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MODULES OVER HEREDITARY NOETHERIAN PRIME RINGS

SURJEET SINGH

Quasi-injective and quasi-projective modules over hereditary noetherian prime rings ((*hnp*)-rings) were studied in [**17**]. In the present paper we give some applications of the results established in [**17**]. Kulikov, Kertesz, Prufer, Szele had made basic contributions to the problem of decomposability of abelian p-groups (Fuchs [**4**]). Kaplansky [**9**] studied analogous problems for modules over (commutative) Dedekind domains. Let R be an (*hnp*)-ring, which is not right primitive. Using the structure of an indecomposable injective torsion R-module, established in [**17**, Theorem 4], some of the basic concepts and results on the decomposability of a torsion abelian group are generalized in Section 2, to modules over R. In particular, it is shown that any non-zero, torsion R-module has a uniform direct summand, which can be chosen to be of finite length if M is not injective (Theorem 10). As a consequence we get that every divisible R-module is injective (Corollary 4). This corollary is a special case of Levy [**11**, Theorem (3.4)]. Some of the analogous results for torsion modules over bounded Dedekind prime rings were established by Marubayashi [**12**; **13**]. His techniques are quite different from those used in this paper. In Section 3, quasi-projective R-modules are investigated. First of all the concept of primary, torsion R-module and primary components of a torsion R-module are introduced. The structure of quasi-projective torsion R-modules is determined in Theorems 12 through 15. These theorems generalize the structure theorem of a torsion quasi-projective, abelian groups proved by Fuchs and Rangaswamy [**5**]. Finally non-torsion quasi-projective R-modules are studied. It is shown in Theorem 16, that any such, reduced R-module has a non-zero finitely generated projective, direct summand. A result of independent interest is Theorem 15, which states that any quasi-projective finitely generated right-module over a prime, right Goldie ring S is projective, whenever it is not a torsion module. All rings considered here are with $1 \neq 0$ and all modules are unital right modules. The notations and terminology are essentially the same as in [**17**] and will be used without comments.

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2. Decomposable modules. Throughout R is an (hnp)-ring, which is not

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right primitive. The following theorem is due to Eisenbud and Griffith [**3**, Corollary (3.2)].

THEOREM 1. *Every factor ring of an hereditary noetherian prime ring is generalized uniserial.*

Since every finitely generated torsion R-module M has $\operatorname{ann}_R(M) \neq (0)$ [**17**, Lemma 2], M is a module over a generalized uniserial ring $R/\operatorname{ann}_R(M)$ and hence M is a direct sum of finitely many uniserial modules [**3**, Proposition (1.1)]. So we obtain:

LEMMA 1. *Any finitely generated torsion R-module is a direct sum of finitely many uniserial modules.*

The following theorem was established in [**17**, Theorem 4].

THEOREM 2. *Let E be an indecomposable, injective, torsion module over an (hnp)-ring R, which is not right primitive. Then in E, there exists an infinite properly ascending chain of submodules:*

$$(1) \qquad 0 = x_0R < x_1R < \ldots . < x_nR < \ldots . < E$$

such that all $x_{i+1}R/x_iR$ are simple modules: the members of the chain are the only submodules of E. Further either all the factor modules $x_{i+1}R/x_iR$ are pairwise non-isomorphic or there exists a positive integer n such that $x_{i+1}R/x_iR \cong x_{j+1}R/x_jR$ if and only if $i \equiv j \pmod{n}$. If such an n exists, it is called the periodicity of E; otherwise E is said to be of periodicity zero.

The series (1) is called the composition series of E. The following lemma is immediate from Theorem 2.

LEMMA 2. (a) *Any uniform torsion R-module is either of finite length and uniserial or is injective and of infinite length.*

(b) *Let U and V be two uniform, torsion right R-modules, and $b(\neq 0) \in U$. If $\sigma : bR \rightarrow V$ is a non-zero R-homomorphism and length, $d(U/bR) \leqq d(V/\sigma(bR))$, then σ can be extended to an R-homomorphism $\eta : U \rightarrow V$ and $U/bR \cong \eta(U)/\eta(bR)$.*

(c) *Any non-zero homomorphic image of a uniform, torsion R-module is uniform.*

Let M be a torsion R-module. Given $x \in M$, x is called a uniform element if the submodule xR is a non-zero uniform R-module. A uniform element x in M is said to be of exponent t, (denoted by $e(x)$) if the length $d(xR) = t$; the supremum of all $d(T/xR)$, where T is a uniform submodule of M containing x, is called the height of x, which is denoted by $H(x)$. A torsion right R-module M is said to be bounded if there exists a positive integer k such that $H(x) \leqq k$ for all uniform elements x of M. Since in an abelian p-group G (p a prime number) any $x(\neq 0) \in G$ generates a finite cyclic subgroup which is uniform as Z-module, the above definitions effectively generalize the concepts of exponent and height of an element in an abelian p-group [**4**, p. 16]. We now prove some lemmas which play central role in this paper.

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LEMMA 3. *Let M be a torsion R-module and $B_1, B_2, \ldots, B_n, A_1, A_2, \ldots, A_m$ be finitely many finite length uniform submodules of M such that $\sum_{i=1}^m A_i = \oplus \sum_{i=1}^m A_i$* and $\sum_{j=1}^n B_j = \oplus \sum_{i=1}^m A_i$. *Then:*

(i) *Every B_j is isomorphic to a submodule of some A_i under the natural projection.*

(ii) *For any i, A_i is a homomorphic image of some B_j.*

Proof. (i) Since $B_j \subset \oplus \sum_{i=1}^m A_i$, intersection of kernels of all natural projections of B_j into various A_i is zero. As by Lemma 2, B_j is uniserial, one of the natural projections of B_j into an A_i, will itself have its kernel zero. So (i) follows.

(ii) Let $p_i : \oplus \sum_{i=1}^m A_i \to A_i$ be the natural projection. Then

$$A_i = p_i(\Sigma_j B_j) = \Sigma_j p_i(B_j)$$

and the fact that A_i is uniserial, implies that $A_i = p_i(B_j)$ for some j.

LEMMA 4. *Let $x_1, x_2, \ldots, x_n$ be finitely many uniform elements of a torsion R-module M such that for some non-negative integer k, $H(x_i) \geqq k$ for all i. Then for every uniform element x of M in $\Sigma_i x_i R$, $H(x) \geqq k$.*

Proof. By the hypothesis there exist uniserial submodules T_i of M containing x_i such that $d(T_i/x_iR) \geqq k$. Now by Lemma 1, $\sum_{i=1}^n T_i = \oplus \sum_{j=1}^m U_j$, for some uniserial R-submodules U_j's of M. Let x in $\Sigma_i x_i R$ be uniform. Then $x = \Sigma_i y_i$; $y_i \in x_i R$. If for some i, $y_i \neq 0$, the definition of height yields that $H(y_i) \geqq H(x_i) \geqq k$. Let $p_j : \oplus \sum U_j \to U_j$ be projections. Consider any non-zero y_i. Suppose for some value t of j, $p_t(y_i) \neq 0$. If p_{it} is the restriction of p_t to T_i, then $y_i \notin \ker p_{it}$ and hence $\ker p_{it} < y_iR$, since T_i is uniserial. Hence

$$T_i/y_iR \cong p_t(T_i)/p_t(y_i)R \subseteq U_t/p_t(y_i)R$$

yields $d(U_t/p_t(y_i)R) \geqq k$, since $d(T_i/y_iR) \geqq k$. Now $x = \Sigma_j u_j$; $u_j \in U_j$. Then $u_j = \Sigma_i p_j(y_i)$ for all j. If for some j, $u_j \neq 0$, then $p_j(y_i) \neq 0$ for some i; then as U_j is uniserial, $\Sigma_i p_j(y_i)R = p_j(y_s)R \neq (0)$ for some s. Consequently $u_j \in \Sigma_i p_j(y_i)R$ yields that $d(U_j/u_jR) \geqq d(U_j/p_j(y_s)R) \geqq k$. Hence whenever $u_j \neq 0$, u_j is a uniform element with $H(u_j) \geqq k$ and $d(U_j/u_jR) \geqq k$.

By Lemma 3, there exists t such that $xR \cong u_tR$ under the correspondence $xr \leftrightarrow u_tr$; $r \in R$. Then for each j, the mapping $\sigma_j : u_tR \to u_jR$ given by $\sigma_j(u_tr) = u_jr$ is a well defined R-epimorphism. Since $d(U_t/u_tR) \geqq k$, we can find some $v_t \in U_t$ such that $d(v_tR/u_tR) = k$. If for some j, $u_j \neq 0$, as $d(U_j/u_jR) \geqq k$, by Lemma 2, σ_j can be extended to an R-homomorphism $\eta_j : v_tR \to U_j$ and $v_tR/u_tR \cong \eta_j(v_t)R/u_jR$. Put $v_j = \eta_j(v_t)$. Consider $y = z_1 + z_2 + \ldots + z_m$ such that for any j, $z_j = v_j$ if $u_j \neq 0$ and $z_j = 0$ if $u_j = 0$. Then $yR \cong v_tR$ under the natural correspondence. Now $u_t = v_ta$ for some $a \in R$. Then the construction yields $u_j = v_ja$ for all $u_j \neq 0$. Hence $x = ya \in yR$. Hence yR is a uniserial submodule of M such that $d(yR/xR) = d(v_tR/u_tR) = k$. Hence $H(x) \geqq k$. This proves the lemma.

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For any torsion R-module M and for any non-negative integer k, let $H_k(M)$ be the submodule of M generated by all those uniform elements of M, which are of height $\geqq k$. The lemma above shows that every uniform element x in $H_k(M)$ is of height at least k.

LEMMA 5. *If $M = U_1 \oplus \ldots \oplus U_n$ is a torsion R-module, where each U_i is uniserial, then for any uniform element x of M, $H(x) \leqq \max(d(U_i)) - 1$ and $e(x) \leqq \max(d(U_i))$.*

Proof. Since for any uniform element y of M, yR is isomorphic to a submodule of some U_j, we have $d(yR) \leqq \max(d(U_j))$, and for any $x(\neq, 0) \in yR$, $d(yR/xR) \leqq \max(d(U_j)) - 1$. Hence the lemma follows.

LEMMA 6. *Let $M = A + B$, be a torsion R-module and A, B be its submodules. Then for any non-negative integer k, $H_k(M) = H_k(A) + H_k(B)$.*

Proof. Since $H_0(M) = M$ the result holds for $k = 0$. To apply induction on k, let $k > 0$ and the result hold for $k - 1$. Thus $H_{k-1}(M) = H_{k-1}(A) + H_{k-1}(B)$. Let T be a submodule of $H_{k-1}(M)$ such that $H_{k-1}(M)/T$ a completely reducible. Suppose $H_k(M) \not\subset T$. By Lemma 1, every non-zero element of M is a sum of finitely many uniform elements. So there exists a uniform element $x \in H_k(M)$ such that $x \notin T$. As xR is uniserial, $xR \cap T = yR$ with $d(xR/yR) \geqq 1$. Since $H(x) \geqq k \geqq 1$, there exists a uniform element z with $H(z) \geqq k - 1$, and $xR < zR$. In that case $zR \cap T = yR$, $d(zR/yR) \geqq 2$ and zR/yR is uniserial. This contradicts the fact that $H_{k-1}(M)/T$ is completely reducible. Hence $H_k(M) \subset T$. It can be seen on similar lines that $H_{k-1}(M)/H_k(M)$ itself is completely reducible. Now

$$H_{k-1}(M)/(H_k(A) + H_k(B)) = \frac{H_{k-1}(A) + H_{k-1}(B)}{H_k(A) + H_k(B)}$$

being a homomorphic image of $H_{k-1}(A)/H_k(A) \oplus H_{k-1}(B)/H_k(B)$, is completely reducible. Consequently $H_k(M) \subset H_k(A) + H_k(B)$. Obviously $H_k(A) + H_k(B) \subset H_k(M)$. Hence $H_k(M) = H_k(A) + H_k(B)$.

We now prove the following generalization of Kulikov's Theorem [**4**, Theorem (11.1)]. It may be noticed that the proof given below has similarity with the corresponding proof in [**4**, Theorem (11.1)].

THEOREM 3. *Let M be a torsion right module over an (hnp)-ring R, which is not right primitive. M is a direct sum of uniserial R-submodules* (hence cyclic) *if and only if M is a union of an ascending sequence M_n, ($n = 1, 2, \ldots\ldots\ldots$) of submodules such that for each n, there exists a positive integer k_n, with the property: $H(x) \leqq k_n$ for all uniform elements x of M_n.*

Proof. Sufficiency: For each n, let P_n be the socle of M_n. Then the socle P of M is $\cup_n P_n$. Lemma 1 yields that P is an essential submodule of M and that every non-zero element of M is a sum of finitely many uniform elements. We

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construct a maximal independent subset S of uniform elements of P as follows. Select in $P_1 \cap H_{k_1}(M)$ a maximal independent set of uniform elements and expand it in turn with uniform elements of

$$P_1 \cap H_{k_1-1}(M), \ldots\ldots, H_0(M) \cap P_1 = P_1,$$

to an independent set S_1, which is at each step maximal. S_1 is a maximal independent subset of uniform elements of P_1. Extend S_1 in $P_2 \cap H_{k_2}(M)$, then in $P_2 \cap H_{k_2-1}(M), \ldots, P_2$, so that the independent set obtained after each step is maximal. This gives a maximal independent subset S_2 of P_2. Repeat this process with S_2 and P_3 to obtain S_3 etc. Then S_n, $(n = 1, 2, \ldots\ldots)$ is an increasing sequence of independent subsets of P and $S = \cup_n S_n$ is a maximal independent subset of P. Let $S = \{c_\lambda | \lambda \in \Lambda\}$. Since each c_λ is in some M_n, and c_λ is uniform with $H(c_\lambda) \leqq k_n$, we can find a uniform element a_λ in M such that $c_\lambda \in a_\lambda R$ and $d(a_\lambda R/c_\lambda R) = m_\lambda = H(c_\lambda) \leqq k_n$. Since $\Sigma_\lambda c_\lambda R$ is direct, we also have $M' = \Sigma_\lambda a_\lambda R = \oplus \Sigma a_\lambda R$. Each $a_\lambda R$ is uniserial by Lemma 1. If we show that $M = M'$, the result follows.

On the contrary let $M \neq M'$. Using Lemma 1, we can find a uniform element g in M such that $g \notin M'$ and the exponent $e(g) = k$, is minimal among all such elements. Naturally $k > 1$, since otherwise $g \in P \subset M'$. Let $yR = \text{socle}(gR)$. Now

(2) $y = c_1 r_1 + c_2 r_2 + \ldots + c_t r_t$

for some $c_i \in S$ and $r_i \in R$ such that $c_i r_i \neq 0$. Then $yR \cong c_i r_i R = c_i R$ under R-isomorphism $\sigma_i : yR \rightarrow c_i r_i R$ given by $\sigma_i(yr) = c_i r_i r$; $r \in R$. If for some value λ of i, $H(c_\lambda r_\lambda) \geqq k - 1$, we can choose $b_\lambda \in a_\lambda R$ satisfying $d(b_\lambda R/c_\lambda R) = k - 1$. Then $e(b_\lambda) = k$ and by Lemma 2 (b), σ_λ can be extended to an isomorphism $\eta_\lambda : gR \rightarrow b_\lambda R$; we can take $b_\lambda = \eta_\lambda(g)$. Then as $y = gs$ for some $s \in R$, we get $c_\lambda r_\lambda = b_\lambda s$. Consider $g' = g - b_\lambda$. Then $g' \notin M'$, and $g'R$ is a homorphic image of gR under the natural mapping. As $e(g)$ is minimal, we get $gR \cong g'R$, and $g's = \sum_{i \neq \lambda} c_i r_i$ is in P. Hence we can suppose that g itself is such that in (2) all $c_i r_i$ satisfy $H(c_i r_i) \leqq k - 2$. Clearly as $H(y) \geqq \mathrm{k} - 1$, $t \geqq 2$. We can find smallest positive integer m such that $c_i \in P_m$ for all i. If $m > 1$, for some value of i say for $i = t$, $c_t \notin P_{m-1}$. Then also $y \notin P_{m-1}$. As $H(y) > H(c_t)$, in the construction of S_m, y is taken into account before c_t. So that y is expressible as a linear combination of c_λ's different from c_t. This violates the independence of S. Similar consideration holds for $m = 1$. Hence $M = M'$.

Necessity: Let $M = \oplus \sum_{i \in I} N_i$, where N_i are uniserial. For each positive integer n, let M_n be the sum of those N_i which have $d(N_i) \leqq n$. Since M_n is a direct summand of M, Lemma 6 yields that the height of every uniform element of M_n, in M_n is the same as its height in M. Then Lemma 5, yields $H(x) \leqq n$ for every uniform elements x of M_n. This proves the theorem.

Since for any uniform element x in a torsion R-module M, if $zR = \text{socle}(xR)$ then $H(x) \leqq H(z)$, we get the following:

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Corollary 1. *Let M be a torsion R-module and P be its socle. Then M is a direct sum of uniserial modules if and only if P is a union of ascending sequence $P_n (n = 1, 2, 3 \ldots \ldots)$ of submodules such that for each n, there exists a positive integer k_n with the property that $H(x) \leqq k_n$ for every uniform element x of P_n.*

The following corollary generalizes the corresponding result of Prüfer for abelian p-groups.

Corollary 2 [**4**, Theorem (11.2)]. *A bounded torsion right R-module is a direct sum of cyclic modules.*

We call a right R-module M to be decomposable if it is a direct sum of cyclic modules and finitely generated torsion free uniform modules [**9**, p. 332]. Since over an (hnp)-ring, every finitely generated module is a direct sum of a projective module and torsion cyclic modules, we get that any decomposable module M over an (hnp)-ring equals $S \oplus T$, where S is projective and T is a direct sum of cyclic torsion modules. With these observations in mind, we obtain the following generalization of Kaplansky [**9**, Theorem 4].

Theorem 4. *Let R be any (hnp)-ring, which is not right primitive. If M is a decomposable R-module, then every submodule of M is decomposable.*

Proof. Now $M = N \oplus T$, N is a projective R-module and T is a direct sum of torsion cyclic R-modules. Let M' be any submodule of M. If $p : M \rightarrow N$ is the natural projection, then $p(M')$ being a submodule of the projective R-module N, is projective. Hence $M' = N' \oplus T'$ where $T' = T \cap M'$ and N' is projective. Since T satisfies the hypothesis of Theorem 3, the same holds for T'. Hence T' is decomposable. By Eisenbud and Robson [**1**, Lemma (1.4)] every projective R-module (in particular N') is a direct sum of uniform right ideals (which are always finitely generated). Hence M' is decomposable.

Definition. Let M be a torsion module over an (hnp)-ring R. A subset B of M is said to be a basis of M if

(i) every member of B is a uniform element of M,

(ii) B is an independent set and it generates M.

Theorem 5 [**4**, Theorem (13.1)]. *Let M be a torsion right module over an (hnp)-ring R, which is not right primitive. Then a subset B of M consisting of uniform elements is a basis of M if and only if*

(i) *B is a maximal independent set,*

(ii) *no element of B can be replaced by a uniform element of exponent greater than that of that element without violating independence.*

Proof. Necessity: Let $B = \{a_\lambda | \lambda \in \Lambda\}$ be a basis of M. Consider any uniform element $b \in M$. We can find a minimal subset $\{a_1, a_2, \ldots, a_k\}$ of B such that $b \in \oplus \sum_{i=1}^{k} a_i R$. We can write $b = \Sigma_i b_i$; $b_i (\neq 0) \in a_i R$. This relation shows that no $a_\lambda \neq a_i (i = 1, 2, \ldots, k)$ can be replaced by b without violating independence. If for some i, say $i = t$, $e(a_t) < e(b)$, then the R-homomorphism

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$\sigma : bR \to a_t R$ given by $\sigma(br) = b_t r; r \in R$ is not a monomorphism. If $cR = \ker \sigma$, then $0 \neq c = bs$ for some $s \in R$ and

$$0 \neq bs = \sum_{i \neq t} b_i s.$$

Hence $b, a_i (i = 1, 2, \ldots, k; i \neq t)$ are not independent. So b cannot replace a_t.

Sufficiency: Let $B = \{a_\lambda | \lambda \in \Lambda\}$ be satisfying (i) and (ii). Let $M' = \oplus \Sigma_\lambda a_\lambda R$. If $M \neq M'$ we can find a uniform element $b \in M$ such that $b \notin M'$ and $e(b)$ is smallest among all such elements. As M' is an essential submodule of M, $e(b) > 1$. Consider $c \in bR$ with $e(c) = e(b) - 1$. Then $c \in M'$, we can uniquely find $a_1, a_2, \ldots, a_k$ in B such that

$$c = c_1 + c_2 + \ldots + c_k; c_i (\neq 0) \in a_i R.$$

On the similar lines as in Theorem 3, we can suppose that $e(a_i) < e(b)$ for all i. However for some i, $cR \cong c_i R$. Consequently $e(c_i) = e(c) = e(b) - 1 \geqq e(a_i)$ yields that $cR \cong a_i R$. So we can choose c in bR such that $c_i = a_i$ for that i. Then $c = bs$ for some s in R yields $a_i = \Sigma c_j - bs$. This then yields that the set B' obtained from B by replacing a_i by b is independent even though $e(b) > e(a_i)$. This violates (ii). Hence $M = M'$.

Using the above theorem, following generalization of Kertèsz Theorem [**4**, Theorem (14.1)] can be proved. We omit the proof.

THEOREM 6. *Let M be a torsion module over an (hnp)-ring R, which is not right primitive, such that M contains no uniform element of infinite height. M is a direct sum of uniserial modules if and only if M contains a principal system.*

Hereby a principal system in M we mean a maximal independent set $L = \{a_\lambda | \lambda \in \Lambda\}$ of uniform elements of M, no element of which can be replaced by a uniform element of greater height without violating independence.

In the following lemma, part (ii) is a weaker version of [**4**, Lemma (22.1)].

LEMMA 7. *Let M be a torsion right R-module, T a submodule of M and K a submodule of M maximal with respect to the property that $T \cap K = (0)$. Then the following hold:*

(i) *If $(xR + K)/K$ is a simple submodule of M/K, then $xR \subset T + K$.*

(ii) *Given a simple submodule $\bar{x}R$ of $M/(T + K)$ there exists a corresponding simple submodule of $[(H_1(M) + K) \cap T]/H_1(T)$.*

Proof (i). Since $(K + xR) \cap T \neq (0)$ we have $y(\neq 0) \in (K + xR) \cap T$. Then $y \notin K$ and $(yR + K)/K = (xR + K)/K$. Hence

$$xR + K = yR + K \subset T + K.$$

This proves (i).

(ii) Consider a minimal submodule $\bar{x}R$ of $M/(T + K)$. By using Lemma 1, we can take x to be a uniform element of smallest exponent among all those uniform elements z of M for which $\bar{x}R = \bar{z}R$. Let $yR = xR \cap (T + K)$. Then

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$yR \subset T + K$ and $\bar{x}R \cong xR/yR$ is a simple module. Hence $y = t + k$; $t \in T$, $k \in K$. If $t = 0$ then $K \cap xR = yR$ and $(xR + K)/K$ is a simple submodule of M/K; so by (i), $xR \subset T + K$. This contradicts the hypothesis that $\bar{x}R \neq (0)$.

We claim $t \notin H_1(T)$. On the contrary, let $t \in H_1(T)$. Now tR being a homomorphic image of yR, is uniform. We can find a uniform element $t_1 \in T$ such that $t \in t_1R$ and t_1R/tR is a simple module. As $H(y) \geqq 1$, $k = y - t$ yields $k \in H_1(M)$. If $k \neq 0$, we can find a uniform element k_1 of M such that $k \in k_1R$ and k_1R/kR is simple. If $k = 0$, put $k_1 = 0$. By Lemma 2, the natural projections of yR onto tR and kR can be extended to homomorphisms of xR onto t_1R and k_1R respectively; we can take t_1 and k_1 as images of x under these extensions. In that case $y = xa$ for some $a \in R$ yields $t = t_1a$, $k = k_1a$, and $xR \cong (t_1 + k_1)R$ under the correspondence $xr \leftrightarrow (t_1 + k_1)r$; $r \in R$. We claim: $xR = (t_1 + k_1)R$. If not, then $(t_1 + k_1)R \cap xR = yR$ and $xR + (t_1 + k_1)R$ is not an essential extension of $(t_1 + k_1)R$, since an essential extension of a uniform module is uniform, and any finitely generated torsion uniform R-module is uniserial. Hence, by using the fact that xR/yR is simple we get $xR + (t_1 + k_1)R = (t_1 + k_1)R \oplus zR$ for some simple submodule zR. Since $kR \subset K \cap k_1R$ and k_1R/kR is a simple module, part (i) yields $k_1 \in T + K$. Hence $\bar{x}R = \bar{z}R$ and $e(z) = 1$. The minimality of $e(x)$ yields $e(x) = 1$. This yields $yR = (0)$ and hence $t = 0$. As seen before $t \neq 0$. This is a contradiction. Hence $xR = (t_1 + k_1)R$ and so $x \in T + K$; which again is a contradiction. Hence $t \notin H_1(T)$. Further $t = y - k$ gives $t \in H_1(M) + K$. As t is uniform, and $T/H_1(T)$ is completely reducible, we get $\bar{t}R$ is a simple submodule of $[H_1(M) + K) \cap T]/K$. Clearly $\bar{t}R$ is determined by $\bar{x}R$.

Remark. The proof given above does not show that $\bar{t}R$ is uniquely determined by $\bar{x}R$.

COROLLARY 3. *Let M, T, K satisfy the hypothesis of the above lemma. If $[(H_1(M) + K) \cap T]/H_1(T) = (0)$, then $M = T \oplus K$.*

Proof. This is immediate.

The following theorem generalizes [**4**, Theorem (24.1)] for torsion abelian group.

THEOREM 7. *Let M be a torsion right module over an (hnp)-ring R, which is not right primitive. Further let N be a submodule of M, such that it is a direct sum of uniserial modules of the same finite length k. Then the following are equivalent:*

(i) *N is a direct summand of M.*

(ii) *$H_n(N) = N \cap H_n(M)$ for all n.*

(iii) *N satisfies $H_k(M) \cap N = (0)$.*

Proof. (i) implies (ii): Let $M = N \oplus T$. By Lemma 6,

$$H_n(M) = H_n(N) \oplus H_n(T).$$

Hence $H_n(M) \cap N = H_n(N) \oplus (H_n(T) \cap N) = H_n(N)$.

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(ii) implies (iii): By Lemma 5, every uniform element in N has height $\leqq k - 1$ in N. So that $H_k(N) = (0)$. Hence by (ii) $N \cap H_k(M) = (0)$.

(iii) implies (i): Let K be a submodule of M maximal with respect to the property that $N \cap K = (0)$ and $H_k(M) \subset K$. Consider $T = (H_1(M) + K) \cap N$. By Lemma 6, $H_{k-1}(T) \subset (H_k(M) + K) \cap H_{k-1}(N) = K \cap H_{k-1}(N) = (0)$, since $K \cap N = (0)$. Hence $H_{k-1}(T) = (0)$. Appealing to Lemma 4, we get $T \subset H_1(N)$. Hence by Corollary 3, $M = N \oplus K$. Hence the result follows.

For torsion abelian groups, the condition (ii) is equivalent to saying that N is a pure subgroup of M. In general it is not known whether any submodule of M satisfying (ii) is a pure submodule.

THEOREM 8. *Let M be a torsion right module over an (hnp)-ring R, which is not right primitive and N be any bounded submodule of M such that $H_n(N) = H_n(M) \cap N$ for all n. Then N is a direct summand of M.*

Proof. The proof is on the same lines as for [**4**, Theorem (24.5)].

The following theorem generalizes Kulikov's Theorem [**4**, Theorem (25.2)] for torsion abelian groups. Since the proof is on the similar lines, it is omitted.

THEOREM 9. *Let M be a torsion module over an (hnp)-ring R, which is not right primitive. Let N be a submodule of M such that $H_n(N) = N \cap H_n(M)$ for all n. If M/N is decomposable, then N is a direct summand of M.*

LEMMA 8. *If in a torsion R-module every uniform element of its socle is of infinite height, then M is an injective module.*

Proof. It is clear from Lemma 6, that the hypothesis is also satisfied by any direct summand of M; further any direct sum of injective R-modules is injective [**14**]. Thus to show that M is injective it is enough to show that if M is non-zero, then M has a non-zero injective submodule. Let x be a uniform element of M. By applying induction on $e(x)$, we show that there exists a uniform element y in M such that $e(y) > e(x)$ and $x \in yR$.

If $x \in$ socle (M), then as $H(x)$ is infinite, there exists a uniform element y of M such that $x \in yR$ and $e(y) > e(x)$. Let $e(x) = n > 1$ and result hold for $< n$. Since xR is uniserial, there exists $z \in xR$, such that zR is a simple submodule. As $H(z)$ is infinite, we can find a uniform element $u \in M$ such that $z \in uR$ and $e(u) > n$. If $x \in uR$ we are finished. Let $x \notin uR$. As $zR \subset xR \cap uR$, Lemma 2, shows that there exists an R-monomorphism $\sigma : xR \rightarrow uR$ such that $\sigma(z) = z$. Then $\eta : xR \rightarrow (x - \sigma(x))R$ given by $\eta(xr) = (x - \sigma(x))r$ is a non-zero R-epimorphism and $z \in \ker \eta$. Hence $e(x - \sigma(x)) < e(x)$. Hence by induction hypothesis $x - \sigma(x) \in vR$ for some uniform element $v \in M$ and $e(x - \sigma(x)) < e(v)$. This all shows that $H(\sigma(x)) \geqq 1$ and $H(x - \sigma(x)) \geqq 1$. Hence as $x = (x - \sigma(x)) + \sigma(x)$, Lemma 4, yields $H(x) \geqq 1$. The definition of height yields a uniform element y satisfying $x \in yR$ and $d(yR/xR) \geqq 1$. Hence $e(y) > e(x)$. This proves the claim.

Since $M \neq (0)$, by Lemma 1, M has a non-zero uniform submodule. So we

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can find a maximal uniform submodule U of M. By Lemma 2 (a) and what we have done, U cannot be of finite length. Hence U is injective. This proves the lemma.

THEOREM 10. *Let M be a (non-zero) torsion module over an (hnp)-ring R, which is not right primitive. Then:*

(i) *If a uniform element x in* socle (M) *is of finite height, x belongs to a uniform finite length, direct summand of M.*

(ii) *M has a uniform direct summand, which can be chosen to be of finite length if M is not injective.*

Proof. (i) Let $H(x) = n$. Then there exists a uniform element y in M such that $x \in yR$ and $e(y) = n + 1$. Consider $N = yR$. We prove that $N \cap H_m(M) = H_m(N)$ for all m. Obviously $H_m(N) \subset N \cap H_m(M)$. Let a uniform element $u \in N \cap H_m(M)$. Then $z \in yR \cap uR$ yields

$$n = H(z) \geqq H(u) + e(u) - e(z) \geqq m + e(u) - 1.$$

Therefore $e(u) \leqq n - m + 1$. Hence $d(yR/uR) = e(y) - e(u) \geqq m$. Hence height of u in N is at least m; so $u \in H_m(N)$. Hence $N \cap H_m(M) = \mathrm{H}_m(N)$. By Theorem 7, N is a direct summand of M. This proves (i).

(ii) If every element in the socle of M is of infinite height, then by Lemma 8, M is injective, and hence by Matlis [**14**], M is a direct sum of uniform modules. If there exists a uniform element in socle (M) of finite height, then (i) yields M has a uniform finite length direct summand. This proves (ii).

As an application of above theorem we get the following special case of Levy [**11**, Theorem (3.4)].

COROLLARY 4. *Every divisible R-module M is injective.*

Proof. Let M be non-zero torsion R-module. If M is not injective, by Lemma 8, M has a uniform element x, in its socle, of finite height. By the above theorem, M has a finite length uniform direct summand U containing x. Thus U is also divisible and hence faithful. However by [**17**, Lemma 1] U is not faithful. This is a contradiction. Hence M is injective.

Let M be not a torsion module, and T be its torsion submodule. Then T is also divisible. Hence T is injective and $M = N \oplus T$ where N is torsion free divisible R-module. By Levy [**11**], every divisible torsion free module over a prime Goldie ring is injective. Hence N is injective. This proves that M is injective.

COROLLARY 5. *Any indecomposable torsion R-module is uniform.*

3. Quasi-projective modules. Throughout all the lemmas R is an (hnp)-ring, which is not right primitive. First of all we determine the structure of torsion quasi-projective (right) R-module. Let E and E' be any two indecomposable injective torsion R-modules. As defined in [**17**], E is said to be equivalent to E' if there exist submodules K and K' of E and E' respectively such

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that $E/K \cong E'/K'$ and $K \neq E$, $K' \neq E'$. If E is of finite periodicity, we saw in [**17**] that E' is equivalent to E if and only if E' is a homomorphic image of E. Let U be a uniform torsion R-module. U is said to be of periodicity n, if and only if its injective hull $E(U)$ is of periodicity n. (This definition differs from one given in [**17**].) Two torsion, uniform R-modules U and V are said to be equivalent if their injective hulls are equivalent. Two uniform elements x and y in a torsion R-module are said to be equivalent if xR and yR are equivalent. A torsion R-module M is said to be primary if every pair of uniform elements of M are equivalent. By using Lemmas 1 and 3 it can be easily proved that given a uniform element x in a torsion R-module M, the submodule N of M generated by all uniform elements equivalent to x, is primary. Such an N is called a primary component of M. Again appealing to Lemmas 1 and 3, we get the following

LEMMA 9. *Any torsion R-module is a direct sum of its primary components.*

The following is a special case of Fuller and Hill [**6**, Theorem (2.3)].

THEOREM 11. *A module M over an artinian ring S is quasi-projective if and only if M is projective as* $S/\text{ann}(M)$*-module.*

Since every finitely generated torsion R-module M has non-zero annihilator [**17**, Lemma 1] and by Theorem 1, every factor ring of an (hnp)-ring is generalized uniserial, Theorem 11 yields the following:

LEMMA 10. *Any finitely generated torsion R-module is quasi-projective if and only if M is projective as* $R/\text{ann}(M)$*-module.*

LEMMA 11. *Let M be a torsion, quasi-projective R-module, and U be a finitely generated, uniform direct summand of M, of finite periodicity n* (i.e., n is the periodicity of $E(U)$). *If V is any finitely generated uniform submodule of M such that* $U \cap V = (0)$, socle $(U) \cong$ socle (V) *and* $U \oplus V$ *is a direct summand of M, then* $|d(V) - d(U)| \leqq n - 1$.

Proof. Let W be any torsion finitely generated, uniform R-module. Let $B = \text{ann}\ (W)$. Then $B \neq (0)$ and W is a faithful module over the generalized uniserial ring $\bar{R} = R/B$. Hence R/B is embeddable in $W^{(m)}$, a direct sum of finitely many, say m, copies of W. So if $\bar{e}$ is a primitive idempotent of $\bar{R}$ then there exist m R-homomorphisms $\sigma_i : \bar{e}\bar{R} \rightarrow W$ $(1 \leqq i \leqq m)$ such that $\bigcap_i \ker \sigma_i = (0)$. However $\bar{e}\bar{R}$ is uniserial. So for some i, $\ker \sigma_i = (0)$. This shows that $\bar{e}\bar{R}$ is embeddable in W. Consequently socle $(\bar{e}\bar{R}) \cong$ socle (W). This shows that $\bar{R} = R/B$ is a generalized uniserial ring with homogeneous socle. Hence by [**17**, Theorem 1] we can find a Kupisch series $\bar{e}_1\bar{R}, \bar{e}_2\bar{R}, \ldots, \bar{e}_t\bar{R}$ of $\bar{R}$ such that $d(\bar{e}_{i+1}\bar{R}) = d(\bar{e}_i\bar{R}) + 1$, and t is the periodicity of any uniserial $\bar{R}$-module.

Since U and V have isomorphic socles. Theorem 2 yields that one of them is embeddable in the other; so if $B = \text{ann}\ (U \oplus V)$, then $B = \text{ann}\ (U)$ or $B = \text{ann}\ (V)$. Consequently by the preceding paragraph, $\bar{R} = \text{R}/B$ has homogeneous

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socle. Since by Lemma 10, U and V are both projective R/B-modules, in the notation of the last paragraph $U = \bar{e}_i\bar{R}$ and $V = \bar{e}_j\bar{R}$ for some i, j and both have periodicity t, as R/B-modules. It is clear that $|d(U) - d(V)| = |d(\bar{e}_iR) - d(\bar{e}_jR)| \leqq t - 1$. However as $E(U)$ is of periodicity n, and all $\bar{e}_k\bar{R}$ are embeddable in $E(U)$, we have $t \leqq n$. Hence $|d(U) - d(V)| \leqq n - 1$.

An R-module M is said to be reduced if it has no non-zero divisible submodule.

THEOREM 12. *Let M be a torsion reduced quasi-projective module over an (hnp)-ring R, which is not right primitive. Let x be a uniform element of M such that $E(xR)$ is of finite periodicity. Then the primary component of M to which x belongs is decomposable, bounded, and is projective as "R modulo its annihilator" module.*

Proof. Let the periodicity of $E(xR)$ be $n > 0$. As M is reduced each of its primary component is reduced; hence by using Theorem 10 and Lemma 9, we get a uniform finite length direct summand U of M such that U is equivalent to xR. Let $\mathscr{F}$ be the family of those direct sums N in M, of uniform submodules, which are such that each of them is equivalent to U and one of them is U itself; further each member N of $\mathscr{F}$ satisfies $H_n(N) = N \cap H_m(M)$ for all m. $\mathscr{F}$ is non-empty, as $U \in \mathscr{F}$. This family is inductive, so has a maximal member say N_U. Now

$$N_U = U \oplus \sum_{i \in I} V_i$$

where each V_i is uniform and equivalent to U. As M is reduced each V_i is of finite length. Since $E(xR)$ is of periodicity n, we can put these V_i into not more than n disjoint classes, such that any two V_i are in the same class if and only if they have isomorphic socles. Let $V_1, V_2, \ldots, V_m$ $(m \leqq n)$ be representatives chosen from each such class. Consider any V_α different from $V_1, V_2, \ldots, V_m$. Now socle $(V_\alpha) \cong$ socle (V_i) for some i with $1 \leqq i \leqq m$. Now $U \oplus V \oplus V_i$ being a direct summand of N, belongs to $\mathscr{F}$. Hence by Theorem 8, $U \oplus V_\alpha \oplus V_i$ is a direct summand of M. Thus by Lemma 11, $d(V_\alpha) \leqq d(V_i) + n$. So if k is the maximum of $d(U) + n$, $d(V_i) + n$, $(1 \leqq i \leqq n)$, then $d(U) \leqq k$ and $d(V_\alpha) \leqq k$ for all $\alpha \in I$. Hence by Theorem 8, N_U is a direct summand of M.

Clearly by construction N_U is decomposable and is contained in the primary component of M, to which x belongs. Now $M = N_U \oplus T$. If we show that T has no uniform submodule equivalent to U, it follows that N_U is the primary component of M to which x belongs. As T is reduced, if T had a uniform submodule equivalent to U, then, as for M, T will have a uniform finite length direct summand V equivalent to U. Then $N_U \oplus V \in \mathscr{F}$ and this contradicts the maximality of N_U. Hence N_U is the primary component of M containing x.

Let $A = \text{ann}\,(N_U)$. We show that $A \neq (0)$. Let the composition series of $E(xR)$ be:

$$0 = x_0R < x_1R < \ldots\ldots < x_lR < \ldots < E(xR).$$

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Each of V_α being equivalent to xR is embeddable in $E(xR)/x_iR$ $(0 \leqq i \leqq n)$. Further as $d(V_\alpha) \leqq k$, we get that each V_α is a submodule of an appropriate homomorphic image of $x_{k+n}R$; same holds for U. Now by [**17**, Lemma 1], $B = \text{ann }(x_{k+n}R) \neq (0)$. Obviously B kills every one of U and V_α. So $B \subset A$ and $A \neq (0)$. Further N_U being a direct summand of M, is a quasi-projective faithful module over the artinian ring R/A. Hence by Theorem 11, N_U is projective R/A-module. Trivially N_U is bounded.

THEOREM 13. *Let M be a reduced torsion R-module having no uniform element x with $E(xR)$ of zero periodicity. Then M is quasi-projective if and only if each of its primary component N is projective as $R/$ann (N)-module. Further if M is quasi-projective, then each of its primary component is bounded, and M is decomposable.*

Proof. Necessity follows from Theorem 12.

Sufficiency: By hypothesis each primary component of M is quasi-projective R-module. If N and N' are two primary torsion R-modules such that no uniform submodule of N is equivalent to a uniform submodule of N', then Lemmas 1 and 2 (c) yield that Hom $(N, N') = (0)$. Using this fact, Lemma 9 and the fact that every primary component of M is fully invarient, we get that M is quasi-projective. The last part follows from Theorem 12.

Let E be an indecomposable injective torsion R-module and

$$0 = x_0R < x_1R < \ldots < x_mR < \ldots < E$$

be its composition series. For each i, $P_i = \text{ann }(x_{i+1}R/x_iR)$ is a maximal ideal of R. Then the infinite sequence $(P_0, P_1, P_2, \ldots, P_m, \ldots)$ is called the prime sequence associated with E. If E is of finite periodicity say n, then $P_0, P_1, \ldots, P_{n-1}$ are all distinct and the above prime sequence is of periodicity n; conversely as R/P_i are simple artinian, we have that if for some distinct i and j, $P_i = P_j$, then E must be of finite periodicity. Thus if R has only finitely many prime ideals (that happens in particular if $J(R) \neq (0)$), there exists no indecomposable, torsion, injective R-module of zero periodicity. The author is not aware of any R, which admits an indecomposable injective torsion module of zero periodicity. We have the following:

LEMMA 12. *Let E be an indecomposable, injective, torsion R-module of finite periodicity. Then E is not quasi-projective.*

Proof. Let E be quasi-projective and be of periodicity $n > 0$. Using the composition series of E, we get that E has a submodule xR of length n such that $E \cong E/xR$. Then by Fuchs and Rangaswamy [**5**, Lemma 4], xR is a direct summand of E. This is a contradiction. Hence E cannot be quasi-projective.

THEOREM 14. *Let R be an (hnp)-ring which is not right primitive and which admits no indecomposable injective, torsion module of zero periodicity. Then any quasi-projective torsion R-module is reduced.*

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Proof. Since every divisible R-module is injective, the result follows from Lemma 12 and the fact that every injective R-module is a direct sum of indecomposable injective R-modules.

Now over a bounded Dedekind prime ring S, every indecomposable injective torsion module is of periodicity one [**17**, Corollary 1] and hence any two equivalent uniform torsion S-modules have isomorphic socles. So if M is a quasi-projective torsion S-module, then by Theorem 14, M is reduced. By Lemma 11 and Theorem 12, each primary component of M is a direct sum of isomorphic, finite length uniform modules. Thus from Theorems 12, 13 and 14, we get the following:

THEOREM 15. *Let S be a bounded Dedekind prime ring and M be a torsion S-module. Then M is quasi-projective S-module if and only if each of its primary component is a direct sum of isomorphic uniserial modules.*

This theorem generalizes the main theorem in [**5**] for torsion quasi-projective abelian groups and the structure theorem for quasi-projective torsion modules over a Dedekind domain, proved by Rangaswamy and Vanaja in [**15**].

We now turn our attention to non-torsion, quasi-projective modules. The complete structure of such modules is not yet known. Here we give some information:

Let S be a prime right Goldie ring. If S contains a direct sum of a uniform right ideals, which is an essential right ideal, then n is called the dimension of S (dim $S = n$). If dim $S = n$, any direct sum in S of n uniform right ideals contains a regular element.

LEMMA 13. (a) *If U and V are uniform right ideals of S, then U and V are embeddable in each other.*

(b) *Any non-zero torsion free right module over a prime right Goldie ring S has an essential submodule which is a direct sum of uniform submodules isomorphic to uniform right ideals of S.*

Proof. (a) By Goldie [7, Lemma (3.3)], given $x \in S$, either $xU = (0)$ or $xu \neq 0$ for every $u\ (\neq 0) \in U$. Consequently if $xU \neq (0)$ then $U \cong xU$. Now $VU \neq (0)$, as S is prime. So for some $v \in V$, $U \cong vU \subseteq V$, this proves (a).

(b) Let Q be the classical right quotient ring of S, which we know is simple artinian. By Levy [**11**], $M \otimes_S Q$ is the injective hull of M. Now $M \otimes_S Q = \oplus \sum_{i \in I} N_i$, $N_i \cong eQ$ for some primitive idempotent e of Q. Since N_i is uniform, we can find a submodule K_i of N_i isomorphic to a right ideal of R contained in $eQ \cap R$. As $K_i \cap M \neq (0)$, we can find such $K_i \subset M$. Then $\oplus \sum K_i \subset' M$. This proves (b).

LEMMA 14. *Let M be a right module over a prime right Goldie ring S such that M is not a torsion module and* dim $S = n$. *Then S is embeddable in a direct sum of n copies of M.*

Proof. Since M is not a torsion module, using the fact that every essential

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right ideal of S contains a regular element [**7**], it follows that torsion submodule of M is not an essential submodule. So by Lemma 13 (b) M has a uniform submodule U which is isomorphic to a uniform right ideal V of S. S has an essential right ideal $V_1 \oplus V_2 \oplus \ldots \oplus V_n$; each V_i uniform. Since by Lemma 13, each V_i is embeddable in V and hence in M, $\oplus \sum V_i$ is embeddable in $M \oplus \ldots \oplus M$ (n copies). However $\oplus \sum_i V_i$ contains a regular element, and so S is embeddable in $\oplus \sum V_i$. Hence S is also embeddable in $M \oplus \ldots \oplus M$ (n copies).

As defined by deRobert [**16**] an S-module (S any ring) is said to be projective relative to an S-module N (or N-projective) if for every submodule K of N, the induced sequence

$$0 \rightarrow \mathrm{Hom}_S(M, K) \rightarrow \mathrm{Hom}_S(M, N) \rightarrow \mathrm{Hom}_S(M, N/K) \rightarrow 0$$

is exact. The class of modules N for which a given module M is N-projective is closed under submodules, quotient modules and finite direct sums [**16**]. Hence in particular a quasi-projective module M is N-projective for all submodules N of M. Further a right R-module M, which is R_R-projective is N-projective for every finitely generated right R-module N; thus in particular if M is also finitely generated, then as M is a homomorphic image of a finitely generated free module F, we get M itself is projective, since M is F-projective. Hence we have the following:

LEMMA 15. *Any finitely generated R_R-projective (R any ring) module M is projective.*

We now prove a result of independent interest.

THEOREM 16. *Any finitely generated quasi-projective right module M over a prime right Goldie ring S is projective, whenever M is not a torsion module.*

Proof. Let dim $S = n$. Now by deRobert [**16**], $M^{(n)}$ is quasi-projective. Since by Lemma 14, S_S is embeddable in $M^{(n)}$, $M^{(n)}$ is also S_S-projective. Hence by Lemma 15, $M^{(n)}$ is projective. Consequently M is projective.

COROLLARY 6. *Let R be a right, left noetherian prime ring. Then R is an (hnp)-ring if and only if every right ideal of R is quasi-projective.*

Proof. Sufficiency: Theorem 16 yields that every right ideal of R is projective. Then by Small [**18**], R is also left hereditary. Hence R is an (*hnp*)-ring. Necessity is obvious.

THEOREM 17. *Let M be a reduced quasi-projective module over an (hnp)-ring R, which is not right primitive. If M is not a torsion module, M has a non-zero finitely generated, projective direct summand.*

Proof. Let dim $R = n$. By Lemma 14 and deRobert [**16**], $M^{(n)}$ is R_R-projective and hence M is R_R-projective. Now M is reduced. So there exists a regular element d in M such that $Md \neq M$, i.e., $MRd \neq M$. By Eisenbud and Robson [**2**, Theorem (4.10)], R is right bounded. So by Lenagan [**10**, Theorem (3.3)], R

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has enough invertible ideals. Consequently by [**2**, Corollary (4.11)] R is left bounded as well. Consequently Rd contains a non-zero ideal and hence contains a product $P_1P_2, \ldots P_n$ of non-zero prime ideals. Thus for some i, $MP_i \neq M$. Thus as R/P_i is simple artinian, we can find a simple direct summand N of M/MP_i and we get a diagram

$$\begin{array}{ccc} & & M \\ & & \downarrow \sigma \\ R_R & \xrightarrow{\pi} & N \to 0 \end{array}$$

where σ and π both are R-epimorphisms. Then R_R-projectivity of M yields that there exists R-homomorphism $f : M \to R$ such that $\pi f = \sigma$. Clearly $f(M) \neq 0$. As R is right hereditary, $f(M)$ is projective R-module. Hence $M = M' \oplus N'$; $M' \cong f(M)$ is a non-zero projective finitely generated submodule of M.

Let M be any right R-module. Since the injective hull of M is a direct sum of uniform modules, M has an essential submodule N, which is a direct sum of uniform submodules, say $N = \oplus \sum_{i \in I} N_i$. Then the cardinality $|I|$ of I is called the rank of M. Since $E(M) = E(N) = \oplus \; \Sigma_{i \in I} E(N_i)$ by the Krull-Schmidt-Azumaya Theorem, $|I|$ is uniquely determined by $E(M)$, and hence by M. This definition generalizes the corresponding definition of rank of an abelian group [**4**, p. 31]. We now prove the following

THEOREM 18. *Let M be a reduced quasi-projective finite rank module over an (hnp)-ring R, which is not right primitive. Then*

(i) *M is projective and finitely generated, or*

(ii) *M is a torsion module.*

Proof. Let M be not a torsion module. As seen in proof of Theorem 17, M is R_R-projective and $M = M_1 \oplus N_1$ where M_1 is projective, finitely generated and non-zero. Clearly rank $(N_1) <$ rank (M). N_1 being a direct summand of M is also R_R-projective. By applying induction on rank (M), we get $M = N \oplus T$, where N is a non-zero finitely generated projective R-module and T is a torsion R-module. As T is R_R-projective and of finite rank, if $T \neq (0)$, then proceeding on the same lines as in Theorem 17, we get that T has a projective direct summand. This is a contradiction. Hence M is a finitely generated, projective R-module.

Remark 1. In Section 2, we essentially used the fact that over an (*hnp*)-ring R, which is not right primitive, no finitely generated torsion module is faithful. All the results in Section 2, can be proved for the class those torsion modules over an arbitrary (*hnp*)-ring, which have no finitely generated torsion faithful submodules.

Remark 2. Let R be an (*hnp*)-ring which is not right primitive and Q be its classical quotient ring. If Q is not quasi-projective as R-module (e.g. the field of rational numbers is not quasi-projective Z-module; Z the ring of integers), then every quasi-projective torsion free R-module is reduced. So for such a ring R,

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any finite rank quasi-projective torsion free R-module is projective, by Theorem 18.

Added in proof. Zaks in [*Some rings are hereditary rings*, Israel J. Math. *10* (1971)] proved that if every proper homomorphic image of a neotherian, bounded, prime ring R is a QF-ring, then R is a Dedekind prime ring. It is of interest to observe that all the results in the present paper which lead to the proof of Corollary 4, viz. every divisible module over a bounded (hnp)-ring R is injective, can be proved for any noetherian prime ring, whose every proper homomorphic image is generalized uniserial. Since a homomorphic image of a divisible module is divisible, so it follows that any such ring must be hereditary, hence an (hnp)-ring. This result generalizes the Zaks theorem mentioned above.

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Ohio University,
Athens, Ohio;
Guru Nanak University,
Amritsar, India

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