$$\frac{1}{2(n+1)} \leqslant \gamma - \gamma_{2n} \leqslant \frac{1}{2}\ln\left(\frac{n+1}{n}\right).$$

Remark

The estimates in the Proposition offer $2(\gamma - \gamma_{2n})$ and $2(\gamma_{2n-1} - \gamma)$ as refinements of inequalities (1). Consequently,

$$\left(1 + \frac{1}{n}\right)^n \leqslant e^{2n(\gamma_{2n-1} - \gamma)} \leqslant e \leqslant e^{2(n+1)(\gamma - \gamma_{2n})} \leqslant \left(1 + \frac{1}{n}\right)^{n+1}.$$

References

1. P. Calabrese, A note on alternating series, *Amer. Mathematical Monthly* **69** (1962) pp. 215-217.
2. M. A. Pinsky, Averaging an alternating series, *Mathematics Magazine*, **51** (1978) pp. 235-237.
3. R. M. Young, Euler's constant, *Math. Gaz.* **75** (June 1991) pp. 187-190.

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PETER R. MERCER
Dept. Mathematics,
SUNY College at Buffalo,
Buffalo NY 14222 USA
e-mail: *mercerpr@buffalostate.edu*

107.21 Proof Without Words: An inverse tangent inequality

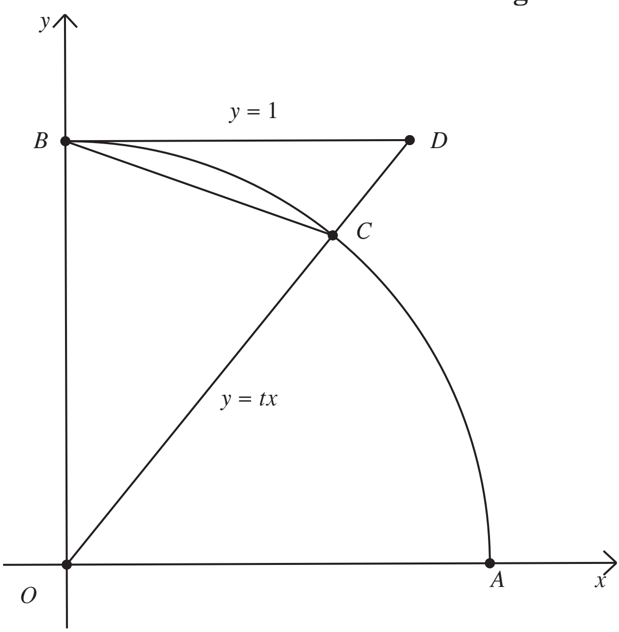

FIGURE 1

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Figure 1 shows a unit quarter circle in the first quadrant with the lines $y = 1$ and $y = tx$, where $t > 0$. Now C is the point $\left(\frac{1}{\sqrt{1 + t^2}}, \frac{t}{\sqrt{1 + t^2}}\right)$ and $\angle COA = \tan^{-1} t$. Now we have the area inequality

$$2\left[\triangle OBC\right] < 2\left[\text{Sector } OBC\right] < 2\left[\triangle OBD\right],$$

and hence

$$\frac{1}{\sqrt{1 + t^2}} < \frac{\pi}{2} - \tan^{-1} t < \frac{1}{t}.$$

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MEHDI HASSANI
Department of Mathematics,
University of Zanjan,
University Blvd., 45371-38791,
ZANJAN, IRAN
e-mail: *mehdi.hassani@znu.ac.ir*
GERRY LEVERSHA
15 Maunder Road, Hanwell,
London W7 3PN
e-mail: *g.leversha@btinternet.com*

107.22 Quick proofs of two inequalities related to the digamma function

We begin with some standard facts and notations which indicate the context in which we are working. References [1, Chapter 2] and [2, p. 334] give the assumed formula (3) and its consequence (4). Let n be a positive integer and consider the harmonic number

$$H_{n-1} = \sum_{j=1}^{n-1} \frac{1}{j}. \tag{1}$$

Recall the Euler-Mascheroni constant $\gamma = \lim_{n \to \infty} \gamma_n$, where

$$\gamma_n = \int_1^n \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x}\right) dx = \sum_{j=1}^{n-1} \left(\frac{1}{j} - \int_j^{j+1} \frac{1}{t} dt\right) = H_{n-1} - \log n. \tag{2}$$

We consider the gamma function $\Gamma(t)$ as a function of the positive real number t. We assume that the digamma function, i.e. the derivative of the log of the gamma function, is represented by the formula

$$\psi(t) = \frac{\Gamma'(t)}{\Gamma(t)} = -\gamma + \sum_{j=0}^{\infty}\left(\frac{1}{j+1} - \frac{1}{j+t}\right) = -\gamma + \sum_{j=0}^{\infty} \frac{t-1}{(j+1)(j+t)}. \tag{3}$$

This series is uniformly convergent for t bounded away from zero. It is noteworthy, as well as obvious from (3), that

$$\psi(n) = -\gamma + H_{n-1}.$$

Thus, $\psi(t)$ interpolates the sequence $-\gamma + H_{n-1}$ and $\log t$ interpolates the