BULL. AUSTRAL. MATH. SOC. 57N35
VOL. 38 (1988) [203–205]

# ON A THEOREM OF LI-BANGHE AND PETERSON ON IMMERSIONS OF MANIFOLDS

TZE-BENG NG

Let $M^n$ and $N^{2n-2}$ be smooth, connected manifolds of dimension $n$ and $2n-2$ respectively with $n \equiv 2 \mod 4$ and $6 \leqslant n \leqslant 26$. Let $f\colon M^n \to N^{2n-2}$ be a continuous map. Under certain suitable conditions on the stable normal bundle of $f$, we give a direct and simpler proof that $f$ is homotopic to an immersion. For the case $6 \leqslant n \leqslant 26$ and $n \neq 18$, the result was proved by Li-Banghe and Peterson by using non-stable obstruction theory and their earlier result.

## 1. INTRODUCTION

Let $M^n$ and $N^{2n-2}$ be smooth, connected, manifolds of dimension $n$ and $2n-2$ respectively with $n \geqslant 6$. Let $f\colon M^n \to N^{2n-2}$ be a continuous map. It is well known [**1**], that $f$ is homotopic to an immersion if, and only if the stable bundle $\nu_f = f^*(\tau_N) + \nu_M$ has geometric dimension $\leqslant n-2$. Thomas [**7**] has shown that, when $n \equiv 2 \mod 4$ with $n \geqslant 6$, $M$ is orientable and $f^*(w_1(N)) = f^*(w_2(N)) = 0$, then $f$ is homotopic to an immersion if $\delta w_{n-2}(\nu_f) = 0, w_n(\nu_f) = 0$ and $w_{n-2}(\nu_f)\cdot w_2(M) = 0$. Li-Banghe and Peterson [**2**] showed that $w_n(\nu_f) = 0$. They proved (for $n \neq 18$) :

THEOREM 1.1. ([**3**, Theorem 2.3 and 2.4]) *Let $f : M^n \to N^{2n-2}$ be a continuous map.*

(i) *Suppose n = 6 or 10 and $\nu_f$ is a stable spin bundle. Then $f$ is homotopic to an immersion.*

(ii) *Suppose n = 14, 18, 22 or 26 and that $\nu_f$ admits a $BO\langle 8\rangle$ structure. Then $f$ is homotopic to an immersion.*

Their proof uses a theorem of [**2**] to lift the classifying map of $\nu_f$ to $BSpin_{n-1}$ or $BO_{n-1}\langle 8\rangle$ and then show that the obstruction to lifting $\nu_f$ further to $BSpin_{n-2}$ or $BO_{n-2}\langle 8\rangle$ is trivial. For n = 18, there is a class $\theta \in H^{18}(BO_n\langle 8\rangle; Z_2)$ not in the image of $i^*$ where $i : BO_n\langle 8\rangle \to BSO_n$ is the projection map of the bundle. Therefore their proof could not give the same conclusion when n = 18.

In this note we shall show that by studying the n-Postnikov towers for the fibration $BSO_{n-2} \to BSO$ for $6 \leqslant n \leqslant 26$, we derive Theorem 1.1 directly without first lifting $\nu_f$ to $BSO_{n-1}$ and also prove an analogous result for the case n = 18. For completeness we have included this result in Theorem 1.1.

Received 3 November 1987

## 2. The Postnikov tower for $\pi : BSO_{n-2} \to BSO$..

According to Mahowald [4], the Postnikov tower of $\pi : BSO_{n-2} \to BSO$ for $n \equiv 2 \bmod 4$ and $n \geqslant 6$ is given as follows :

$$
\begin{aligned}
k_1^1 &= \delta w_{n-2}, \\
k^2 &= \left(p_1^* w_n, k_1^2\right)
\end{aligned}
$$

where $k_1^2$ is defined by the relation $k_1^2 : \left(Sq^2 + w_2\cdot\right)\delta w_{n-2} = 0$ and $p_1 : E_1 \to BSO$ is the principal fibration with classifying map $\delta w_{n-2} : BSO \to K(\mathbb{Z}, n-1)$.

$$
\begin{array}{ccccc}
 & & BSO_{n-2} & & \\
 & & \downarrow & & \\
 & & E_1 & \xrightarrow{(p_1^* w_n, k_1^2)} & K(\mathbb{Z}_2, n) \times K(\mathbb{Z}_2, n) \\
 & & \downarrow{\scriptstyle p_1} & & \\
BSpin & \xrightarrow{\eta} & BSO & \xrightarrow{(\delta w_{n-2})} & K(\mathbb{Z}, n-1)
\end{array}
$$

Let $\eta$: $BSpin \to BSO$ and $\tilde{\eta}$: $BO\langle 8\rangle to BSO$ be the obvious inclusion map. Since $H^*(BSpin; \mathbb{Z})$ has only order 2-torsion, $\delta w_4$ and $\delta w_8$ are trivial in $H^*(BSpin; \mathbb{Z})$, and $\delta w_{12}, \delta w_{16}, \delta w_{20}$ and $\delta w_{24}$ are all trivial in $H^*(BO\langle 8\rangle; \mathbb{Z})$ (this can be easily derived by looking at a truncated Poincaré series for the $Sq^1$-cohomology of $H^*(BO\langle 8\rangle; \mathbb{Z}_2)$) the fibration $\eta$ lifts to $E_1$ for n = 6 or 10 and the fibration $\tilde{\eta}$ lifts to $E_1$ for n = 14, 18, 22 or 26.

## 3. Spin-Structure.

For n = 6 or 10 let $l : BSpin \to E_1$ be a lifting of $\eta : BSpin \to BSO$.

THEOREM 3.1. *Let $M^n$ be a manifold of dimension n = 6 or 10. Let $\xi$ be a stable spin bundle over M with $w_n(\xi) = 0$. Then the geometric dimension of $\xi \leqslant n-2$.*

PROOF: Now $H^6(BSpin; \mathbb{Z}_2) \approx \langle w_6\rangle$ so that for n = 6, $l^*\left(k_1^2\right) = \alpha w_6$ for some $\alpha \in \mathbb{Z}_2$. Thus if $\xi$ is a stable spin bundle over $M^6$ with $w_6(\xi) = 0, 0 \in k^2(\xi)$. Also $H^{10}(BSpin; \mathbb{Z}_2) \approx \langle w_{10}, w_4 \cdot w_6\rangle$. Assume now n = 10. An exact sequence of Thomas [8], shows that $\left(Sq^2 + w_2\cdot\right)k_1^2 \in p_1^*(Ker\, \pi^*) \cap H^{12}(E_1)$.

Since $\left(Sq^2 + w_2\cdot\right)H^{10}(BSpin; \mathbb{Z}_2) \cap \eta^*(Ker\, \pi^*) = \{0\}$, $l^*\left(Sq^2 + w_2\cdot\right)k_1^2 = 0$. Hence $l^* k_1^2 = \alpha w_{10}$ for some $\alpha \in Z_2$. As before if $\xi$ is a stable spin bundle with $w_{10}(\xi) = 0$, $0 \in k^2(\xi)$. ∎

## 4. BO⟨8⟩-Structures.

For $14 \leqslant n \leqslant 26$, let $\tilde{l}\colon BO\langle 8\rangle \to E_1$ be a lifting of $\tilde{\eta}\colon BO\langle 8\rangle \to BSO$.

THEOREM 4.1. *Let $\xi$ be a $BO\langle 8\rangle$ – bundle over $M^n$ of dimension $n \equiv 2(4)$ with $14 \leqslant n \leqslant 26$. Suppose $w_n(\xi) = 0$. Then $\xi$ has geometric dimension $\leqslant n-2$.*

PROOF: $H^{14}(BO\langle 8\rangle;\mathbb{Z}_2) \approx \langle w_{14}\rangle, H^{18}(BO\langle 8\rangle;\mathbb{Z}_2) \approx 0$, $H^{22}(BO\langle 8\rangle;\mathbb{Z}_2) \approx \langle w_{22}, w_8 \cdot w_{14}\rangle$ and $H^{26}(BO\langle 8\rangle;\mathbb{Z}_2) \approx \langle w_{26}, w_{12} \cdot w_{14}\rangle$. For n = 14 and 26, the proof is similar to that of Theorem 3.1. For n = 18, it is trivial. Now $w_8$ and $w_{12}$ in $H^*(BO\langle 8\rangle;\mathbb{Z}_2)$ are integral. That is there are classes $Q_2 \in H^8(BO\langle 8\rangle;\mathbb{Z})$ and $Q_3 \in H^{12}(BO\langle 8\rangle;\mathbb{Z})$ with $\rho_2 Q_2 = w_8$ and $\rho_2 Q_3 = w_{12}$ where $\rho_2$ is reduction mod 2. Thus $(Sq^2 + w_2\cdot)(Q_2 \cdot Q_3) = Sq^2(Q_2 \cdot Q_3) = Sq^2(w_8 \cdot w_{12}) = w_8 \cdot w_{14}$ in $H^*(BO\langle 8\rangle;\mathbb{Z}_2)$. Thus for $n = 22, w_8 \cdot w_{14} \in Indet^{22}\big(k^2(\tilde{\eta}), BO\langle 8\rangle\big)$. Thus if $w_{22}(\xi) = 0$ and n = 22, $0 \in k^2(\xi)$. Therefore $\xi$ lifts to $BSO_{n-2}$ and so the geometric dimension of $\xi \leqslant n-2$. ∎

## 5. Proof of Theorem 1.1.

Let $f : M^n \to N^{2n-2}$ be a continuous map. Take $\xi = \nu_f$. Then part (i) follows from Theorem 3.1 and part (ii) follows from Theorem 4.1.

### REFERENCES


[1] M. Hirsch, 'Immersion of manifolds', *Trans. Amer. Math. Soc.* **93** (1959), 242–276.

[2] B.H. Li and F.P. Peterson, 'On immersions of k-manifolds in (2k-1)-manifolds"', *Proc. Amer. Math. Soc.* **83** (1981), 159–162.

[3] B.H. Li and F.P. Peterson, 'On immersions of n-manifolds in (2n-2)-manifolds"', *Proc. Amer. Math. Soc.* **97** (1986), 532–538.

[4] M. Mahowald, 'On obstruction theory in orientable fiber bundles', *Trans. Amer. Math. Soc.* **110** (1964), 315–349.

[5] Ng Tze-Beng, 'A note on the mod 2 cohomology of $B\widehat{SO}_n\langle 16\rangle$', *Canad. J. Math.* **37** (1985), 893–907.

[6] D. Quillen, 'The mod 2 cohomology rings of extra-special 2-groups and the spinor groups', *Math. Ann.* **194** (1971), 197–212.

[7] E. Thomas, 'On the existence of immersions and submersions', *Trans. Amer. Math. Soc.* **132** (1968), 387–394.

[8] E. Thomas, *Seminar in fiber spaces: Lecture Notes in Math. No. 13* (Springer–Verlag, 1966).



Department of Mathematics
National University of Singapore
Lower Kent Ridge Rd
Singapore 0511