it follows, by the maximality theorem on cluster sets ([4], Theorem 4.9, p. 79) that the set $I(w)$ is residual on C, and from this, by another theorem of Meier ([4], Theorem 8.8, p. 154), it follows that $F(w)$ is of first category on C. This does not, however, enable us to conclude that under the conditions of Theorem 2 the set $J(w)$ is residual on C. Indeed, the following example, for which I am indebted to Professor Piranian, shows that, even if the set of Fatou points of a Tsuji function is empty, the set of Julia points may be of first category. Corresponding to each natural number j, we write

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$$r_j = 1 - 2^{-j},$$
$$z_{jq} = (r_j - 1/(j+q)!) \exp (2\pi iq/2^j) \ (q = 1, \ldots, 2^j).$$

Since no two of the z_{jq} have the same modulus, we can define disks D_{jq} with centers z_{jq} such that no circle $|z| = r$ meets more than one of these disks. For appropriate constants a_{jq} the function

$$w(z) = \sum_{j,q} a_{jq}/(z - z_{jq})$$

is a Tsuji function [3, p. 247] and is bounded outside of the union of the D_{jq}. Since each Stolz angle of sufficiently large aperture contains infinitely many of the points z_{jq}, no value w_0 on the Riemann sphere can be a Fatou value of the function w. On the other hand, the set of points $e^{i\theta}$ whose symmetrically placed Stolz angle of aperture $\pi/4$ meets only finitely many of the disks D_{jq} is residual, and therefore the Julia points of w form a set of first category.

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Added in proof: A closely related theorem was proved in the author's paper Tsuji Functions with Julia Points printed in the volume Contemporary Problems in the Theory of Analytic Functions (Russian): International Conference on the Theory of Analytic Functions, Erevan 1965.