symplectic geometry will find in this book a lot of useful information which has never been given in such detail before.

Here is an outline of the contents.

Part 1. Multiplication on the Tangent Bundle

(1) Introduction.

(2) Definition and first properties of F-manifolds (finite-dimensional algebras; vector bundles with multiplication; decomposition of F-manifolds; examples; potentiality).

(3) Massive F-manifolds and Lagrange maps (Lagrange property; existence of Euler fields; Lyashko–Looijenga maps and graphs of Lagrange maps; miniversality).

(4) Discriminants and modality of F-manifolds (two-dimensional F-manifolds; logarithmic vector fields; discriminants and modality of germs of F-manifolds; analytic spectrum).

(5) Singularities and Coxeter groups (hypersurface singularities; boundary singularities; Coxeter groups and F-/Frobenius manifolds; three-dimensional F-manifolds).

Part 2. Frobenius Manifolds, Gauss–Manin Connections, and Moduli Spaces for Hypersurface Singularities

(6) Introduction.

(7) Connections over the punctured plane (flat vector bundles on the punctured plane; (saturated) lattices; Riemann–Hilbert–Birkhoff problem; spectral numbers).

(8) Meromorphic connections (logarithmic vector fields and differential forms; logarithmic poles along divisors).

(9) Frobenius manifolds and second structure connections (this chapter reviews some basic properties of Frobenius manifolds).

(10) Gauss–Manin connections for hypersurface singularities (semi-universal unfoldings and F-manifolds; Gauss–Manin connections; higher residue pairings; polarized mixed Hodge structures and opposite filtrations; the Brieskorn lattice).

(11) Frobenius manifolds for hypersurface singularities.

(12) μ-constant stratum (canonical complex structure; period map and infinitesimal Torelli theorem).

(13) Moduli spaces for singularities (compatibilities, symmetries, and global moduli spaces for singularities).

(14) Variance of the spectral numbers.

S. MERKULOV

Peller, V. V. *Hankel operators and their applications* (Springer, 2003), 784 pp., 0 387 95548 8 (hardback), £63 (US$99.95).

A Hankel operator on a Hilbert space with orthonormal basis $(e_n)_{n=0}^{\infty}$ is a linear mapping T such that $\langle Te_m, e_n\rangle = a_{m+n}$ for some sequence of numbers $(a_n)_{n=0}^{\infty}$. Thus, if the operator is represented as an infinite matrix, then the entries are constant on stripes going from the bottom left to the top right—this should be contrasted with a Toeplitz matrix, where the stripes go from top left to bottom right.

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Hankel operators have been around for over 120 years: one of the earliest results in the subject is the 1881 theorem of Kronecker, characterizing the finite-rank Hankel operators. However, it is the 1957 result of Nehari characterizing the boundedness of these operators which leads to the use of the Hardy spaces of analytic functions in the unit disc, and brings out the link between the operator theory and function theory that is so important in this area.

There have already been several short texts and survey articles written about Hankel operators, and these mappings have played a key part in more general works on operator theory, function theory and control theory. However, this is the first major monograph on the subject, and it covers far more ground.

The first chapter of this book, which is introductory, already shows some of the many applications of Hankel operators. For example, they appear inevitably in the solution of classical interpolation problems associated with Nevanlinna, Pick, Carathéodory and Fejér; they are also seen in the Hamburger moment problem, and as integral operators on the half-line. The theory of Hankel operators is closely linked with work on Toeplitz operators, to which a chapter is devoted, giving their basic properties.

Beyond that, Peller has chosen a wide selection of applications, in most of which he also appears as a major contributor to the theory: over 30 of the author's own papers are cited. The celebrated Adamjan–Arov–Krein theory, which links approximation by meromorphic functions with the approximation of a Hankel operator by operators of a fixed rank, leads into topics such as membership of the Schatten–von Neumann classes, continuity of the (nonlinear) best uniform approximation mapping $\mathcal{A}$ (and the notion of a 'decent' function space, which is a Banach algebra satisfying certain axioms that guarantee continuity of $\mathcal{A}$), and then applications to rational approximation.

The book covers two important applications outside what is normally regarded as mainstream functional analysis. First, there is the theory of stationary Gaussian processes, and the intriguing notion usually expressed poetically as the 'angle between past and future', which is well known in harmonic analysis: the past is the span of a sequence of random variables $(X_j)_{j<0}$, while the future is the span of $(X_j)_{j\geqslant 0}$. It turns out that problems involving regularity of processes may be formulated in the language of Hankel operators, as explained in a complex 80-page paper of Peller and Khruschev published in 1982.

Second, Hankel operators appear throughout modern control theory, since various fundamental problems to do with designing stabilizing controllers for linear systems can be reformulated as approximation problems, which have their solution in terms of Hankel operators. Peller's treatment is not exhaustive, but he gives a good introduction to some of the main questions considered.

Other topics covered in some detail include spectral theory (and inverse spectral problems, where the idea of a realization, taken from control theory, plays an important role), Wiener–Hopf factorizations, and superoptimal approximation of analytic matrix-valued functions—it is certainly pleasing that one now has a complete account of some of these subjects in one place for the first time.

The final chapter contains one recent and unexpected application of Hankel operators. The Halmos problem from 1970 asked whether a *power-bounded* Hilbert space operator (i.e. one such that whenever p is a polynomial, $\|p(T)\|$ is dominated by the supremum of $|p(z)|$ on the unit disc, at least to within an absolute constant) is necessarily similar to a contraction. (The condition of power-boundedness is necessary because of the von Neumann inequality.) The question stimulated a great deal of research, and the answer was eventually shown to be 'no', with the first counterexamples constructed using the theory of Hankel operators.

The book has two brief appendices: the first gives the necessary background on Hilbert space operators; the second presents the function spaces (Hardy spaces, BMO, Besov spaces, etc.) that are required in the study of Hankel operators.

For reasons of space (the book is already nearly 800 pages long), the author deliberately restricts himself to 'classical' Hankel operators on the Hardy space H^2, although vector-valued spaces are included in this scheme. Thus much fine work on Hankel operators on Bergman spaces and multivariable Hardy spaces is necessarily omitted. Indeed, this lengthy book contains a huge amount of interesting material, but even so there are other major topics which one is sorry not to see here: the reviewer's own personal choice would have included the very influential L^∞ 'sum of the tail' model reduction results of Glover, the explicit formulae for Hankel singular values of delay systems obtainable by solving two-point boundary-value problems, and the recent striking link between Bonsall's theorem on boundedness of Hankel operators and the admissibility of control and observation operators for semigroups. Further applications in non-commutative geometry and perturbation theory must also be sought elsewhere. Nonetheless, this is a very clear and well-written book, which will be a major source of reference for many years to come.

J. R. PARTINGTON

Peletier, L. A. and Troy, W. C. *Spatial patterns: higher order models in physics and mechanics* (Birkhäuser, 2001), 320 pp., 0 8176 4110 6 (hardback), £47.

One of the best-understood objects in the theory of nonlinear partial differential equations is the scalar reaction–diffusion equation in one space dimension:

$$u_t = du_{xx} + f(u), \tag{1.1}$$

where either $x \in I$, a bounded interval, in which case boundary conditions would be prescribed, or on the whole real axis $\mathbb{R}$.

Work of Kolmogorov, Fife, Henry, Hale, Temam and others (see, for example, [**1**, **2**] or, for a short overview, Chapter 1 of this book) has provided decisive answers to questions such as the asymptotic behaviour of solutions; dynamics of transients; structure of the set of equilibria and, remarkably, of stable and unstable manifolds of equilibria; existence and structure of compact attractors; existence of inertial manifolds, of travelling waves, etc.

In obtaining these results, one is helped by two facts. First of all, for stationary solutions of (1.1) phase-plane techniques are available, and, secondly, the maximum principle severely constrains the possibilities of dynamical behaviour and of static patterns.

While models of the type of (1.1) crop up in a number of applications (simplified models of combustion, population dynamics, etc.), one is often naturally led to parabolic equations more complicated than (1.1). Frequently, systems of reaction–diffusion equations are encountered. In addition, many different types of consideration, such as of viscous effects, lead to higher-order parabolic equations. The book under review is, to the present reviewer's knowledge, the first monograph dedicated to the analysis of fourth-order scalar parabolic equations in one space dimension.

From the above it should be clear that analysis of this class of equations, which include the Kuramoto–Sivashinsky, the Swift–Hohenberg and the elastic beam equations (the two latter ones are briefly covered in Chapters 9 and 10), as well as the Cahn–Hilliard equations and the thin-film equations, among many others (for a review, see Chapter 1), is much more difficult than that of (1.1), since the phase space of stationary solutions is now four dimensional and the maximum principle does not apply. Hence one should not expect universally applicable tools, and at this stage analysis has to be conducted on a case-to-case basis. How this can be done in practice is shown in an admirable way in this book.

The 'canonical equation' considered in detail in the present book is

$$u_{xxxx} + qu_{xx} + f(u) = 0, \quad x \in \mathbb{R}. \tag{1.2}$$

To see how (1.2) arises, consider, for example, the extended Fisher–Kolmogorov (EFK) equation:

$$u_t = -\gamma u_{xxxx} + u_{xx} + u - u^3.$$