# COMPARING RISK ADJUSTED PREMIUMS FROM THE REINSURANCE POINT OF VIEW

By

João Manuel Andrade e Silva and Maria de Lourdes Centeno [1]

*ISEG, Technical University of Lisbon, Portugal*

## Abstract

In this paper we compare, from the point of view of reinsurance, the several risk adjusted premium calculation principles considered in Wang (1996b). We conclude that, with the exception of the proportional hazard (PH) premium calculation principle, all the others behave in a way similar to the expected value principle. We prove that the stop loss reinsurance premium when calculated using the PH premium principle gives a higher premium than any of the other transforms, provided that the priority is big enough. We observe a similar behaviour with respect to excess of loss reinsurance in all the examples given.

We also study the behaviour of the adjustment coefficient, both from the insurer's and the reinsurer's point of view as functions of the priority, when the PH principle is used as opposed to the expected value principle.

## Keywords

PH-transform; risk adjusted premium; reinsurance; adjustment coefficient.

## 1 The Risk Adjusted Premium Calculation Principles

A principle of premium calculation is a rule, say $H$, that assigns a non-negative number to every risk defined by its probability distribution of loss. It can be regarded as a functional from the class of distribution functions into $\Re^+$.

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We denote an insurance risk by a non-negative loss random variable $Y$. Let $S_Y(t) = \Pr\{Y > t\}$ be the survival function of $Y$. Given any survival function $S_Y(t)$, the equation

$$S_Z(t) = (S_Y(t))^{1/\rho} \tag{1}$$

defines another survival function $S_Z(t)$. The mapping $\Pi_\rho(Y): S_Y(t) \to S_Z(t)$ is called the *proportional hazard (PH) transform*. Wang (1995) has proposed a new premium calculation principle, based on PH-transforms, and defined in the following way,

**Definition**: For a risk $Y$, with survival function $S_Y(t)$, the premium calculated according to the *PH-transform principle* is defined as

$$\pi_\rho(Y) = \int_0^\infty (S_Y(t))^{1/\rho} dt, \; \rho \geq 1, \tag{2}$$

where $\rho$ is called the *risk aversion index*.

When $\rho = 1$ the pure premium is obtained, that is, $\pi_1(Y) = E(Y)$.

Wang (1996b) has proposed a more general class of premium calculation principles, called *risk adjusted premium principles* by using other transforms to distort the survival function, i.e.

$$S_Z(t) = g[S_Y(t)], \tag{3}$$

where $g$ is an increasing, continuous and concave function, with $g(0) = 0$ and $g(1) = 1$. The corresponding premium calculation principle is

$$H(Y) = \int_0^\infty g[S_Y(t)] dt. \tag{4}$$

The PH-transform arises as a special case of (3) when $g(x) = x^{1/\rho}$.

The common point of risk adjusted premium calculation principles is that they all consist of taking the expected value of a distorted distribution of the original risk. They have many desirable properties, (see Wang (1996b)), namely:

- $E[Y] \leq H(Y) \leq \max(Y)$ (*positive loading and no ripoff*);
- $Pr\{Y = b\} = 1 \Rightarrow H(Y) = b$ (*no unjustified risk loading*);
- $H(aY + b) = aH(Y) + b, a \geq 0, b \geq 0$ (*linearity*);
- $H(Y)$ preserves first stochastic dominance [1]: $Y_1 \prec_{1st} Y_2 \Rightarrow H(Y_1) \leq H(Y_2)$;
- $H(Y)$ preserves the order of dangerousness [2]: $Y_1 \prec_D Y_2 \Rightarrow H(Y_1) \leq H(Y_2)$;
- For any two non-negative random variable $Y_1$ and $Y_2$, regardless of dependence, $H(Y_1 + Y_2) \leq H(Y_1) + H(Y_2)$ (*sub-additivity*);

[1] $Y_1 \prec_{1st} Y_2$ if and only if $S_{Y_1}(t) \leq S_{Y_2}(t)$, $\forall t \geq 0$.

[2] $Y_1$ is less dangerous than $Y_2$ ($Y_1 \prec_D Y_2$) if $E(Y_1) \leq E(Y_2)$ and there exists a unique crossing point $t_0$ such that $S_{Y_1}(t) \geq S_{Y_2}(t)$, $\forall t < t_0$ and $S_{Y_1}(t) \leq S_{Y_2}(t)$, $\forall t \geq t_0$.

- If $Y_1$ and $Y_2$ are comonotonic [1] then $H(Y_1+Y_2)=H(Y_1)+H(Y_2)$ (*additivity for comonotonic risks*);

**Definition**: A layer $h$ in excess of $b$ of a risk $Y$, which we will denote $I_{(b,b+h]}(Y)$, is defined as the loss from the stop loss cover [2]

$$I_{(b,b+h]}(Y)=\begin{cases} 0 & 0\leq Y<b, \\ Y-b & b\leq Y<b+h, \\ h & Y\geq b+h, \end{cases} \tag{5}$$

where $b$ is called the retention or the priority and $h$ is called the limit. We shall use $I_{(b,b+h]}$ instead of $I_{(b,b+h]}(Y)$ when only one risk is under consideration.

As a layer is always an increasing function of the original risk, the last property implies that, for any division $0=y_0<y_1<...<y_n<...$

$$H(Y)=\sum_{i=1}^{\infty} H(I_{(y_{i-1},y_i]}), \tag{6}$$

i.e., layer premiums are additive.

Based on the concavity of $g$, Wang (1996b) shows that the relative loading for an infinitesimal small layer at $(t,t+dt]$,

$$\phi(t)=\frac{H(I_{(t,t+dt]})}{E(I_{(t,t+dt]})}=\frac{S_Z(t)dt}{S_Y(t)dt}=\frac{g[S_Y(t)]}{S_Y(t)}, \tag{7}$$

is an increasing function of $t$. [3]

The minimum rate on line is an empirical reinsurance phenomenon. The rate on line is the premium divided by the limit $h$, and most reinsurers establish a minimum for this ratio, whatever is the priority. Based on this idea Wang (1996b) considers as desirable that $g'(0)=\infty$, in such a way that $E(I_{t,t+h]})$ converges to zero faster than $H(I_{t,t+h]})$ when $t$ goes to infinity. This will be clarified in the next section.

Wang (1996b) gives some examples of the risk adjusted premium principles, by specifying different functions $g$, namely:

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[1] $Y_1$ and $Y_2$ are comonotonic if there exists a risk $Y_3$ and weakly increasing functions $f_1$ and $f_2$ such that $Y_1 = f_1(Y_3)$ and $Y_2 = f_2(Y_3)$.

[2] As in this paper we are dealing with reinsurance, we follow the most common terminology for excess of loss and stop loss reinsurance, i.e. stop loss is an aggregate type of cover while excess of loss stands for individual claim amounts, see for instance Daykin, Pentikäinen and Pesonen (1994) and Gerathewohl (1980).

[3] Let $u = S_Y(t)$ and let us choose two points $t_1$ and $t_2$ such that $0 < t_1 < t_2$. As $S_Y(t)$ is a decreasing function of $t$ then $u_1 = S_Y(t_1) > S_Y(t_2) = u_2$. But $u_2$ can be written as a convex linear combination of the points 0 and $u_1$ with weights equal to $(u_1 - u_2)/u_1$ and $u_2/u_1$ respectively, from which it follows from $g(0) = 0$ and the concavity of $g$ that $g(u_2) \geq (u_2/u_1)g(u_1)$, this is $g[S_Y(t_2)]/S_Y(t_2) \geq g[S_Y(t_1)]/S_Y(t_1)$, which proves that $\phi(t)$ is an increasing function of $t$.

**PH-transform principle (PH)** $g(x) = x^{1/\rho},\ \rho \geq 1,\ (g'(0) = \infty)$;
**Dual power function principle (DP)** $g(x)=1-(1-x)^{\alpha}, \alpha \geq 1, (g'(0)=\alpha)$;
**Denneberg's absolute deviation principle (AD)**

$$g(x)=\begin{cases} (1+\alpha)x & 0 \leq x < 0.5 \\ \alpha+(1-\alpha)x & 0.5 \leq x \leq 1 \end{cases} 0 \leq \alpha \leq 1, (g'(0)=1+\alpha);$$

**Gini principle (GP)** $g(x)=(1+\alpha)x-\alpha x^2, 0 \leq \alpha \leq 1, (g'(0)=1+\alpha)$;
**Square-root function principle (SR)**

$$g(x)=\begin{cases} \frac{\sqrt{1+\alpha x}-1}{\sqrt{1+\alpha}-1}, & \alpha > 0 \\ x & \alpha=0, \end{cases} \left(g'(0)=\frac{0.5\alpha}{\sqrt{1+\alpha}-1}<\infty\right);$$

**Exponential function principle (E)**

$$g(x)=\begin{cases} \frac{1-e^{-\alpha x}}{1-e^{-\alpha}}, & \alpha > 0 \\ x & \alpha=0, \end{cases} \left(g'(0)=\frac{\alpha}{1-e^{-\alpha}}<\infty\right);$$

**Logarithmic function principle (L)**

$$g(x)=\begin{cases} \frac{\log(1+\alpha x)}{\log(1+\alpha)}, & \alpha > 0 \\ x & \alpha=0, \end{cases} \left(g'(0)=\frac{\alpha}{\log(1+\alpha)}<\infty\right)$$

As we have already mentioned, Wang (1996b) has considered as desirable that $g'(0)=\infty$, which, among the functions considered, is only true for the PH transform.

## 2 Applying the risk adjusted premium principles to reinsurance

These premium calculation principles seem to have all the nice properties we could think about. This is particularly important in relation to reinsurance. In this section we shall compare these premium principles, for some examples, as functions of the retention, for different forms of reinsurance.

### 2.1 Quota-share reinsurance

Consider a risk for which the aggregate claim amount in some fixed time interval is denoted by a random variable $Y$ with distribution function $F_Y(.)$ and survival function $S_Y(.)$.

Let $a$ be, for quota-share reinsurance, the insurer retention level. As the risk adjusted premiums are scale invariant, the reinsurance premium is in this case $(1-a)H(Y)$, where $H(Y)$ is the premium that should be charged if the whole risk was ceded. This is exactly what happens in practice for proportional reinsurance.

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## 2.2 Stop Loss

Let us consider a stop loss contract, such that the cedent retains $\min(Y, M)$ of risk $Y$, and reinsures $\max(0, Y-M)$, which is the same as $I_{(M,\infty)}$. The distribution function of the ceded part is then $F_M^R(t) = F_Y(t+M)$ and the corresponding survival function $S_M^R(t) = S_Y(t+M)$, both for $t \geq 0$. Denoting by $P(M)$ the reinsurer's premium with respect to this contract, we have that, if a risk adjusted premium calculation principle is used,

$$P(M) = \int_0^\infty g[S_Y(t+M)]\,dt = \int_M^\infty g[S_Y(t)]\,dt \tag{8}$$

Let $\psi(M) = P(M)/E(I_{(M,\infty)})$ and let $\phi(M)$ be defined as in (7), i.e. $\phi(M) = g[S_Y(M)]/S_Y(M)$. Note that as $P'(M) = -g[S_Y(M)]$ and $E'(I_{(M,\infty)}) = -S_Y(M)$ and using L'Hospital's rule we can say that

$$\lim_{M\to\infty} \psi(M) = \lim_{M\to\infty} \phi(M) = g'(0). \tag{9}$$

Hence if the PH-transform is applied we can say that

$$\lim_{M\to\infty} \psi(M) = \lim_{M\to\infty} \frac{P(M)}{E(I_{(M,\infty)})} = \infty. \tag{10}$$

Condition (9) clearly shows why it is desirable $g'(0)$ to be infinity. Although both $P(M)$ and $E(I_{(M,\infty)})$ converge to zero as $M$ goes to infinity, the later will go faster when $g'(0) = \infty$.

If the aggregate claim amount is limited, i.e. if there exists a finite $t_1$ for which $S_Y(t_1) = 0$, then the limit in (10) should be substituted by the limit when $M \to t_1$ which is still infinity.

When the PH-transform is applied, the stop loss premium is particularly easy to compute for some distributions as we will see in the next section.

### *2.2.1 The stop loss premium when the PH premium calculation principle is used*

1. If $Y$ is exponential distributed with mean $1/c$, i.e. if $S_Y(t) = e^{-ct}$ for $t \geq 0$, then:

$$P(M) = \frac{\rho}{c} e^{-\frac{c}{\rho}M}. \tag{11}$$

2. If $Y$ is Pareto distributed with parameters $(\alpha, \beta)$, i.e. if $S_Y(t) = \left(\frac{\beta}{\beta+t}\right)^\alpha$, for $t \geq 0$, then

$$P(M) = \frac{\beta+M}{\frac{\alpha}{\rho}-1}\left(\frac{\beta}{\beta+M}\right)^{\frac{\alpha}{\rho}}, \ \alpha > \rho. \tag{12}$$

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3. If $Y$ has a Weibull distribution with parameters $(c,\tau)$, i.e. if $S_Y(t)=e^{-ct^\tau}$, for $t\geq 0$, then

$$P(M)=\Gamma\left(\frac{1}{\tau}+1\right)\left(\frac{\rho}{c}\right)^{1/\tau}\left(1-J\left(\frac{1}{\tau},\frac{c}{\rho}M^\tau\right)\right), \tag{13}$$

where $J(a,y)$ denotes the incomplete gamma function, i.e.

$$J(a,y)=\frac{1}{\Gamma(a)}\int_0^y t^{a-1}e^{-t}dt,\ y>0.$$

4. If $Y$ has a Burr distribution with parameters $(\alpha,\beta,\tau)$, i.e. if $S_Y(t)=\left(\frac{\beta}{\beta+t^\tau}\right)^\alpha$, for $t\geq 0$, then

$$P(M)=\beta^{1/\tau}\frac{\Gamma\left(\frac{\alpha}{\rho}-\frac{1}{\tau}\right)\Gamma\left(\frac{1}{\tau}+1\right)}{\Gamma\left(\frac{\alpha}{\rho}\right)}B\left(\frac{\alpha}{\rho}-\frac{1}{\tau},\frac{1}{\tau},\frac{\beta}{\beta+M^\tau}\right),\ \alpha\tau>\rho, \tag{14}$$

where $B\ (a,\ b,\ y)$ denotes the incomplete Beta function, i.e.

$$B(a,b,y)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^y x^{a-1}(1-x)^{b-1}dx,\ 0<y<1.$$

For these distributions it is easy to deduce $P(M)$ because these distributions are closed under the PH-transform. For some other distributions we have to use numerical techniques to calculate $P(M)$.

If instead of an unlimited cover, only a limited layer, $h$ in excess of $M$, of the aggregate claim amounts is covered by the stop loss treaty, then the reinsurance premium is

$$H(I_{(M,M+h]})=\int_M^{M+h} g[S_Y(t)]\,dt=P(M)-P(M+h).$$

### 2.2.2 *Comparing different risk adjusted premium principles*

We are interested in comparing the stop loss premium when different transforms are applied. Assuming that the total premium for an entire risk is fixed, we will use various risk adjusted premium functionals to allocate the total premiums to layers.

Let $g_1$ and $g_2$ be two continuous, increasing and concave functions defined in $[0,1]$, with $g_1(0)=g_2(0)=0$ and $g_1(1)=g_2(1)=1$. Let us assume that

$$\int_0^\infty g_1[S_Y(t)]\,dt=\int_0^\infty g_2[S_Y(t)]\,dt, \tag{15}$$

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i.e. that the reinsurance premium for full coverage is the same for the two principles. $g_1(x)$ has to cross $g_2(x)$ at least once for $x \in (0,1)$, otherwise (15) could not hold. Let $U$ be a random variable with survival function $g_1[S_Y(t)]$ and $V$ another random variable with survival function $g_2[S_Y(t)]$. If $g_1(x)$ and $g_2(x)$ cross exactly once in (0, 1) and if $g_1'(0) > g_2'(0)$ we can say that $V$ is less dangerous than $U$. As in general the order in dangerousness implies stop loss order [1] (see e.g. Wang (1996a) or Kaas, Van Heerwaarden and Goovaerts (1994)) we can conclude, under the above conditions, that the stop loss premium is greater, for all the values of the retention limit, using transform $g_1$ than transform $g_2$. This together with the fact that, among the transforms considered in section 2, the PH-transform is the only one for which $g'(0) = \infty$, has the obvious implication: if $g_1$ is the PH-transform and $g_2$ any of the other transforms, if condition (15) holds and if $g_1$ and $g_2$ only cross once, then the PH principle gives a higher stop loss premium than the premium based on transform $g_2$, for all the values of the retention limit $M$. When $g_1$ and $g_2$ cross more than once we can still say, attending to (9) that there is an $\hat{M} \geq 0$ from where onwards the stop loss premium using $g_1$ is greater than when $g_2$ is used. This implies that, in general, the PH premium calculation principle, gives a higher premium than when any of the other transforms is used, provided that the retention is big enough.

We have calculated the stop loss premiums for the different principles considered in section 2, for two loss distributions: the Pareto (2,1) truncated at 1000 and the exponential with parameter 1 truncated at the point 8.33. The last value was chosen in such a way that both have the same expected value. $\rho$ was chosen equal to 1.15.

**Truncated Pareto loss distribution**

Table 1 shows the values of $\alpha$ (see the premium calculation principles in section 1) for the different risk adjusted premium principles, as well as for the expected value principle, calculated in such a way that the premium for full coverage is the same and equal to $\int_0^\infty (S_Y(t))^{1/\rho} dt$, with $\rho = 1.15$.

Figure 1 shows *P(M)*, as function of the retention limit $M$, when the several risk adjusted premium calculation principles are used, as well as for the expected value principle. The curves between the expected value principle and the PH principle curves are ordered by increasing order of the premiums for every $M$ greater than 0,4, what is to say according to the order of magnitude of $g'(0)$ ($1 + \alpha$ for the expected value principle).

As we have already seen, if two functions $g_1$ and $g_2$, with $g_1'(0) > g_2'(0)$ cross exactly once then, $P_1(M) > P_2(M)$, for all $M$. This is the case, in the example, for the PH, the square root, the exponential and the logarithmic

[1] $V$ is smaller than $U$ in stop loss order ($V \prec_{sl} U$) if and only if $\int_x^\infty S_V(t)dt \leq \int_x^\infty S_U(t)dt$, $\forall x \geq 0$.

TABLE 1

LOADING COEFFICIENTS FOR $\rho = 1.15$; STOP LOSS REINSURANCE; TRUNC. PARETO DISTRIBUTION.
($g'(0) = \infty$ for the PH principle)

| *Principle* | $\alpha$ | $g'(0)$ |
|---|---|---|
| EV | 0.343105 | $(1 + \alpha =)$ 1.343105 |
| AD | 0.414264 | 1.414264 |
| DP | 1.480544 | 1.480544 |
| Gini | 0.515115 | 1.515115 |
| E | 0.949823 | 1.548985 |
| L | 1.371120 | 1.588116 |
| SR | 4.157265 | 1.635481 |

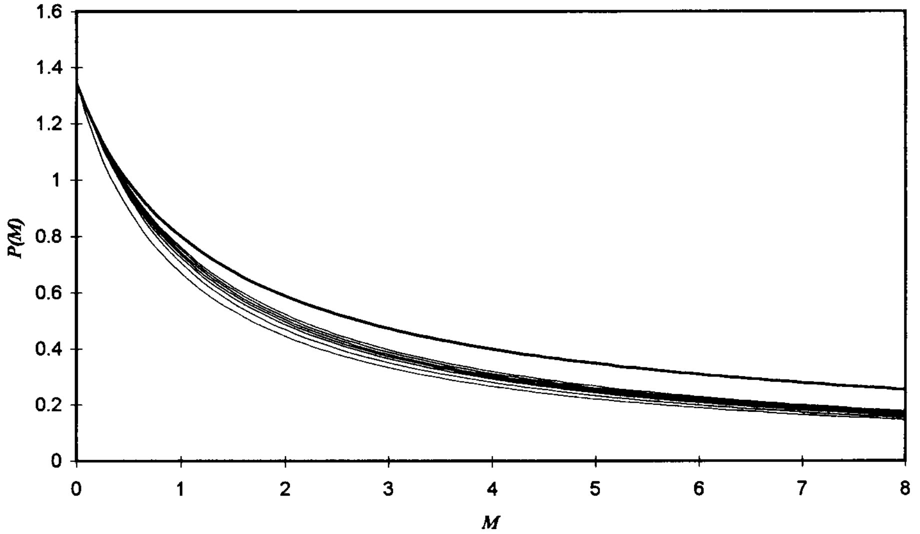


FIGURE 1: Stop loss reinsurance – truncated Pareto distribution

transforms, which cross all the others exactly once. The conclusion is simple: they are all very similar, with one singularity: the PH-transforms gives much higher premiums for not very small values of the retention.

## Truncated exponential loss distribution

Table 2 and Figure 2 are in all similar to Table 1 and Figure 1, respectively, but using the truncated exponential distribution function. The ordering of the premiums, between the expected value principle and the PH principle, for $M > 0.9125$ is the same as for the Pareto distribution, because the order of $g'(0)$ was the same.

TABLE 2

LOADING COEF. FOR $\rho = 1.15$; STOP LOSS REINSURANCE; TRUNC. EXPONENTIAL DISTRIBUTION

| ***Principle*** | $\alpha$ | $g'(0)$ |
|---|---|---|
| EV | 0.148521 | $(1 + \alpha =)$ 1.148521 |
| AD | 0.214432 | 1.214432 |
| DP | 1.248052 | 1.248052 |
| Gini | 0.297496 | 1.297496 |
| E | 0.579275 | 1.317446 |
| L | 0.751445 | 1.340810 |
| SR | 2.020900 | 1.369037 |

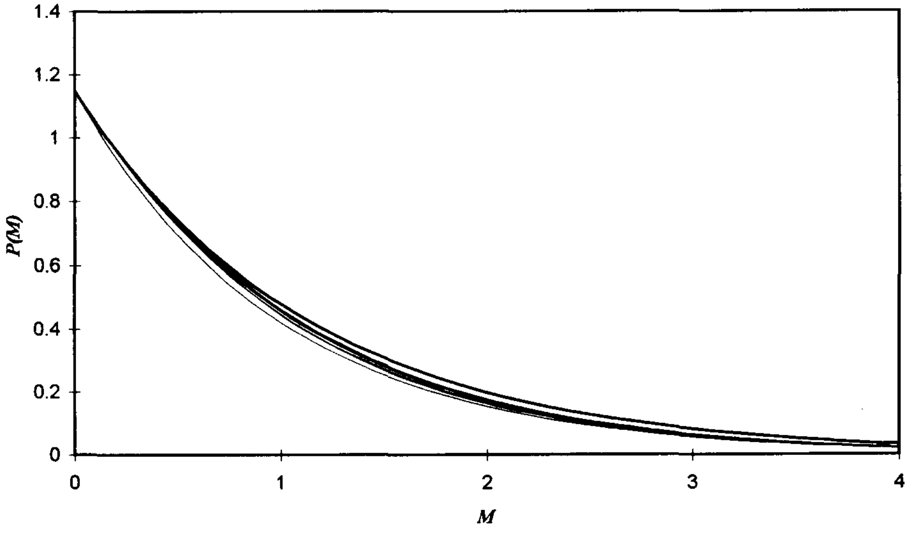


FIGURE 2: Stop loss reinsurance – truncated exponential distribution

Figure 3 shows $P(M)$ when the PH-transform principle and the expected value principle are used, for the truncated exponential and Pareto loss functions. As it is natural the Pareto loss distribution implies a higher premium than the exponential distribution, for the same priority. The same happens for the other transforms, but the difference is higher when the PH premium principle is applied.

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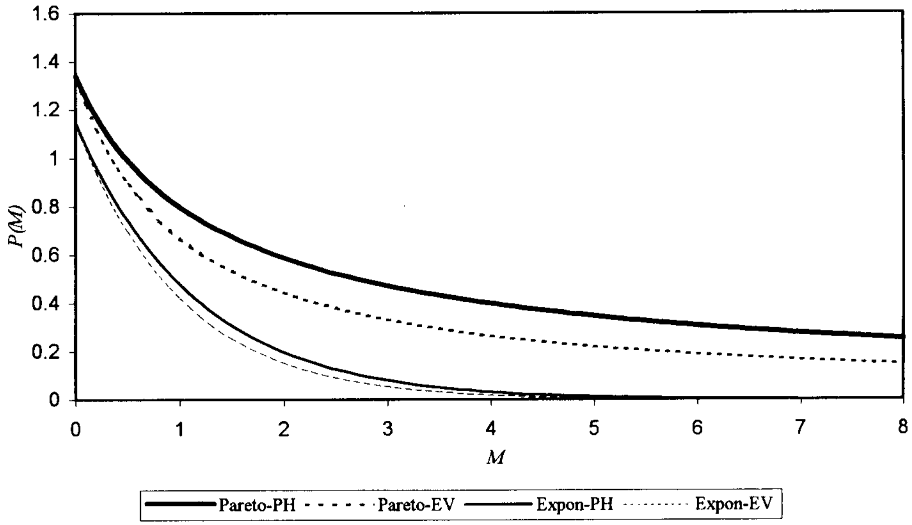


FIGURE 3: Stop loss reinsurance – PH-principle; Pareto versus exponential

### *2.2.3 Excess of Loss*

We assume that $Y$ has a compound distribution, so that

$$Y = \sum_{i=0}^{N} X_i$$

where $X_0 \equiv 0, \{X_i\}_{i=1,2,\ldots,N}$ is a sequence of i.i.d. non-negative random variables, denoting the individual claim sizes, with common distribution $G$ independent of the number of claims $N$ which is a counting random variable. Let $F$ be the distribution function of $Y$. We shall consider the case where $N$ is either a Poisson, or a negative binomial random variable. Let us consider an excess of loss arrangement, such that the aggregate ceded claims are $\sum_{i=0}^{N} \max(0, X_i - M)$, and the aggregate retained claims are $\sum_{i=0}^{N} \min(X_i, M)$.

To obtain in this case $P(M)$ we have to calculate first, using numerical techniques, the survival compound function, in second place to distort it and finally to calculate the mean with respect to the distorted distribution. This procedure is very time consuming, specially for very skewed claim amounts, as for the Pareto (2,1), even when truncated (in a big value).

For the calculations we have used Panjer's algorithm, after discretising the individual claim amount distribution, by matching the first moment, in steps of 1/30, for the truncated Pareto, and 0.0025 for the exponential case. The survival functions were calculated until they reached the value $10^{-9}$.

Table 3 gives the coefficient $\alpha$ for each premium calculated in such a way that the premiums are equal to the PH premium for the retention limit $M$ = 0, with $\rho$=1.15, for a compound Poisson distribution with $\lambda$=1 and individual claim amounts Pareto distributed with parameters (2,1) truncated at 1000. Figure 4 shows the reinsurance risk adjusted premiums, as well as the reinsurance premium calculated according to the expected value principle.

TABLE 3

LOADING COEFFICIENTS FOR $\rho$ = 1.15; XL-REINSURANCE; POISSON-TRUNCATED PARETO

| *Principle* | $\alpha$ | $g'(0)$ |
|---|---|---|
| EV | 0.380533 | (1 + $\alpha$ =) 1.380533 |
| AD | 0.391033 | 1.391033 |
| DP | 1.465864 | 1.465864 |
| Q | 0.494443 | 1.494443 |
| E | 0.910526 | 1.523415 |
| L | 1.293763 | 1.558387 |
| SR | 3.856743 | 1.601901 |

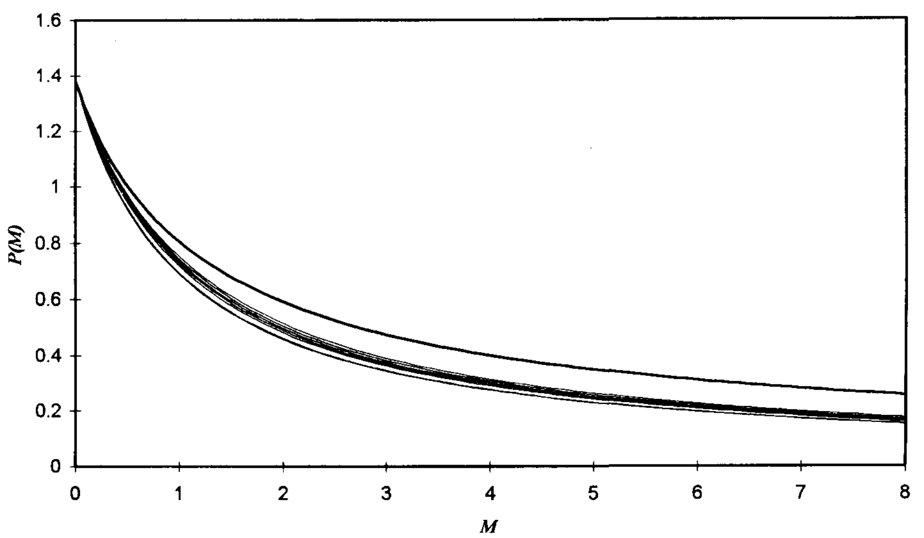


Figure 4: XL reinsurance: Poisson claim numbers and trunc. Pareto claim amounts.

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Although we are not able to take any conclusions in theoretical terms, we can say from all the cases that we have considered that the behaviour is very similar to the stop loss case. The order of magnitude of the risk adjusted premiums for big enough values of the retention is the same as the order of the $g'(0)$'s.

When the individual claims are truncated exponential instead of truncated Pareto, with the same expected value, the relative behaviour of the risk adjusted premiums is similar, but the figures are lower in the exponential case, as they should be.

When we use a different Poisson parameter the conclusions are all similar.

Table 4 gives the reinsurance premiums calculated according to the PH transform principle and the ratio $\psi(M)$, both for the truncated Pareto and truncated exponential individual claim amounts.

The relative behaviour of the risk adjusted premiums is similar when a compound negative binomial distribution is used for the aggregate claim amounts.

Figure 5 shows the PH premium as function of the retention for the compound Poisson with $\lambda = 1$ and for the compound negative binomial with mean equal to 1 and variance equal to 10, in the Pareto case. An interesting feature is that for not small values of the retention the premiums are almost identical for both the compound Poisson and the compound negative binomial.

TABLE 4

XL-REINSURANCE; COMPOUND POISSON DISTRIBUTION

| | *Trunc. Pareto claim amounts* | | *Trunc. exponential claim amounts* | |
|---|---|---|---|---|
| ***M*** | ***PH*** | $\psi(M)$ | ***PH*** | $\psi(M)$ |
| 0 | 1.377767 | 1.380524 | 1.208096 | 1.210526 |
| 1 | 0.804207 | 1.614856 | 0.489168 | 1.336667 |
| 2 | 0.590210 | 1.781285 | 0.200338 | 1.499517 |
| 3 | 0.474030 | 1.911361 | 0.081743 | 1.693323 |
| 4 | 0.399763 | 2.018928 | 0.032626 | 1.915186 |
| 5 | 0.347647 | 2.111116 | 0.012341 | 2.166502 |
| 6 | 0.308790 | 2.192076 | 0.004122 | 2.457970 |
| 7 | 0.278550 | 2.264447 | 0.000993 | 2.833274 |
| 8 | 0.254257 | 2.330012 | 0.000053 | 3.600732 |
| 9 | 0.234253 | 2.390040 | | |
| 10 | 0.217457 | 2.445466 | | |
| 15 | 0.161810 | 2.673750 | | |
| 20 | 0.130073 | 2.849837 | | |
| 25 | 0.109260 | 2.994291 | | |
| 30 | 0.094427 | 3.117334 | | |

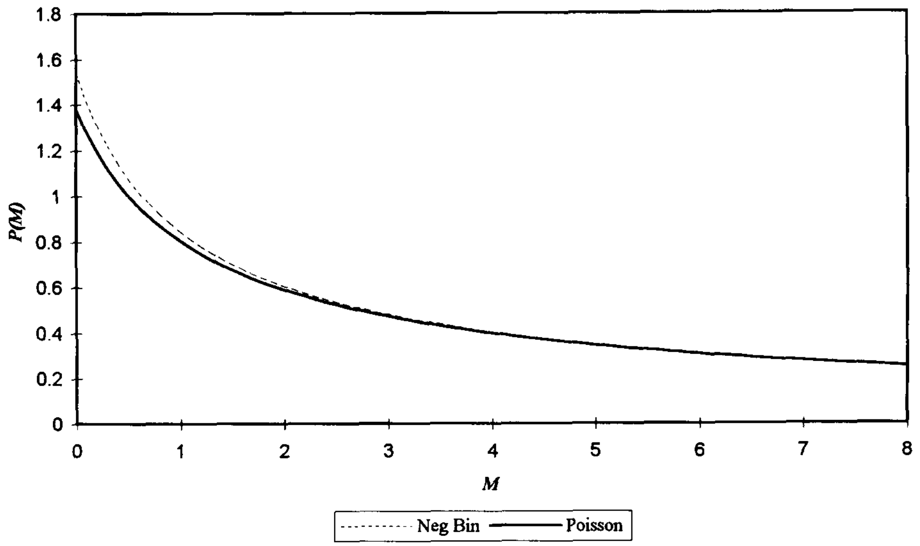


FIGURE 5: XL reinsurance; Poisson versus negative binomial claim numbers; trunc. Pareto

Figure 6 shows the ratio $\psi$ (PH premium divided by the expected value) as a function of the Poisson parameter $\lambda$ and truncated Pareto claim amounts, for three retention levels: 0, 20 and 100. The calculations were performed using a precision of $10^{-6}$ in Panjer's algorithm.

## 3 THE ADJUSTMENT COEFFICIENT AS FUNCTION OF THE RETENTION

Let $\theta$ be the retention level [1], taking values in the interval from 0 to $L$, with $L = 1$ for quota-share reinsurance and $L=\infty$ for stop loss or excess of loss reinsurance. Let $Y(\theta)$ be the aggregate net (of reinsurance) claims and $P(\theta)$ the reinsurance premium. $Y$ and $P$ denote the aggregate gross claims and the gross (but net of expenses) premium, respectively.

Let $C(\theta)=P(\theta)-E[Y-Y(\theta)]$ be the loading of the reinsurance premium.

The adjustment coefficient of the retained risk is, for each $\theta$, as it is well known, the unique positive root $R=R(\theta)$, when it exists, of

$$E[\exp[R[Y(\theta)-(P-P(\theta))]]]=1. \tag{16}$$

The adjustment coefficient of the reinsurer is, for each $\theta$, the unique positive root $\tilde{R}=\tilde{R}(\theta)$ of

$$E\left[\exp\left[\tilde{R}[Y-Y(\theta)-P(\theta)]\right]\right]=1. \tag{17}$$

[1] We shall denote by $a$ the quota-share retention level, and by $M$ the excess of loss or stop loss retention limit.

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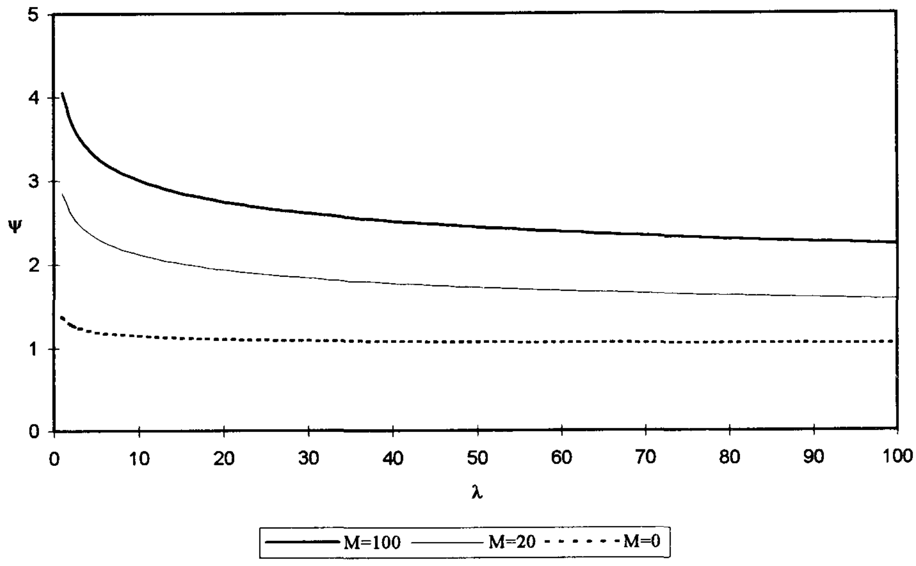


FIGURE 6: $\psi$ as function of $\lambda$

Waters (1983) studied the behaviour of the adjustment coefficient of the retained risk, for quota-share, excess of loss and stop loss reinsurance.

In all the examples considered we have chosen $P$ in such a way that $P=0.95P(0)$. In this way the insurer will not make a profit with full coverage.

## 3.1 Quota-share

Under very weak assumptions Waters (1983) proved that the adjustment coefficient of the retained risk $R(a)$ is a unimodal function of $a$ attaining its maximum value at $a = 1$ if and only if

$$P'(1)e^{R(1)P}+E\left[Ye^{R(1)Y}\right]\leq 0. \tag{18}$$

As we have already mentioned $P(a)=(1-a)P(0)=(1-a)H(Y)$ when a risk adjusted premium is applied, which implies that

$$C(a)=(1-a)[H(Y)-E(Y)]. \tag{19}$$

As it is easy to verify the risk adjusted premiums satisfy all the assumptions made by Waters (1983) if we consider that

$$H(Y)>P, \tag{20}$$

(i.e. that the ceding insurer does not make an expected profit by means of reinsurance) in which case we can say that his result applies.

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When we consider the adjustment coefficient from the reinsurers point of view, i.e. the unique positive root $\tilde{R}=\tilde{R}(a)$ of

$$E\left[\exp\left[\tilde{R}(1-a)(Y-P(0))\right]\right]=1, \tag{21}$$

the adjustment coefficient is such that $\tilde{R}(a)=\tilde{R}(0)/(1-a)$, which is an increasing function of $a$, going to infinity when $a$ goes to 1.

Although a bit puzzling this is a good point in favour of both the adjustment coefficient and of the risk adjusted premiums: let $\psi(u)$ be the probability of ultimate ruin associated to $(Y, P(0))$, which is equal to the probability of ultimate ruin associated to $((1-a)Y, P(a))$, and an initial reserve of $(1-a)u$ (note that for risk adjusted premium principles $P(a)=(1-a)P(0)$). This is exactly what happens with the upper bound provided by Lundberg's inequality so far as the reinsurance premium calculation principle used is scale invariant, which is the case for the risk adjusted premiums (this is also the case for the expected value and the standard deviation principles). This implies that it does not make sense to maximize the reinsurer's adjustment coefficient, by itself, when quota-share reinsurance is considered. Note that if the insurer's adjustment coefficient goes to zero as the expected profit goes to zero, the reinsurer's adjustment coefficient goes to infinity as the expected profit goes to zero. The reinsurer share should only depend on the size of the company (through the reserve) and on the risk aversion index (through $P(0)$).

### 3.2 Stop loss

When the expected value principle is applied, it is not possible to prove, in general, that the insurer's adjustment coefficient is unimodal with the retention. The same happens when the risk adjusted premiums are applied. We were not able, however, of finding a counterexample.

Figure 7 shows the insurer's adjustment coefficient, assuming that the aggregate claim amounts are either truncated exponential or truncated Pareto as in section 2.2.2, and the reinsurer premium is calculated according to the expected value or the PH principle. In the exponential case the adjustment coefficient attaints its maximum value at 0.805 and 5.27, when the reinsurance premium is calculated according to the expected value and the PH principle respectively. In the Pareto case the respective values are 0.5 and 4.0.

The most interesting feature is that when the expected value principle is applied the adjustment coefficient for the Pareto case is greater than for the exponential case, but the order is reversed if the PH principle is applied. This is quite a point in favour of the PH principle and is due to the fact that the PH principle differentiates much better the most dangerous risks than the expected value principle.

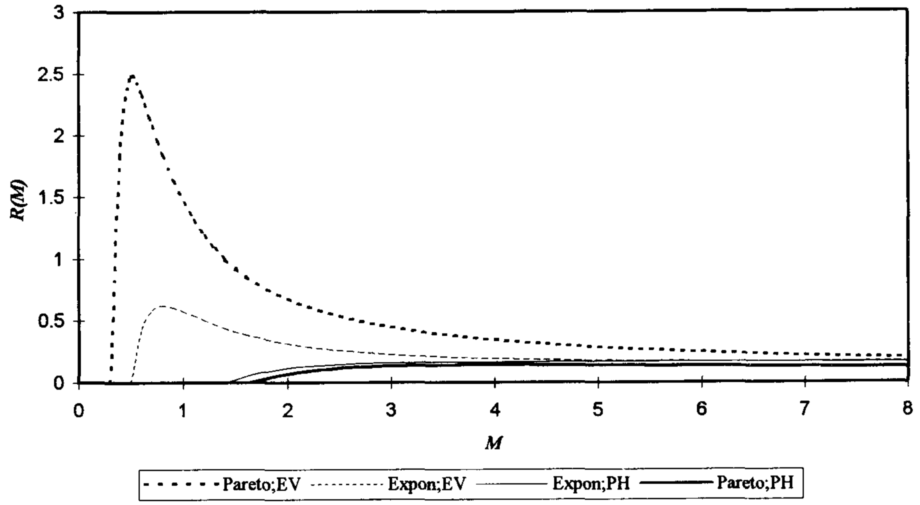


FIGURE 7: Stop loss reinsurance: the insurer's adjustment coefficient

Figure 8 shows the reinsurer's adjustment coefficient for the same examples. This function does not obey to a specific pattern. When the premium used is calculated according to the PH principle it is a decreasing function of $M$ for very skewed distributions, unless $\rho$ takes extremely high values, where it increases.

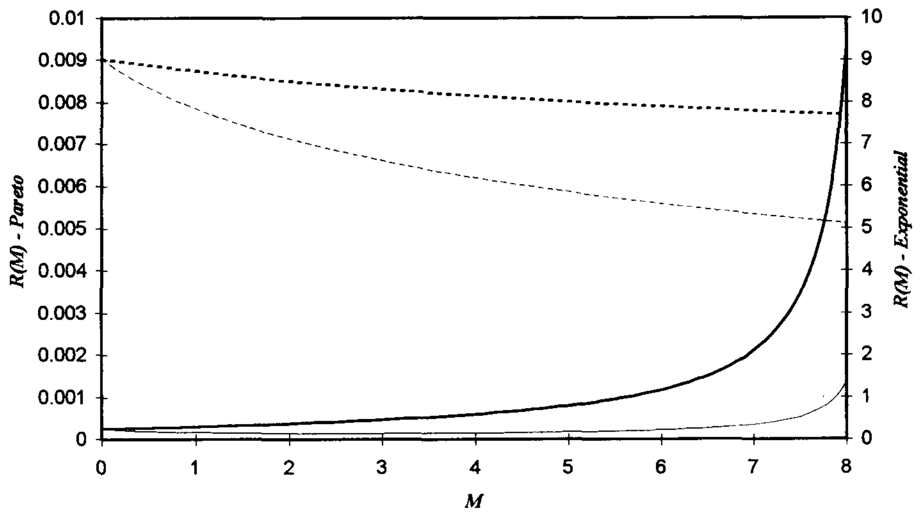


FIGURE 8: Stop loss reinsurance: the reinsurer's adjustment coefficient

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In the exponential case it increases with $M$, going to infinity as $M$ goes to the truncation point, both for the expected value and PH principle. Note that when the random variable is a non truncated exponential and the expected value principle is applied, the reinsurer's adjustment coefficient is independent of $M$.

### 3.3 Excess of Loss

When the expected value principle is applied to calculate the reinsurer's premium and the aggregate claims are compound Poisson, the insurer's adjustment coefficient is, as it is well known, a unimodal function of the retention. We were not able of proving a similar result when the PH principle applies, but we could not find a counterexample either.

The insurer's and the reinsurer's adjustment coefficients, behave in a very similar way to the stop loss case, when $\lambda$ is 1. Figure 9 is equivalent to Figure 7, but for excess of loss reinsurance.

In the truncated exponential case, when the premium principle is the expected value the maximiser of the adjustment coefficient is 0.69 and 3.73 when the principle used is the PH premium principle. For the Pareto case the corresponding values are 0.63 and 3.37 respectively.

Figure 10 shows the insurer's adjustment coefficient when the aggregate claims are compound negative binomial, as opposed to the compound Poisson, with individual claim amounts truncated Pareto (2,1) when the premium used is calculated according to the PH principle. The expected

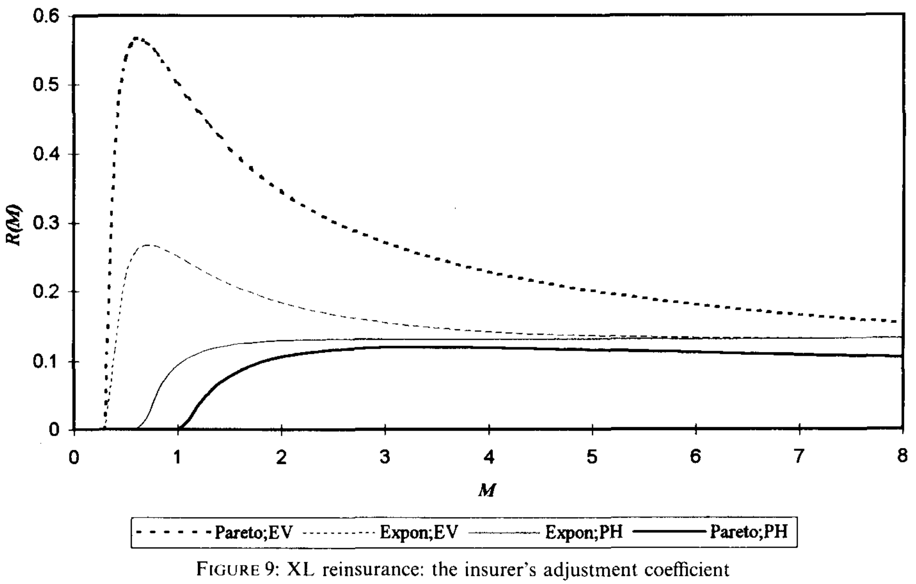


FIGURE 9: XL reinsurance: the insurer's adjustment coefficient

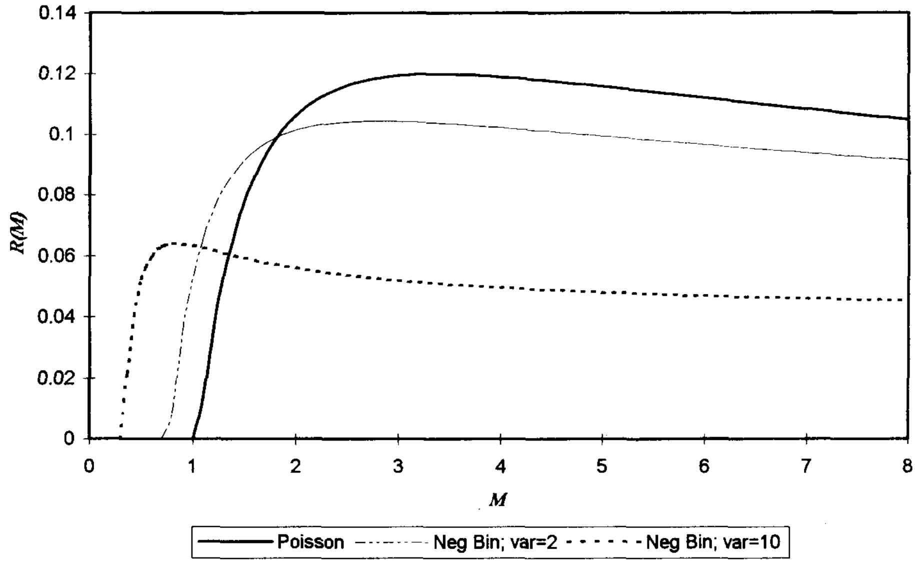


FIGURE 10: XL reinsurance: the insurer's adjustment coefficient; compound negative binomial

value of the claim numbers is 1, and the variance is in one case 2 and in the other 10. The conclusion is expected, i.e. for the same priority the retained adjustment coefficient increases as the variance decreases.

## 4 CONCLUSIONS

The main conclusion to be taken when applying risk adjustment premium principles to non-proportional reinsurance is that with the exception of the proportional hazard (PH) premium calculation principle, all the other behave in a way similar to the expected value principle. Hence although all of the risk adjusted premium principles considered in the paper share common and very interesting properties, just the PH premium principle provides, from the practical point of view, significant differences when compared to the classical expected value principle. This is due to the fact that it is the only principle, among the principles studied, for which $g'(0)=\infty$. Using this property we have also proved that the stop loss reinsurance premium when calculated using the PH premium principle gives a higher premium than any of the other transforms, provided that the priority is big enough.

Also, we have mentioned in section 3, when using the Pareto distribution versus the exponential distribution for modelling the claim size, the PH principle discriminates much better the most dangerous risks than the expected value principle. We obtain a higher value for the insurer's adjustment coefficient for the Pareto case when the reinsurance premium is calculated according to the expected value principle, but the order is reversed if the PH principle is applied.

Although the examples presented in the paper assumed that $\rho=1.15$, the main conclusions are independent of this particular value. The difference between the PH principle and the expected value principle increase with the value of the risk aversion index $\rho$.

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JOÃO ANDRADE E SILVA
*Instituto Superior de Economia e Gestão*
*Rua do Quelhas, 6*
*1200 Lisboa*
*Portugal*
*email:joaoas@iseg.utl.pt*