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Realizations of Regular Toroidal Maps

Dedicated to our teacher, colleague and friend Donald Coxeter

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Abstract. We determine and completely describe all pure realizations of the finite regular toroidal polyhedra of types $\{3,6\}$ and $\{6,3\}$.

1 Introduction

Regular figures and their symmetries have been studied since antiquity and from a great many points of view. Recently there has been a renewed and fresh interest in the combinatorial properties of regular figures, a renewal greatly inspired by the many contributions of Donald Coxeter. For example, his 1937 paper on regular skew polyhedra [2], concerns one of the first truly significant generalizations of the classical regular polytopes (convex or starry) and honeycombs. Also his work on regular (and chiral) maps, which we can now view as key examples of regular (respectively, chiral) abstract polyhedra, is a crucial component of *Generators and Relations for Discrete Groups* [5], written jointly with Willy Moser.

Indeed, from these investigations and from the work of many others, the notion of a combinatorially regular polytope has developed in several subtly different ways over the last thirty years. (See for example [6], [7], [8] and [13], as well as the brief history in [10, pp. 97–100]. In Section 2 we give an overview of the basic theory of regular (abstract) polytopes, referring to the survey article [10] and forthcoming monograph [13] for details.)

Although the 'abstract' side of the theory of polytopes is interesting in its own right, and does of course clarify many general properties and constructions, one is always drawn to the 'real' part of the theory. The link is provided by McMullen's work on the cone of realizations for a regular polytope (see [9], which incidentally was dedicated to Donald Coxeter on his eightieth birthday).

In this paper, we investigate the pure realizations of finite regular toroidal polyhedra (or maps) of type $\{3,6\}$ and $\{6,3\}$. Burgiel and Stanton have described elsewhere the pure realizations of these maps, essentially by examining the action of the automorphism group on a unitary space whose basis is identified with the vertex set of the map [1]. Here we take a somewhat different approach, which allows us to explicitly describe real representations of the group. In another paper [14] we have similarly dealt with the regular toroidal polyhedra of type $\{4,4\}$.

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2 Regular Toroidal Polyhedra of Types $\{3, 6\}$ and $\{6, 3\}$

An *(abstract) n-polytope* $\mathcal{P}$ is a partially ordered set with a strictly monotone rank function having range $\{-1, 0, \ldots, n\}$. An element $F \in \mathcal{P}$ with $\mathrm{rank}(F) = j$ is called a *j-face*; naturally, faces of ranks 0, 1 and $n-1$ are called *vertices*, *edges* and *facets*, respectively. We also require that $\mathcal{P}$ have two improper faces: a unique least face F_{-1} and a unique greatest face F_n. Furthermore, each maximal chain or *flag* in $\mathcal{P}$ must contain $n+2$ faces, and $\mathcal{P}$ should be strongly flag-connected. Finally $\mathcal{P}$ must have a homogeneity property: whenever $F < G$ with $\mathrm{rank}(F) = j-1$ and $\mathrm{rank}(G) = j+1$, there are exactly two j-faces H with $F < H < G$.

We shall mostly be concerned with the case $n = 3$. Thus the facets of the polyhedron $\mathcal{P}$ are 2-faces, or polygons.

The symmetry of $\mathcal{P}$ is, of course, exhibited by its *automorphism group* $\Gamma(\mathcal{P})$. In particular, $\mathcal{P}$ is *regular* if $\Gamma(\mathcal{P})$ is transitive on flags, as we henceforth assume. Now fix a base flag $\Phi = \{F_{-1}, F_0, \ldots, F_{n-1}, F_n\}$, with $\mathrm{rank}(F_j) = j$. For $0 \le j \le n-1$, there is a unique flag Φ^j differing from Φ in just the rank j face; so let ρ_j be the (unique) automorphism with $(\Phi)\rho_j = \Phi^j$. In this case, $\Gamma(\mathcal{P})$ is generated by the involutions $\rho_0, \rho_1, \ldots, \rho_{n-1}$, which satisfy at least the relations

$$(\rho_i \rho_j)^{p_{ij}} = 1, \quad 0 \le i,\ j \le n-1, \tag{1}$$

where $p_{ii} = 1, 2 \le p_{ij} \le \infty$ for $i \ne j$, and $p_{ij} = 2$ for $|i-j| \ge 2$. Furthermore, an *intersection condition* on standard subgroups holds:

$$\langle \rho_i : i \in I \rangle \cap \langle \rho_i : i \in J \rangle = \langle \rho_i : i \in I \cap J \rangle \tag{2}$$

for all $I, J \subseteq \{0, \ldots, n-1\}$. In short, $\Gamma(\mathcal{P})$ is a certain quotient of a Coxeter group with linear diagram, and we call $\Gamma(\mathcal{P})$ a *string C-group*.

Conversely, given any group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ generated by involutions and satisfying (1) and (2), one may construct a polytope $\mathcal{P}$ with $\Gamma(\mathcal{P}) = \Gamma$ (see [10, Theorem 2.9]).

As a first example, we consider the regular tessellation $\{3, 6\}$ of the plane (by congruent equilateral triangles). Indeed, $\{3, 6\}$ is an infinite regular 3-polytope; and the full symmetry group $[3, 6]$ is generated by the reflections ρ_0, ρ_1, ρ_2 indicated in Figure 1. The unit translations $\tau_x = \rho_1\rho_2\rho_1\rho_2\rho_1\rho_0$ and $\tau_y = \rho_1\tau_x\rho_1 = \rho_2\rho_1\rho_2\rho_1\rho_0\rho_1$, along lines inclined at $\pi/3$, generate an abelian subgroup of $[3, 6]$; and we may regard $\tau_x^b\tau_y^c$ as translating the origin $(0, 0)$ to the point (b, c). For a fixed pair of non-negative integers (b, c), consider the translation subgroup $\langle \tau_x^b\tau_y^c, \tau_x^{-c}\tau_y^{b+c} \rangle$, whose fundamental region is the rhombus with vertices

$$(0, 0), (b, c), (b-c, b+2c), (-c, b+c).$$

Identifying opposite edges of this rhombus, we obtain the finite toroidal polyhedron (or map) $\mathcal{P} = \{3, 6\}_{(b,c)}$, having $v = b^2 + bc + c^2$ vertices, $3v$ edges and $2v$ faces. In fact, $\mathcal{P}$ is itself a regular 3-polytope when $b \ge 2$, $c = 0$ (or vice versa), or when $b = c \ge 1$ ([5, Section 8.4]). Moreover, for $\{3, 6\}_{(b,0)}$, the automorphism group, of order $12b^2$, has the presentation

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^3 = (\rho_1\rho_2)^6 = (\rho_0\rho_2)^2 = 1$$

$$(\rho_2\rho_1\rho_0)^{2b} = 1, \tag{3}$$

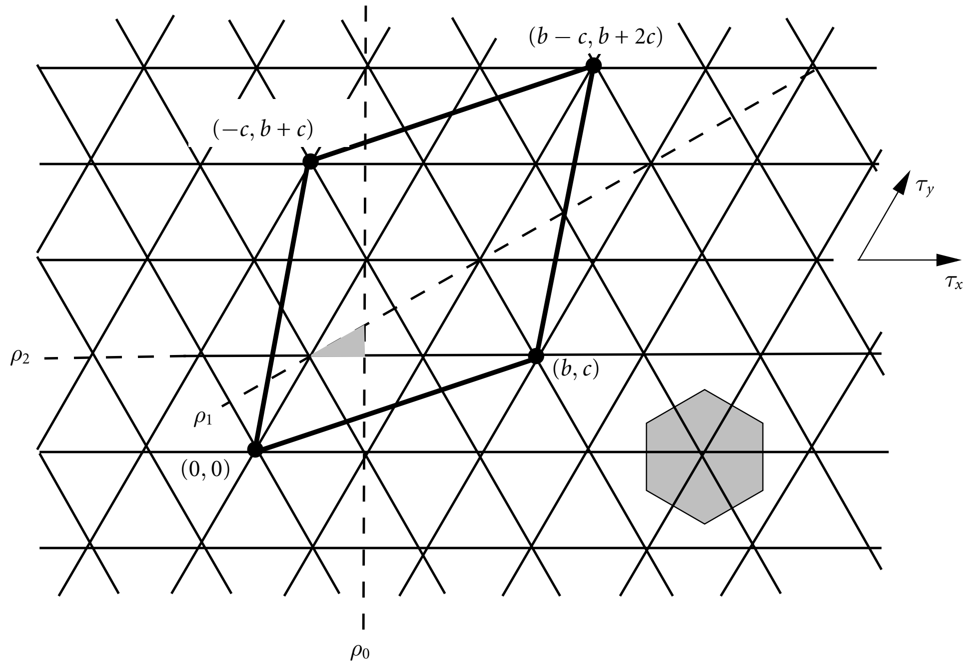

Figure 1: The tessellation $\{3, 6\}$.

([5, Section 8.6]). Similarly, the automorphism group for $\{3, 6\}_{(c,c)}$ has order $36c^2$ and the presentation

$$
\begin{gathered}
\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^3 = (\rho_1\rho_2)^6 = (\rho_0\rho_2)^2 = 1 \\
(\rho_2\rho_1\rho_2\rho_1\rho_0)^{2c} = 1.
\end{gathered} \tag{4}
$$

(For simplicity we also use ρ_j to indicate the generators of these finite groups; and it is convenient, though not geometrically accurate, to still speak of τ_x, τ_y as 'translations'.)

We next consider the dual tessellation $\{6, 3\}$, that is the tessellation of the plane by regular hexagons (one of which is indicated in Figure 1). Factoring out by the same translation subgroup as before, we thereby obtain the (topological) dual of $\{3, 6\}_{(b,c)}$, namely the toroidal map $\{6, 3\}_{(b,c)}$, with $v = b^2 + bc + c^2$ faces, $3v$ edges and $2v$ vertices. The automorphism groups for the two dual maps are isomorphic, so they are both regular (or both chiral).

When $bc(b-c) = 0$ and $b^2 + bc + c^2 \geq 3$, we may view the dual maps as dual regular abstract polyhedra. In this case, the distinguished generators ρ_0, ρ_1, ρ_2 for $\{3, 6\}_{(b,c)}$ (see (1)), are simply reversed in order so as to provide the distinguished generators for $\{6, 3\}_{(b,c)}$.

Although it is fruitful, even necessary at times, to abandon concrete geometric figures (such as a torus) when thinking of an abstract polytope $\mathcal{P}$, it is nevertheless interesting to model $\mathcal{P}$ in a natural way in Euclidean d-space E, as is done in McMullen's theory of *realizations* of $\mathcal{P}$. (We assume that $\mathcal{P}$ is finite and thus modify slightly the discussion in [10, Section 3].)

Fixing an origin $o \in E$, we consider any homomorphism

$$f\colon \Gamma(\mathcal{P}) \to O(E)$$

(into the orthogonal group). Taking $R_j := (\rho_j)f$, we define the *Wythoff space* for f as

$$W := \{p \in E : pR_j = p, 1 \leq j \leq n-1\}.$$

A *realization* $P := [f, p]$ is then defined by the homomorphism f, together with a base vertex $p \in W_P := W$.

Now consider the vertex set of $\mathcal{P}$, namely $\mathcal{P}_0 := \{F_0\gamma : \gamma \in \Gamma(\mathcal{P})\}$. Then the map

$$\beta\colon \mathcal{P}_0 \to E$$
$$F_0\gamma \mapsto p(\gamma f)$$

is well-defined, and each $\gamma \in \Gamma(\mathcal{P})$ thereby induces an isometric permutation on $V(P) := (\mathcal{P}_0)\beta$ (the vertex set of the realization). If $E' = \operatorname{aff}\big(V(P)\big)$, then the *dimension* of the realization is $\dim(P) = \dim(E')$. Note that the linear group $G(P) := \big(\Gamma(\mathcal{P})\big)f$ leaves E' invariant.

We naturally say that two realizations of $\mathcal{P}$, say $P_j = [f_j, p_j]$ in E_j, $(j = 1, 2)$, are *congruent* if there is an isometry $g\colon E_1 \to E_2$ such that $(p_1)g = p_2$ and $(\gamma f_1)g = g(\gamma f_2)$, $\forall \gamma \in \Gamma$. It is known that the congruence classes of realizations have the structure of a convex r-dimensional cone, where r is the number of diagonal classes in $\mathcal{P}$ [10, Theorem 3.8]. (A *diagonal* is an unordered pair of distinct vertices in $\mathcal{P}_0$.) If the j^{th} diagonal class is represented by $p, q_j \in V(P)$, and $||p - q_j||^2 = \delta_j$, then P is determined by the diagonal vector $\triangle(P) = (\delta_1, \ldots, \delta_r)$.

Now if $G(P)$ acts reducibly on E', then in a natural way P is congruent to a *blend* of lower dimensional realizations, say Q and R, and we write $P \equiv Q\#R$ (*cf.* [10, Section 3.1.4]). On the other hand, if this does not happen, *i.e.* if $G(P)$ acts irreducibly on E', then P is said to be a *pure* realization. The fact that diagonal vectors of pure realizations span the extreme rays in the realization cone is crucial to McMullen's proof of the fundamental numerical results outlined below.

For $v = |\mathcal{P}_0|$, let $\bar{E}$ be $(v-1)$-dimensional Euclidean space. Clearly, $\mathcal{P}$ has a *simplex realization* T in $\bar{E}$, obtained by letting $\Gamma(\mathcal{P})$ act in a natural way on the vertex set $V(T)$ of a regular simplex in $\bar{E}$. Let $\bar{w} = \dim(W_T)$.

Take d_G to be the degree and w_G to be the Wythoff space dimension for each of the (finitely many) distinct, irreducible representations G of $\Gamma(\mathcal{P})$, excluding cases with $w_G = 0$. And let $\Gamma_0 = \langle \rho_1, \ldots, \rho_{n-1} \rangle$ in $\Gamma(\mathcal{P})$.

Theorem 2.1 **([10, Section 3.2])** *With the notation above and summing over distinct irreducible representations of* $\Gamma(\mathcal{P})$*, we have:*

(a) $\sum_G w_G d_G = v - 1 = |\{\Gamma_0\sigma : \sigma \in \Gamma \setminus \Gamma_0\}|$.

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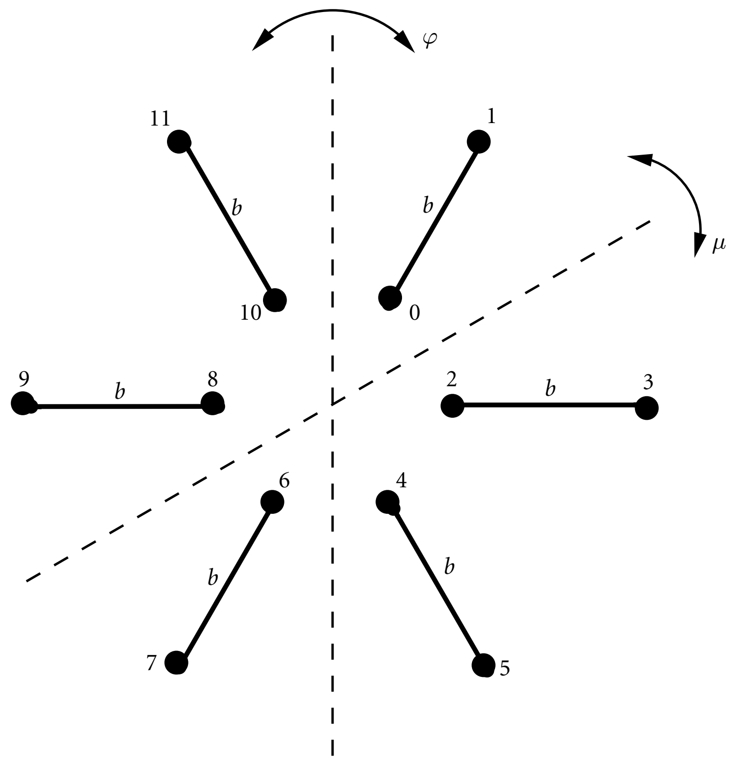

Figure 2: The group K and certain automorphisms.

(b) $\sum_G \frac{1}{2} w_G(w_G + 1) = r = |\{\Gamma_0\sigma\Gamma_0 \cup \Gamma_0\sigma^{-1}\Gamma_0 : \sigma \in \Gamma \setminus \Gamma_0\}|$.
(c) $\sum_G w_G^2 = \bar{w} = |\{\Gamma_0\sigma\Gamma_0 : \sigma \in \Gamma \setminus \Gamma_0\}|$.

These character-like results are extremely useful in classifying the full range of pure realizations for a finite regular polytope $\mathcal{P}$.

3 General Pure Realizations for $\{3,6\}_{(b,0)}$ and $\{6,3\}_{(b,0)}$

Throughout this section, $b \geq 2$ is a fixed positive integer, and $\mathcal{P} = \{3,6\}_{(b,0)}$; it is convenient to let $\mathcal{Q} = \{6,3\}_{(b,0)}$ denote the dual polyhedron $\mathcal{P}^*$. Thus $\mathcal{P}$ and $\mathcal{Q}$ both have automorphism group $\Gamma = \langle\rho_0, \rho_1, \rho_2\rangle$, whose 'translation' subgroup is generated by $\tau_x = \rho_1\rho_2\rho_1\rho_2\rho_1\rho_0$ and $\tau_y = \rho_1\tau_x\rho_1$.

We shall soon produce an exhaustive list of pure realizations for these polyhedra, essentially by mimicking our treatment of the toroidal maps of type $\{4,4\}$ in [14]. The group representations employed there were suggested—after considerable experimentation—by certain well-known 4-dimensional tori [4, Section 4.5].

Here, an analogous construction begins with the Coxeter group K whose diagram is displayed in Figure 2. (The generators $r_0, r_1, \ldots, r_{11}$ of K correspond to the nodes as labelled.

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As usual, when $b = 2$ the branches in the diagram are removed.) Thus K is a direct product of dihedral groups. We also require two outer automorphism φ, μ for K, as suggested in Figure 2: μ interchanges r_0 and r_2, *etc.*, while φ interchanges r_0 and r_{10}, r_1 and r_{11}, *etc.* By adjoining φ, μ to K we obtain a group K' of order $6(2b)^6$.

We can faithfully represent K' as a group of orthogonal transformations on the Euclidean space $E = \mathbb{R}^{12}$, endowed with the usual inner product. In fact, if $e_0, \ldots, e_{11}$ is the usual basis for E, we may suppose for $j = 0, 2, 4, 6, 8, 10$ that r_j has root e_j, and for $j = 1, 3, 5, 7, 9, 11$ that r_j has root $\cos(\pi/b)e_{j-1} + \sin(\pi/b)e_j$. Likewise, μ is the linear map interchanging e_0 and e_2, e_1 and e_3, e_4 and e_{10}, *etc.*; φ acts similarly on basis vectors.

Our search for just the right subgroups of K' was assisted by the fact that K' can also be represented as a group of monomial, unitary (or sesquilinear) transformations on $\mathbb{C}^6$. In this connection, we are indebted to the referee for noting that the unitary reflection group $[1, 1, 1]^b$ is isomorphic to a subgroup of index two in the automorphism group Γ for $\{3, 6\}_{(b,0)}$. Indeed, Coxeter anticipated the 6-dimensional realizations described below in [3, page 263]; see also [12, Section 3].

In any case, we can now define the group of main interest to us:

Definition For integers ℓ, k satisfying $0 \leq \ell, k \leq b-1$, suppose $m := -(\ell + k)$ and let $G_{\ell,k}$ be the subgroup of K' generated by

$$g_0 = \mu(r_0r_1)^\ell r_0(r_2r_3)^\ell r_2(r_4r_5)^k r_4(r_6r_7)^m r_6(r_8r_9)^m r_8(r_{10}r_{11})^k r_{10}$$
$$g_1 = \varphi r_0r_2r_4r_6r_8r_{10}$$
$$g_2 = \mu.$$

Noting that we compose linear mappings left to right, it is easy to verify that

$$g_0^2 = g_1^2 = g_2^2 = (g_0g_1)^3 = (g_1g_2)^6 = (g_0g_2)^2 = (g_2g_1g_0)^{2b} = I,$$

where I denotes the identity on E. Hence $G_{\ell,k}$ is the image of Γ under the homomorphism

$$f\colon \Gamma \to G_{\ell,k},$$

which sends each $\rho_j \to g_j$. Since $r_{j-1}r_j$ has period b (j odd), we now see why we may treat ℓ, k as residues (mod b).

We require basic 'translations' $x := (\tau_x)f = g_1g_2g_1g_2g_1g_0$ and $y = (\tau_y)f = g_2g_1g_2g_1g_0g_1$. For any integers p, q, we may write x^py^q as a 12×12 block diagonal matrix (with respect to the usual basis):

$$x^py^q = [R(pl-qk)R(pl-qm)R(pk-qm)R(pm-qk)R(pm-ql)R(pk-ql)]. \tag{5}$$

(Recall that $\ell + k + m = 0$.) Here, $R(t)$ denotes the rotation matrix

$$R(t) = \begin{bmatrix} \cos(2\pi t/b) & -\sin(2\pi t/b) \\ \sin(2\pi t/b) & \cos(2\pi/b) \end{bmatrix}.$$

Having described the crucial group $G_{\ell,k}$, we now determine its Wythoff spaces. Starting with the polyhedron $\mathcal{P} = \{3,6\}_{(b,0)}$, we find that the corresponding Wythoff space $W_{\ell,k} = \{u \in E : (u)g_1 = u = (u)g_2\}$ is one-dimensional and is spanned by

$$v := e_1 + e_3 + e_5 + e_7 + e_9 + e_{11}. \tag{6}$$

We therefore have the ingredients necessary to define a realization $P_{\ell,k} := [f, v]$ (depending on (ℓ, k)) for $\mathcal{P} = \{3,6\}_{(b,0)}$.

We are mainly interested in the action of $G_{\ell,k}$ on E', the *linear* subspace of E spanned by the vertex set $V(P_{\ell,k})$. Notice that g_0 fixes v only if $\ell = k = 0$, in which case $V(P_{0,0}) = \{v\}$, so that $P_{0,0}$ is a trivial realization for $\mathcal{P}$. Hence $\dim(P_{0,0}) = 0$, although it is still useful to consider $G_{0,0}$ as a trivial linear group of degree $\deg(G_{0,0}) = 1$ acting irreducibly on $E' = \mathbb{R}v$.

Otherwise, when $0 < \ell$ or $0 < k$, $o \in E$ is the unique point fixed by $G_{\ell,k}$ and so E' is actually the affine hull of $V(P_{\ell,k})$. In this case, $\dim(P_{\ell,k}) = \dim(E') = \deg(G_{\ell,k}) > 0$. (The dimension of the realization here coincides with the degree of the *induced* representation in E', which, as we shall soon see, is always irreducible.)

Somewhat in contrast, we find for the dual polyhedron $\mathcal{Q} = \{6,3\}_{(b,0)}$ that the Wythoff space $W^*_{\ell,k} = \{u \in E : (u)g_0 = u = (u)g_1\}$ is two-dimensional, being spanned by the (orthogonal) rows of

$$A_{\ell,k} = \begin{bmatrix} 1 & 0 & -C(\beta\ell) & S(\beta\ell) & C(\beta k) & -S(\beta k) & -C(\beta k) & -S(\beta k) & C(\beta\ell) & S(\beta\ell) & -1 & 0 \\ 0 & 1 & S(\beta\ell) & C(\beta\ell) & S(\beta k) & C(\beta k) & -S(\beta k) & C(\beta k) & -S(\beta\ell) & C(\beta\ell) & 0 & 1 \end{bmatrix}$$

where $\beta = 2\pi/b$, and (for brevity) we let $C(\theta) = \cos(\theta)$, $S(\theta) = \sin(\theta)$. Up to similarity, the essentially distinct choices of base vertices in $W^*_{\ell,k}$ are thus given by

$$u(\theta) = [\cos\theta, \sin\theta]A_{\ell,k}, \quad 0 \le \theta < \pi.$$

We thereby obtain a realization $Q_{\ell,k}(\theta) := [f, u(\theta)]$, or just $Q_{\ell,k}$, for $\mathcal{Q} = \mathcal{P}^* = \{6,3\}_{(b,0)}$.

The description of the linear subspace spanned by the vertex set $V\big(Q_{\ell,k}(\theta)\big)$ is a little more complicated, since $W^*_{\ell,k}$ may intersect non-trivially with two minimal $G_{\ell,k}$-invariant subspaces of E. Note, however, that $W_{\ell,k} \cap W^*_{\ell,k} \neq \{o\}$ only when $\ell = k = 0$. Indeed, $Q_{0,0}(\pi/2)$ is the trivial realization for $\{6,3\}_{(b,0)}$. To distinguish from the E' used in the dual case, we let $F' = F'(\theta)$ denote the linear hull of $V\big(Q_{\ell,k}(\theta)\big)$.

After first assembling some general properties of these realizations, we shall separately consider the particulars in the dual cases:

Theorem 3.1 *For any integers ℓ, k and $b \ge 2$, let $P_{\ell,k}$ and $Q_{\ell,k}(\theta)$ be the realizations described above for $\{3,6\}_{(b,0)}$ and $\{6,3\}_{(b,0)}$, respectively. Then*

(a) Each such realization is congruent, for unique (ℓ, k), to some realization satisfying

$$0 \le k \le \ell \le (b-k)/2$$

(which we henceforth assume). More specifically, even when the Wythoff space is two-dimensional (i.e. for $\mathcal{Q} = \{6,3\}_{(b,0)}$), the realizations $Q_{\ell,k}(\theta)$ are never similar for different θ satisfying $0 \le \theta < \pi$ (apart from two exceptional cases: $Q_{0,0}(\theta)$ and $Q_{\frac{b}{3},\frac{b}{3}}(\theta)$, described in Theorem 3.3 below.)

(b) Let $d = \gcd(b, \ell, k)$ *and set*

$$\epsilon = \begin{cases} 3, & \textit{if } (\ell/d) \equiv (k/d) \equiv \pm 1 \pmod 3, \textit{ and } (b/d) \equiv 0 \pmod 3; \\ 1, & \textit{otherwise} \end{cases}.$$

(i) For the realizations $P_{\ell,k}$ *of* $\{3,6\}_{(b,0)}$*, the vertex set has size* $|V(P_{\ell,k})| = \frac{b^2}{\epsilon d^2}$.

(ii) For the realizations $Q_{\ell,k}(\theta)$ *of* $\{6,3\}_{(b,0)}$*, and for any* θ*, the vertex set usually consists of two translation orbits, and so has size*

$$|V\big(Q_{\ell,k}(\theta)\big)| = \frac{2b^2}{\epsilon d^2}.$$

However, in two cases the translation orbits coalesce; indeed,

$$|V\big(Q_{0,0}(\pi/2)\big)| = 1 \quad \text{(the trivial realization);}$$
$$|V\big(Q_{\frac{b}{2},0}(\pi/2)\big)| = 4 \quad \text{(when } b \text{ is even).}$$

Proof (a) It is useful to think of (ℓ, k) as describing oblique coordinates based on the unit translations τ_x and τ_y (see Figure 1). Hence integer points (ℓ, k) describe the vertices of the tessellation $\{3, 6\}$. Regarding the group $G_{\ell,k}$, we have already observed that we may take the integers $(\ell, k) \pmod b$, so that (ℓ, k) lies in the Dirichlet fundamental region for the sublattice generated by $\{\tau_x^b, \tau_y^b\}$, namely in the regular hexagon defined by

$$\max\{|2\ell + k|, |\ell + 2k|, |\ell - k|\} \le b.$$

(See Figure 3; indeed, by suitably identifying opposite edges of this hexagon, we again obtain the map $\{3, 6\}_{(b,0)}$.)

Consider now the isometries χ, η defined on $\mathbb{R}^{12}$ by permuting the standard basis vectors according to $(0, 10)(2, 4)(6, 8)(1, 11)(3, 5)(7, 9)$ and $(0, 6)(2, 8)(4, 10)(1, 7)(3, 9)(5, 11)$, respectively. (These also correspond to automorphisms of the Coxeter diagram in Figure 3.) Conjugating $G_{\ell,k}$ by χ (resp. $\eta r_0 r_2 r_4 r_6 r_8 r_{10}$), we find that

$$P_{\ell,k} \equiv P_{k,\ell}, \quad P_{\ell,k} \equiv P_{\ell+k,-k}$$

and

$$Q_{\ell,k}(\theta) \equiv Q_{k,\ell}(\pi - \theta), \quad Q_{\ell,k}(\theta) \equiv Q_{\ell+k,-k}\big(\theta - (2\pi k/b)\big).$$

But the transformations $(\ell, k) \to (k, \ell)$ and $(\ell, k) \to (\ell + k, -k)$ define reflection symmetries for the hexagon in Figure 3, in mirrors inclined at $\pi/6$. We may therefore suppose (ℓ, k) lies in the indicated triangle, with vertices $(0, 0)$, $(b/2, 0)$ and $(b/3, b/3)$.

Finally, consider the parametrized family of realizations $Q_{\ell,k}(\theta)$. At least when $G_{\ell,k}$ acts irreducibly on the affine hull of $V\big(Q_{\ell,k}(\theta)\big)$, it follows from the general theory ([10, Section 3.1.6]), that the realizations $Q_{\ell,k}(\theta)$ are inequivalent for distinct θ, with $0 \le \theta < \pi$.

Here we can verify this directly, for example by computing the angle between the base vertex $u(\theta) \in W^*_{\ell,k}$ and its immediate neighbour $\big(u(\theta)\big)g_2$. (The angle fails to vary as desired only when $(\ell, k) = (0, 0)$ or $(b/3, b/3)$.)

(b) From (5) we see that the translation subgroup $\langle x, y\rangle$ of $G_{\ell,k}$ is isomorphic to the subgroup of $\mathbb{Z}_b^6$ generated by the rows of

$$\begin{bmatrix} \ell & \ell & k & m & m & k \\ -k & -m & -m & -k & -\ell & -\ell \end{bmatrix}. \tag{7}$$

But $\langle x, y\rangle$ acts transitively on $V(P_{\ell,k})$. After finding the elementary divisors of (7), we easily compute the stabilizer of a vertex and thus determine $|V(P_{\ell,k})|$ as claimed.

Moving to $Q_{\ell,k}(\theta)$, we similarly find that the translation orbit for each $u(\theta) \in W^*_{\ell,k}$ has this same size. But another calculation with congruences shows that the two translation orbits (containing $u(\theta)$ and $\big(u(\theta)\big)g_2$) fuse only when $(\ell, k) = (0, 0)$ or $(b/2, 0)$, with $\theta = \pi/2$. ∎

A remarkable consequence of the previous and the next theorem is that much of the data for the pure realizations $P_{\ell,k}$ is neatly encoded in a picture of the polyhedron $\{3, 6\}_{(b,0)}$ as a toroidal map. The general situation is indicated in Figure 3, where we interpret (ℓ, k) as coordinates (mod b) for a typical vertex of the map, here represented by a hexagon with opposite edges appropriately identified. Also indicated are two adjacent lines of symmetry for the hexagon. The upshot of Theorem 3.1 (a) is that

- each distinct pure realization $P_{\ell,k}$ corresponds to exactly one vertex in the right-triangular fundamental domain enclosed by the two mirrors of symmetry. (The vertices in the shaded domain are indicated by a circle or box.)

(Compare [14, Section 3].) In fact, even more is true, as we observe below for Figure 4.

Let us now consider in more detail the realizations $P_{\ell,k}$; for each of these there is an essentially unique base vertex v.

Theorem 3.2 *For integers $b \ge 2$, and $0 \le k \le \ell \le (b-k)/2$, let $P_{\ell,k}$ be the realization in E' described above for $\{3, 6\}_{(b,0)}$. Then*

(a) Each $P_{\ell,k}$ is a pure realization in E' with the properties detailed in Table 1 below.

(b) The realizations enumerated in part (a) are mutually incongruent.

(c) Every pure realization for $\{3, 6\}_{(b,0)}$ is similar to just one of the realizations listed in (a).

Proof (a) From character theory we know that a (complex) representation $f : \Gamma \to O(E)$ is irreducible if and only if its character norm equals 1 ([15, page 69]). In the present (real) case, it follows that $G_{\ell,k} = (\Gamma)f$ acts irreducibly on the 12-dimensional space E *if*

$$S := \sum_{\gamma\in\Gamma} \Big[\operatorname{trace}\big((\gamma)f\big)\Big]^2 = |\Gamma| = 12b^2. \tag{8}$$

Now the relations in (3) imply that every $g \in G_{\ell,k}$ can be written as

$$g = hx^p y^q$$

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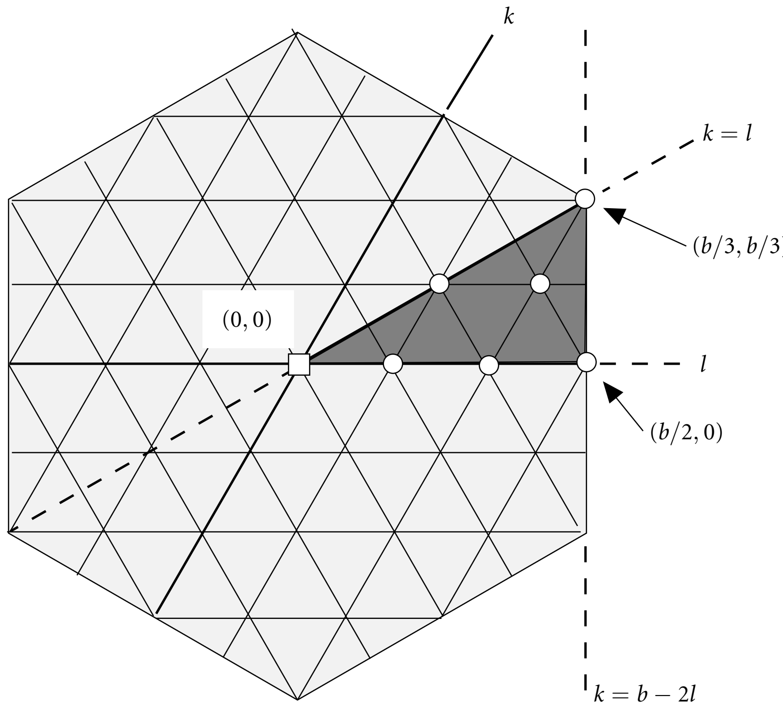

Figure 3: A fundamental region for equivalent realizations.

for some $h \in \langle g_1, g_2 \rangle$ and $0 \leq p, q \leq b-1$. It is then easily verified that *all* $g \in G_{\ell,k}$ other than 'translations' have trace 0. On the other hand, taking $\xi = \exp(2\pi i/b)$, it follows at once from (5) or (7) that

$$(9) \qquad \begin{aligned} s_{p,q} &:= \operatorname{trace}(x^p y^q) \\ &= (\xi^{p\ell-kq} + \xi^{kq-p\ell}) + (\xi^{p\ell-km} + \xi^{km-p\ell}) + \cdots + (\xi^{pk-q\ell} + \xi^{q\ell-pk}). \end{aligned}$$

Thus, with a little patience, we find that for $0 < k < \ell < (b-k)/2$, the sum in (8) reduces to

$$S = \sum_{p,q=0}^{b-1} (s_{p,q})^2 = 12b^2,$$

thereby verifying case (i) in Table 1. (Note that $\sum_{p=0}^{b-1} (\xi^j)^p = 0$ if $j \not\equiv 0 \pmod{b}$.)

For other ('boundary') values of the parameters (ℓ, k), the group $G_{\ell,k}$ does act reducibly on E. For example, in case (ii), with $0 = k < \ell < \frac{b}{2}$, we easily find by inspection that $V(P_{\ell,0})$ spans the invariant 6-dimensional subspace E' of E whose basis is

$$\{e_0 - e_6, e_1 + e_7, e_2 - e_8, e_3 + e_9, e_4 - e_{10}, e_5 + e_{11}\}.$$

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(Note that the base vertex $v \in E'$.) Furthermore, g_0, g_1 act with trace 2 on E', g_2 with trace 0, so that some non-translations in the induced representation have non-vanishing trace. Nevertheless, a similar and straightforward calculation of (8) shows that $G_{\ell,0}$ acts irreducibly on E'. Thus $P_{\ell,0}$, $0 < \ell < \frac{b}{2}$, is a pure 6-dimensional realization.

The details in the remaining cases (ii) to (vii) are similar.

(b) Given the information concerning dimension and trace, it is clear that realizations from different classes (i)–(vii) are incongruent—except perhaps for (iii) and (iv). So consider, for example, two realizations in class (i), for pairs (ℓ_1, k_1), (ℓ_2, k_2). If these realizations were congruent, corresponding translations $x^p y^q$ would have equal traces, as described in (9). Taking $(p, q) = (1, 0), (2, 0), (3, 0)$, we find that all symmetric polynomials in $\cos(2\pi\ell/b)$, $\cos(2\pi k/b)$, $\cos(2\pi m/b)$ are equal for $(\ell, k, m) = (\ell_1, k_1, m_1)$, (ℓ_2, k_2, m_2) respectively. Thus one set of cosines must actually be a permutation of the other; with the given constraints, this implies $\ell_1 = \ell_2$, $k_1 = k_2$. Cases (ii) to (vii) follow similarly and more easily. Also, we observe that realizations in cases (iii) and (iv) do—in some sense—belong to one family, but can only be congruent for identical values of ℓ. However, the parameter ℓ lies in non-overlapping intervals for the two cases.

(c) Finally, we must show that the realizations $P_{\ell,k}$ exhaust all possibilities. Referring to Theorem 2.1 (a), we recall that

$$\sum_G w_G d_G = v - 1 = b^2 - 1. \tag{10}$$

Now each non-trivial $G_{\ell,k}$ must occur somewhere in this sum over distinct, irreducible representations of $\Gamma(\mathcal{P})$, and always with $w_{G_{\ell,k}} = 1$. Furthermore, as described in Theorem 3.1, these $G_{\ell,k}$ are indexed by those vertices of $\{3, 6\}_{(b,0)}$ which lie in the fundamental domain indicated in Figure 3. Comparing part (a) of this theorem we now observe and use a remarkable fact: for each $G_{\ell,k}$ the degree $d_{G_{\ell,k}}$ equals the size of the orbit of the corresponding map vertex, under hexagonal symmetry. In short, the sum in (10) is exhausted by the (non-trivial) $G_{\ell,k}$'s. ∎

Further Remarks (see Table 1) In case (v), for b even, we find that $\{3, 6\}_{(b,0)}$ collapses onto the regular tetrahedron $\{3, 3\}$; and $G_{b/2,0}$ acts on E' as the Coxeter group $[3, 3]$.

In (vi), when $b \equiv 0 \pmod 3$, $\{3, 6\}_{(b,0)}$ collapses onto an equilateral triangle. Here g_0, g_1 act as reflections, and g_2 as the identity on the 2-space E'.

We can now readily compute the total number r of distinct pure realizations $P_{\ell,k}$; in fact, suppose $b \equiv q \pmod 6$, $0 \le q \le 5$. If $q = 0$, then $r = (b^2 + 6b + 12)/12$. Otherwise, $r = (b + q)(b - q + 6)/12$ (*cf.* [1]). The cases $b = 6$ and $b = 7$, which are typical enough, are indicated in Figure 4.

We emphasize again our observation that

• the dimension of each non-trivial pure realization $P_{\ell,k}$ equals the size of the orbit of the corresponding vertex of the map $\{3, 6\}_{(b,0)}$, under symmetries of the hexagonal diagram (with opposite edges identified). (In Figure 4, this dimension appears as a label inside each circle. The central box, which indicates the trivial realization, also fits into this scheme, if we replace dimension by degree.)

Let us next investigate further the realizations $Q_{\ell,k}(\theta)$ of the polyhedron $\{6, 3\}_{(b,0)}$. We recall that for $0 \le \theta < \pi$, the vertex set $V\big(Q_{\ell,k}(\theta)\big)$ spans the subspace $F' = F'(\theta)$ of E.

Type	(ℓ, k)	$\mathrm{Dim}(E')$ (degree)	Traces of g_0, g_1, g_2 (restricted to E')	Comments
(i) $P_{\ell,k}$	$0 < k < \ell < (b-k)/2$	12	0,0,0	The generic case.
(ii) $P_{\ell,0}$	$0 < \ell < b/2$	6	2,2,0	
(iii) $P_{\ell,\ell}$	$0 < \ell < b/3$	6	0,0,2	
(iv) $P_{\ell,b-2\ell}$	$b/3 < \ell < b/2$	6	0,0,2	
(v) $P_{b/2,0}$	b even	3	1,1,1	See remarks below.
(vi) $P_{b/3,b/3}$	$b \equiv 0 \pmod 3$	2	0,0,2	See remarks below.
(vii) $P_{0,0}$		0 (degree $= 1$)	1,1,1	The trivial realization.

Table 1: The pure realizations $P_{\ell,k}$ for $\{3,6\}_{(b,0)}$.

Theorem 3.3 *For integers $b \geq 2$, and $0 \leq k \leq \ell \leq (b-k)/2$, let $Q_{\ell,k}(\theta)$ be the realization in $F'(\theta)$ described above for $\{6,3\}_{(b,0)}$.*

(a) When the parameter θ is restricted as indicated in Table 2 below, each $Q_{\ell,k}(\theta)$ is a pure realization in $F'(\theta)$.

(b) The different types of realizations described in part (a) are mutually incongruent.

(c) Each pure realization for $\{6,3\}_{(b,0)}$ is similar to just one of the realizations described in (a).

Proof The details here are much like those in the proof of Theorem 3.2, so we mention only a few points.

We still have $G_{\ell,k}$ generated by g_0, g_1, g_2, which for realizations of $\{6,3\}_{(b,0)}$ should be distinguished in reversed order. Indeed, the Wythoff space $W^*_{\ell,k}$ now consists of points fixed by g_0, g_1. Note that $G_{\ell,k}$ still acts irreducibly on E' (the realization space for $P_{\ell,k}$); the orthogonal complement $(E')^\perp$ is also invariant. Moreover, the induced Wythoff space in any $G_{\ell,k}$-invariant subspace F must equal $W^*_{\ell,k} \cap F$. With these remarks, it is a straightforward matter to verify all the details in part (a). It is useful to view cases (v), (vi), (vii) in Table 2 as specializations of two previous 6-dimensional cases, each with its invariant subspaces. Taking intersections and orthogonal complements, it is then quite easy to determine the orthogonal decomposition of E into minimal invariant subspaces.

In case (vi), $G_{b/3,b/3}$ actually leaves invariant a second 4-dimensional subspace of E; but the corresponding realization, with $\theta = \pi/3$, is congruent to that for $\theta = -\pi/6$

Parts (b), (c) follow in essentially the same way as in Theorem 3.2. ■

Further Remarks (see Table 2) The fact that $W^*_{\ell,k}$ is 2-dimensional complicates any sort of nice diagrammatic description for the realizations $Q_{\ell,k}(\theta)$ analogous to that in Figure 4. Therefore we shall simply conclude this section with a closer look at some special cases. Recall that E' denotes the corresponding realization space for $P_{\ell,k}$.

Case (i) is 'generic'. Only here and in case (ii) is the Wythoff space 2-dimensional. In

Type (See remarks following theorem.)	(ℓ, k)	θ	$\mathrm{Dim}(F'(\theta))$ (degree)	Traces of g_0, g_1, g_2 (rest. to $F'(\theta)$)	Wythoff space dimension
(i) $Q_{\ell,k}(\theta)$	$0 < k < \ell < (b-k)/2$	$0 \leq \theta < \pi$	12	0,0,0	2
(ii) $Q_{\ell,0}(\theta)$	$0 < \ell < b/2$	$0 \leq \theta < \pi$	6	2,2,0	2
(iii) $Q_{\ell,\ell}(\frac{\pi}{2})$	$0 < \ell < b/3$	$\theta = \pi/2$	6	0,0,2	1
$Q_{\ell,\ell}(0)$	$0 < \ell < b/3$	$\theta = 0$	6	$0, 0, -2$	1
(iv) $Q_{\ell,b-2\ell}(\theta_1)$	$b/3 < \ell < b/2$	$\theta_1 = \pi/2 - (2\pi\ell)/b$	6	0,0,2	1
$Q_{\ell,b-2\ell}(\theta_2)$	$b/3 < \ell < b/2$	$\theta_2 = -2\pi\ell/b$	6	$0, 0, -2$	1
(v) $Q_{b/2,0}(0)$	b even	$\theta = 0$	3	$1, 1, -1$	1
$Q_{b/2,0}(\pi/2)$	b even	$\theta = \pi/2$	3	1,1,1	1
(vi) $Q_{b/3,b/3}(-\pi/6)$	$b \equiv 0 \pmod 3$	$\theta = -\pi/6$	4	0,0,0	1
(vii) $Q_{0,0}(0)$		$\theta = 0$	1	$1, 1, -1$	1
$Q_{0,0}(\pi/2)$		$\theta = \pi/2$	0 (degree = 1)	1,1,1	0 (trivial)

Table 2: The pure realizations $Q_{\ell,k}(\theta)$ for $\{6, 3\}_{(b,0)}$.

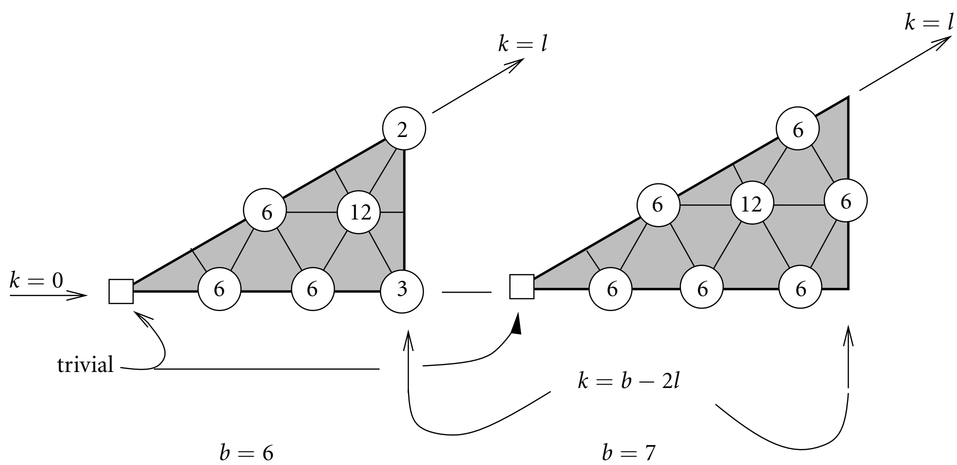

Figure 4: Pure realizations of $\{3, 6\}_{(b,0)}$ encoded in the polyhedron itself.

fact, in case (ii) we have for all θ that $F'(\theta) = E'$ (*i.e.* the same 6-dimensional realization space as for $P_{\ell,0}$).

Cases (iii) and (iv) are really different portions of one family. In case (iii), for example, there are inequivalent 6-dimensional realizations in $F'(\pi/2) = E'$ and in $F'(0) = (E')^{\perp}$, each with a 1-dimensional Wythoff space.

In (vii) we observe that besides the trivial realization, the polyhedron $\{6, 3\}_{(b,0)}$ always collapses onto the 1-polytope (or segment).

Let us turn now to the remaining realizations. In case (v), where $b = 2\ell$ is even, $\ell \geq 1$, the group $G_{\ell,0}$ acts on the 3-space E' as the Coxeter group $[3, 3]$; and the realization $P_{\ell,0}$ induces a collapse of $\mathcal{P} = \{3, 6\}_{(2\ell,0)}$ onto the regular tetrahedron $\{3, 3\}$. At the same time, the realization $Q_{\ell,0}(\pi/2)$ induces a collapse of the dual polyhedron $\mathcal{P}^* = \{6, 3\}_{(2\ell,0)}$ onto a dual regular tetrahedron $\{3, 3\}$. In a natural way $\mathcal{P}$ and $\mathcal{P}^*$ are therefore simultaneously realized by the *stella octangula* depicted in Figure 5. In a different 3-space, the same group $G_{\ell,0}$ provides the realization $Q_{\ell,0}(0)$, in which $\{6, 3\}_{(2\ell,0)}$ collapses in a natural way onto $\{6, 3\}_4$, the Petrie dual of an ordinary cube ([11, Section 4.1]); see Figure 6.

Turning finally to case (vi), where $b = 3\ell$, $\ell \geq 1$, the group $G_{\ell,\ell}$ acts on the 4-dimensional space $F' = F'(-\pi/6)$, as an irreducible linear group of order 36 generated by half-turns. From Theorem 3.1(b) we note that the corresponding realization $Q_{\ell,\ell}(-\pi/6)$ for $\{6, 3\}_{(3\ell,0)}$ has 6 vertices. In fact, these vertices belong to two congruent equilateral triangles lying in totally orthogonal planes. Each face of $\{6, 3\}_{(3\ell,0)}$ is now realized as one of the three hexagons in F' whose alternate vertices belong to the two triangles. In Figure 7 we give the most symmetrical orthogonal projection of this configuration, together with a distorted view which separates the faces.

In fact, it follows from Theorem 4.1 below that $Q_{\ell,\ell}(-\pi/6)$ can also be viewed as a realization for the regular polyhedron $\{6, 3\}_{(\ell,\ell)}$; in particular, this realization is faithful when $\ell = 1$.

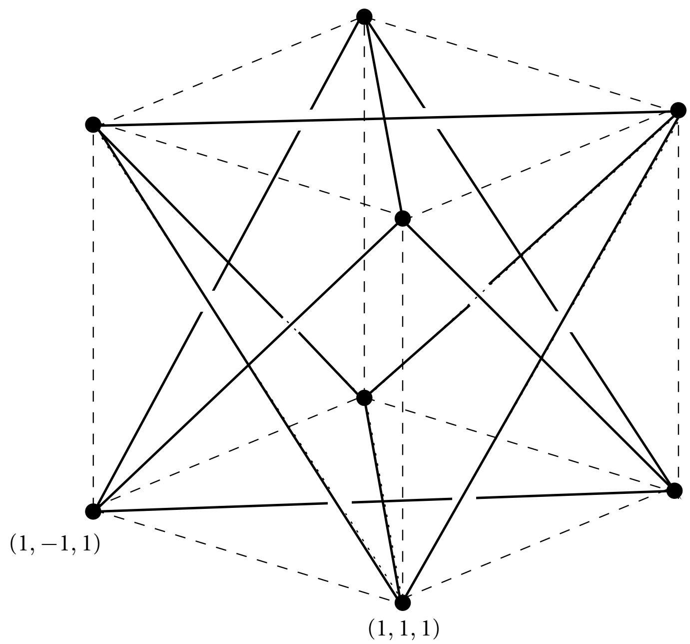

Figure 5: The *stella octangula*, a realization for the duals $\{3,6\}_{(2\ell,0)}$ and $\{6,3\}_{(2\ell,0)}$.

4 General Pure Realizations for $\{3,6\}_{(c,c)}$ and $\{6,3\}_{(c,c)}$

The key 'extra' relation in the presentation (4) for the automorphism group $\Gamma(\{3,6\}_{(c,c)})$ asserts that $\tau_y^c = \tau_x^{-c}$. But $\tau_x\tau_y = \tau_y\tau_x$, and so

$$1 = (\rho_2\tau_x^c\rho_2)(\rho_2\tau_y^c\rho_2) = \tau_x^c(\tau_x\tau_y^{-1})^c = \tau_x^{2c}\tau_y^{-c} = \tau_x^{3c}.$$

Thus there is an epimorphism

$$\Gamma(\{3,6\}_{(3c,0)}) \to \Gamma(\{3,6\}_{(c,c)}),$$

which preserves the distinguished generators, so that $\{3,6\}_{(3c,0)}$ *covers* $\{3,6\}_{(c,c)}$ ([10, Section 2.1.2]). Moreover, every pure realization of $\{3,6\}_{(c,c)}$ is therefore some pure realization $P_{\ell,k}$ of $\{3,6\}_{(3c,0)}$, by Theorem 3.2. Indeed, this happens precisely when $x^c y^c = I$ in $G_{\ell,k}$. Using the matrix (7), and the fact that $b = 3c$, we conclude that his happens exactly when $\ell \equiv k \pmod 3$. Evidently the same conclusions hold for the polyhedra $\{6,3\}_{(c,c)}$:

Theorem 4.1 *For $c \geq 1$, the pure realizations of $\{3,6\}_{(c,c)}$ are precisely those realizations $P_{\ell,k}$, with $\ell \equiv k \pmod 3$. Likewise, each pure realization of $\{6,3\}_{(c,c)}$ is of type $Q_{\ell,k}(\theta)$, with $\ell \equiv k \pmod 3$.*

■

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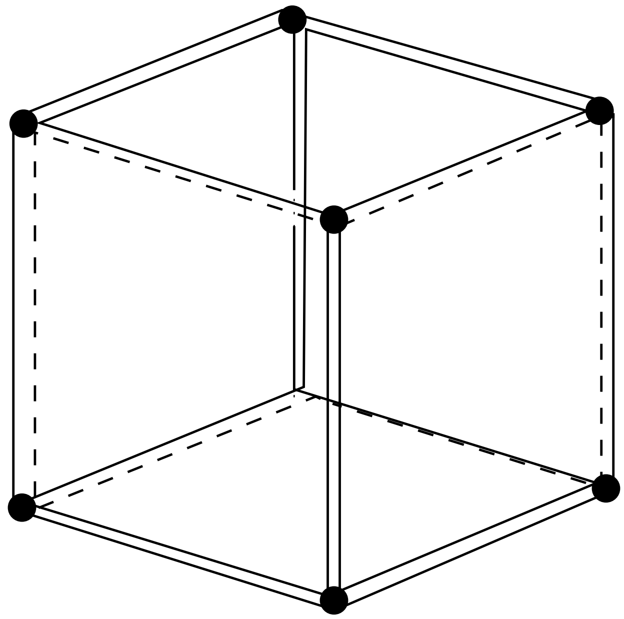


Figure 6: $\{6,3\}_{(2\ell,0)}$ realized by the Petrie dual of the cube.

Referring to Figure 4, we find that the pure realizations of $\{3,6\}_{(c,c)}$ are encoded in a diagram of the polytope $\{3,6\}_{(3c,0)}$. They correspond to the vertices of a sublattice of index 3 in the $\{3,6\}$ lattice, as indicated in Figure 8. Thus $\{3,6\}_{(4,4)}$ is the first such map to have a pure 12-dimensional realization.

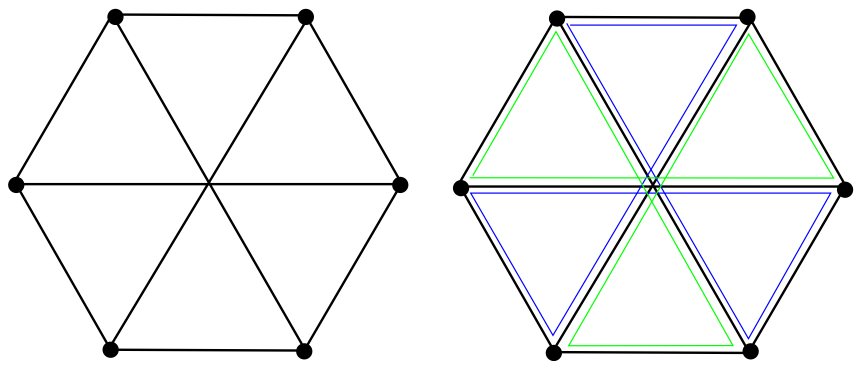
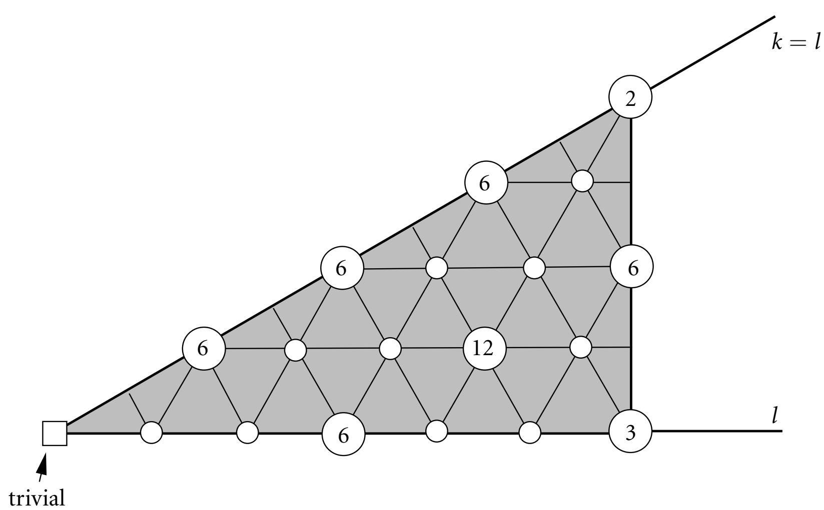


Figure 7: $\{6,3\}_{(3\ell,0)}$ realized in 4-space.



$c = 4 \quad$ (*i.e.*, $b = 12$)

Figure 8: Pure realizations of $\{3,6\}_{(c,c)}$ are encoded in the polyhedron $\{3,6\}_{(3c,0)}$.

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