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Discrete Series of Classical Groups

Yuanli Zhang

Abstract. Let G_n be the split classical groups $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n+1)$ and $\mathrm{SO}(2n)$ defined over a p-adic field F or the quasi-split classical groups $U(n,n)$ and $U(n+1,n)$ with respect to a quadratic extension E/F. We prove the self-duality of unitary supercuspidal data of standard Levi subgroups of $G_n(F)$ which give discrete series representations of $G_n(F)$.

1 Introduction

Let F be a non-archimedean local field of characteristic zero. Let G_n be a classical split group $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n+1)$, $\mathrm{SO}(2n)$ defined over F, or a unitary group $U(n,n)$, $U(n+1,n)$ with respect to a quadratic extension E/F. We fix a Borel subgroup B defined over F. For a standard parabolic subgroup P of G_n defined over F with $P = MN$ as its Levi decomposition, we have

$$M \simeq H_{m_1} \times \cdots \times H_{m_r} \times G_{m_0}, \quad \sum_{i=1}^{r} m_i + m_0 = n.$$

Here $H_{m_i} = \mathrm{GL}_{m_i}$ when G_n is a split classical group, and $H_{m_i} = \mathrm{Res}_{E/F}\,\mathrm{GL}_{m_i}$ when G_n is a unitary group. We allow the possibility that $m_0 = 0$, in which case we let $G_{m_0} = 1$. Note $H_{m_i}(F) \simeq \mathrm{GL}_{m_i}(E)$, when G_n is a unitary group. Suppose that

$$\tau = \rho_1 \otimes \cdots \otimes \rho_r \otimes \sigma$$

is a unitary supercuspidal representation of $M(F)$. When $m_0 = 0$, we let $\sigma = \{1\}$.

In this paper we study the problem of when τ is special with respect to G_n. By τ being special with respect to G_n, we mean that there exists a $\lambda \in \mathfrak{a}^*$ such that the parabolically induced representation $I(\lambda, \tau)$ has a discrete series composition factor. The notation $\mathfrak{a}^*$ will be explained in the next section. In general, for a connected reductive group defined over F, Silberger [13] gave a criterion for this question. What we will do here is to give a more concrete criterion on the specialty of τ, for the classical groups G_n by applying Silberger's result. Let $m(\rho)$ be the number of the times with which ρ appears among $\{\rho_1, \ldots, \rho_r\}$. The following is our result.

Theorem *Let*

$$M \simeq H_{m_1} \times \cdots \times H_{m_r} \times G_{m_0}, \quad \sum_{i=1}^{r} m_i + m_0 = n$$

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be the Levi factor of a standard parabolic subgroup of G_n, and

$$\tau = \rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma$$

be a unitary supercuspidal representation of $M(F)$. Then τ is special with respect to G_n if and only if

(1) $\tilde{\rho}_i \simeq \rho_i$, (resp. $\tilde{\bar{\rho}}_i \simeq \rho_i$, for unitary groups), for $1 \leq i \leq r$, and
(2) if $m(\rho_i) = 1$, then $\rho_i \otimes \sigma$ is special with respect to $G_{m_i+m_0}$.

There have been several approaches to classifying discrete series representations of classical groups (*e.g.* [14], [6]). A similar result was obtained by Tadic for $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$ in [14], and by Ban for $\mathrm{SO}(2n)$ in [1]. Casselman's criterion in [3] for discrete series representations was applied in their proof, which is a different approach from ours. Here we shall pursue the original approach of Harish-Chandra and Silberger [11], [12], [13]. From the construction theory of admissible representations of a connected reductive p-adic group (see [3]), every irreducible admissible representation is isomorphic to a composition factor of the parabolically induced representation from a supercuspidal representation of a Levi factor of a standard parabolic subgroup. Moreover every supercuspidal representation is a twist of a unitary supercuspidal representation by a element λ of $\mathfrak{a}^*$. So now the problem of classifying the discrete representations of a classical group $G_n(F)$ is to find all $\lambda \in \mathfrak{a}^*$ such that the induced representation indeed has a discrete series representation as a composition factor of it. Note that the cardinality of the set of the candidates λ who may give a discrete series representation is only finite. So theoretically, Casselman's criterion on discrete series representations can be applied to test which λ indeed gives a discrete series representation. We are planning to pursue this as our future work. Moreover, part (1) of the Theorem is among the ingredients necessary for the proof of the holomorphy of certain L-functions [9] attached to tempered representations of classical groups which is given in [4] (Theorem 4.1 of [4]) and one significance of this paper is to provide a proof of this following Harish-Chandra's work. Recently Zhang [15], using the results here, proved some results about the support of a supercuspidal representation of $G_n(F)$ in the general linear group.

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2 Silberger's Criterion

In this section, we recall Silberger's criterion in [13] on the specialty of a unitary supercuspidal representation.

Let G be a connected reductive quasi-split group defined over a p-adic field F, and $G(F)$ be the group of the F-rational points of G. We fix a Borel subgroup $B = TU$ of G defined over F, where T is a maximal torus of G and U is the uniponent radical of B. Let Φ be the F-roots of G with respect to a maximal F-split subtorus A_o contained in T. We denote by Φ^+ the set of positive roots of Φ with respect to B, and Δ the set of simple roots of it. Now any standard parabolic subgroup G defined over F with respect to B (*i.e.*, the one containing B) is of the form $P = MN$ with M the centralizer of a subtorus A of A_o and N the uniponent radical of P. Let $X(A)$ be the abelian group of F-rational characters of A, *i.e.*, the morphisms of A to the multiplicative group $\mathbb{G}_m$ which are defined over F. Then one defines

$$\mathfrak{a}^* = X(A) \otimes_{\mathbb{Z}} \mathbb{R},$$

and denote by $\mathfrak{a}^*_{\mathbb{C}}$ the complexification of $\mathfrak{a}^*$.

For a given tempered representation τ of $M(F)$, one can associate a function $\mu(\nu, \tau)$ on $\mathfrak{a}^*_{\mathbb{C}}$, which is called the Plancherel measure. See [11] for this. Let $\sum(G, A)$ denote the set of the roots of A in G. Its elements can be identified with the elements of Φ which do not vanish on A modulo the restriction to A. Let $\sum_r(G, A)$ be the reduced roots of $\sum(G, A)$. In general, $\sum_r(G, A)$ may not be a root system, but it generates $\mathfrak{a}^*/\mathfrak{z}^*$, where $\mathfrak{z}^* = X(Z) \otimes_{\mathbb{Z}} \mathbb{R}$ and Z is the connected component, where the identity element lies, of the center of G. We call the dimension of A/Z which is also the dimension of $\mathfrak{a}^*/\mathfrak{z}^*$, the rank of $\sum_r(G, A)$. For $\beta \in \sum_r(G, A)$, let A_β be the maximal subtorus contained in the kernel of β, and M^β be the centralizer of A_β in G. Then $M(N \cap M^\beta)$ is a maximal parabolic subgroup of M^β over F with its Levi factor M. Then for this rank one case, we have the Plancherel measure $\mu_\beta(\nu, \tau)$ associated to these data.

Suppose that τ is a unitary supercuspidal representation of $M(F)$. We say τ is special with respect to G, if there is a $\lambda \in \mathfrak{a}^*$ perpendicular to $\mathfrak{z}^*$, such that the induced representation $I_P^G(\lambda, \tau)$ has a discrete series representation as a composition factor. When P is a maximal standard parabolic subgroup of G, let β be the unique simple root in N. Then we know that τ is special if and only if $\mu_\beta(\lambda, \tau)$ vanishes at zero, and that if τ is special, then there is a unique positive $\lambda \in \mathfrak{a}^*$ such that $I(\lambda, \tau)$ has a discrete series composition factor. For a given $\beta \in \sum_r(G, A)$, we say β is special, if $\mu_\beta(\nu, \tau)$ vanishes at zero. Let $\sum''(\tau)$ be the set of all special roots in $\sum_r(G, A)$. Observe that $\sum''(\tau)$ is a root system in the space spanned by the elements of $\sum''(\tau)$. Then Silberger's result says that τ is special if and only if there exist enough special roots. We record the his result in the following theorem.

Theorem 2.1 **(Silberger, [13])** *τ is special if and only if there exist r linear independent special roots, where r is the rank of $\sum_r(G, A)$.*

The proof of this theorem is based on a result in [11] and by exhibiting a $\lambda \in \mathfrak{a}^*$ which gives a discrete series subquotient. Such a particular λ used by Silberger corresponds to the set of simple roots of $\sum''(\tau)$. As Silberger remarked in his paper, there is a serious question that if every discrete series representation comes from one of these λ corresponding to the set of simple roots in $\sum''(\tau)$. For general linear groups, it does from Berstein's result on the classification of discrete series representations of $\mathrm{GL}_n(F)$.

3 The Special Roots

From now on, we let $G = G_n$ be a split classical group $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n+1)$, $\mathrm{SO}(2n)$ defined over F, or the unitary group $U(n, n)$ or $U(n+1, n)$ with respect to a quadratic extension

E/F. Let $P = MN$ be a standard parabolic subgroup of G_n defined over F, with its Levi factor M. In this section we first study $\sum_r(G,A)$, as well as the Levi embedding of M into M^β for every $\beta \in \sum_r(G,A)$ and determine the specialty of the root β.

We call the roots in $\sum_r(G,A)$ which are restrictions of simple roots in Δ simple roots of $\sum_r(G,A)$, and use $\sum_r^o(G,A)$ for the set of simple roots of $\sum_r(G,A)$. Some times later we may identify $\sum_r^o(G,A)$ with a subset of Δ, instead of saying "the restriction of a subset of Δ". Let

$$\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$

indexed as in the corresponding Dynkin diagram. Suppose that $P = P_\theta$ corresponds to the subset

$$\theta = \Delta - \{\alpha_{n_1}, \alpha_{n_2}, \dots, \alpha_{n_r}\}, \quad \text{with } n_i < n_j \text{ for } 1 \le i < j \le r.$$

Let β_i be the restriction of α_{n_i} on A, $1 \le i \le r$. Then

$$\Sigma_r^o(G,A) = \{\beta_1, \beta_2, \dots, \beta_r\},$$

and the rank of $\sum_r(G,A)$ is r.

In this section we always assume that $\beta_r \neq \alpha_{n-1}$ when $G_n = \mathrm{SO}(2n)$. The situation of $\beta_r = \alpha_{n-1}$ and $G_n = \mathrm{SO}(2n)$ can be deduced to the situation here. This will be explained in the end of Section 5. Now we assume

$$M \simeq H_{m_1} \times \cdots \times H_{m_r} \times G_{m_0}.$$

Here $H_m = \mathrm{GL}_m$ when G_n is a split classical group, and $H_m = \mathrm{Res}_{E/F}\,\mathrm{GL}_m$ when G_n is a unitary group with respect to E/F.

To apply Silberger's criterion, we need to study the linear relations among the elements of $\sum_r(P,A)$. For simplicity, we define the following notations. Set

$$\beta_i' = 2(\beta_i + \cdots + \beta_{r-1}) + \beta_r, \quad 1 \le i \le r-1$$
$$\beta_i'' = 2(\beta_i + \cdots + \beta_{r-2}) + \beta_{r-1} + \beta_r, \quad 1 \le i \le r-2$$
$$\beta_{i,j} = \beta_i + \beta_{i+1} + \cdots + \beta_j, \quad 1 \le i < j \le r$$
$$\beta_{i,j,r} = \beta_i + \cdots + \beta_{j-2} + \beta_r, \quad 1 \le i < j \le r-2$$
$$\beta_{i,j,r-1,r} = \beta_i + \cdots + \beta_j + \beta_{r-1} + \beta_r, \quad 1 \le i < j \le r-2$$
$$\beta_{i,j}' = \beta_i + \beta_{i+1} + \cdots + \beta_{j-1} + 2(\beta_j + \cdots + \beta_r), \quad 1 \le i < j \le r$$
$$\beta_{i,j,r}' = \beta_i + \beta_{i+1} + \cdots + \beta_{j-1} + 2(\beta_j + \cdots + \beta_{r-1}) + \beta_r, \quad 1 \le i < j \le r-1$$
$$\beta_{i,j,r-1,r}' = \beta_i + \cdots + \beta_{j-1} + 2(\beta_j + \cdots + \beta_{r-2}) + \beta_{r-1} + \beta_r, \quad 1 \le i < j \le r-2.$$

When $G_n = \mathrm{Sp}(2n), \mathrm{SO}(2n+1), \mathrm{SO}(2n)$, Φ is of the type C_n, B_n, D_n, respectively. When $G_n = U(n,n)$, the unrestricted root system $\tilde{\Phi}$ is of the type ${}^2A_{2n-1}$, and the F-root system Φ is of the type C_n. When $G_n = U(n+1,n)$, the unrestricted root system $\tilde{\Phi}$ is of the type

${}^2A_{2n}$, and the F-root system Φ is of the type BC_n which is not a reduced root system, the reduced roots Φ_r of Φ is of the type B_n. Since all the roots we will consider are reduced F-roots, it is enough to only consider the situations of $G_n = \mathrm{Sp}(2n)$ or $\mathrm{SO}(2n+1)$ or $\mathrm{SO}(2n)$. Actually, in the rest of this paper, we will read ${}^\tau g^{-1}$ and $\tilde{\rho}$ as ${}^\tau \bar{g}^{-1}$ and $\tilde{\bar{\rho}}$, respectively for the situation of unitary groups, in the all conclusions and their proofs in this paper. Here ${}^\tau g$ is the transpose of g with respect to the second diagonal, and $\bar{g}$ is the conjugation of g under the action of the nontrivial element of the Galois group of E/F.

Lemma 3.1

(1) When $G_n = \mathrm{SO}(2n+1)$, or $G_n = \mathrm{Sp}(2n)$ and $\beta_r \neq \alpha_n$, or $G_n = \mathrm{SO}(2n)$ and $\beta_r \neq \alpha_n$ and $\beta_{r-1} \neq \alpha_{n-1}$, then we have

$$\Sigma_r(P,A) = \{\beta_i; 1 \leq i \leq r\} \cup \{\beta_{i,j}; 1 \leq i < j \leq r\} \cup \{\beta'_{i,j}; 1 \leq i < j \leq r\}.$$

(2) When $G_n = \mathrm{Sp}(2n)$ and $\beta_r = \alpha_n$, we have

$$\Sigma_r(P,A) = \{\beta_i; 1 \leq i \leq r\} \cup \{\beta_{i,j}; 1 \leq i < j \leq r\} \cup \{\beta'_{i,j,r}; 1 \leq i < j \leq r-1\} \cup \{\beta'_i; 1 \leq i \leq r-1\}.$$

(3) When $G_n = \mathrm{SO}(2n)$ and $\beta_r = \alpha_n$ and $\beta_{r-1} = \alpha_{n-1}$, we have

$$\begin{aligned}\Sigma_r(P,A) = \{\beta_i; 1 \leq i \leq r\} & \\ \cup \{\beta_{i,j}; 1 \leq i < j \leq r, (i,j) \neq (r-1,r)\} & \cup \{\beta_{r-1,r}; m_{r-1} > 1\} \\ \cup \{\beta_{i,r-2,r}; 1 \leq i < r-2\} & \cup \{\beta'_{i,j,r-1,r}; 1 \leq i < j \leq r-2\} \\ \cup \{\beta''_i; 1 \leq i \leq r-2, m_i > 1\}. &\end{aligned}$$

(4) When $G_n = \mathrm{SO}(2n)$ and $\beta_r = \alpha_n$ and $\beta_{r-1} \neq \alpha_{n-1}$, we have

$$\Sigma_r(P,A) = \{\beta_i; 1 \leq i \leq r\} \cup \{\beta_{i,j}; 1 \leq i < j \leq r\} \cup \{\beta'_{i,j,r}; 1 \leq i < j \leq r-1\} \cup \{\beta'_i; 1 \leq i \leq r-1, m_i > 1\}.$$

Proof We compute all the positive roots of Φ_r which do not vanish on A, and look at their restriction on A. Then we use the tables in [2] to get the lemma. ■

For each $\beta \in \sum_r(P,A)$ we can compute A_β and $M^\beta = Z_{G_n}(A_\beta)$. Then by comparing M and M^β, we can see the following lemmas on the embedding of $M(F)$ as a maximal Levi into $M^\beta(F)$, as long as we fix a certain form defining G_n (see [5]). Remember that we assumed in this section that $\beta_r \neq \alpha_{n-1}$ when $G_n = \mathrm{SO}(2n)$.

Lemma 3.2 *Let $\beta \in \sum_r(P,A)$.*

(1) If $\beta = \beta_i$ with $1 \leq i \leq r-1$, then β corresponds to $\mathrm{GL}_{m_i+m_{i+1}}$ and its maximal Levi $\mathrm{GL}_{m_i} \times \mathrm{GL}_{m_{i+1}}$ with the embedding

$$(g_i, g_{i+1}) \mapsto \begin{pmatrix} g_i & \\ & g_{i+1} \end{pmatrix}.$$

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(2) *If* $\beta = \beta_{i,j}$ *with* $1 \le i < j \le r-1$, *then* β *corresponds to* $\mathrm{GL}_{m_i+m_{j+1}}$ *and its maximal Levi* $\mathrm{GL}_{m_i} \times \mathrm{GL}_{m_{j+1}}$ *with the embedding*

$$(g_i, g_j) \mapsto \begin{pmatrix} g_i & \\ & g_{j+1} \end{pmatrix}.$$

(3) *If* $\beta = \beta'_{i,j}$ *with* $1 \le i < j \le r$, *then* β *corresponds to* $\mathrm{GL}_{m_i+m_j}$ *and its maximal Levi* $\mathrm{GL}_{m_i} \times \mathrm{GL}_{m_j}$ *with the embedding*

$$(g_i, g_j) \mapsto \begin{pmatrix} g_i & \\ & {}^{\tau}g_j^{-1} \end{pmatrix}.$$

Lemma 3.3 *Let* $\beta = \beta_r$.

(1) *If* $G_n \ne \mathrm{SO}(2n)$ *and* $\beta_r \ne \alpha_n$, *or* $G_n = \mathrm{SO}(2n)$ *and* $\beta_r \ne \alpha_n$ *and* $\beta_{n-1} \ne \alpha_{n-1}$, *then* β *corresponds to* $G_{m_r+m_0}$ *and its maximal Levi* $\mathrm{GL}_{m_r} \times G_{m_0}$ *with the embedding*

$$(g_r, g_0) \mapsto \begin{pmatrix} g_r & & \\ & g_0 & \\ & & {}^{\tau}g_r^{-1} \end{pmatrix}.$$

(2) *If* $G_n \ne \mathrm{SO}(2n)$ *and* $\beta_r = \alpha_n$, *or* $G_n = \mathrm{SO}(2n)$ *and* $\beta_r = \alpha_n$ *and* $\beta_{r-1} \ne \alpha_{n-1}$, *then* β *corresponds to* G_{m_r} *and its maximal Levi* GL_{m_r} *with the embedding*

$$g_r \mapsto \begin{cases} \begin{pmatrix} g_r & \\ & {}^{\tau}g_r^{-1} \end{pmatrix}, & \text{for } G_n \ne \mathrm{SO}(2n+1) \\ \begin{pmatrix} g_r & & \\ & 1 & \\ & & {}^{\tau}g_r^{-1} \end{pmatrix}, & \text{for } G_n = \mathrm{SO}(2n+1). \end{cases}$$

(3) *If* $G_n = \mathrm{SO}(2n)$ *and* $(\beta_{r-1}, \beta_r) = (\alpha_{n-1}, \alpha_n)$, *then* β *corresponds to* $\mathrm{GL}_{m_{r-1}+m_r}$, *(here* $m_r = 1$*), and its maximal Levi* $\mathrm{GL}_{m_{r-1}} \times \mathrm{GL}_{m_r}$ *with the embedding*

$$(g_{r-1}, g_r) \mapsto \begin{pmatrix} g_{r-1} & \\ & g_r^{-1} \end{pmatrix}.$$

Lemma 3.4 *Let* $\beta = \beta_{i,r}$, $1 \le i \le r-1$.

(1) *If* $G_n \ne \mathrm{SO}(2n)$ *and* $\beta_r \ne \alpha_n$, *then* β *corresponds to* $G_{m_i+m_0}$ *and its Levi* $\mathrm{GL}_{m_i} \times G_{m_0}$ *with the embedding*

$$(g_i, g_0) \mapsto \begin{pmatrix} g_i & & \\ & g_0 & \\ & & {}^{\tau}g_i^{-1} \end{pmatrix}.$$

(2) *If $G_n = \mathrm{Sp}(2n)$ and $\beta_r = \alpha_n$, or $G_n = \mathrm{SO}(2n)$ with $\beta_r = \alpha_n$ and $\beta_{r-1} \neq \alpha_{n-1}$, then β corresponds to $\mathrm{GL}_{m_i+m_r}$ and its Levi $\mathrm{GL}_{m_i} \times \mathrm{GL}_{m_r}$ with the embedding*

$$(g_i, g_r) \mapsto \begin{pmatrix} g_i & \\ & {}^{\tau}g_r^{-1} \end{pmatrix}.$$

(3) *If $G_n = \mathrm{SO}(2n+1)$ and $\beta_r = \alpha_n$, then β corresponds G_{m_i} and its Levi GL_{m_i} with the embedding*

$$g_r \mapsto \begin{pmatrix} g_i & & \\ & 1 & \\ & & {}^{\tau}g_i^{-1} \end{pmatrix}.$$

(4) *Suppose that $G_n = \mathrm{SO}(2n)$ and $(\beta_{r-1}, \beta_r) = (\alpha_{n-1}, \alpha_n)$. Then for i, $1 \leq i \leq r-2$, β corresponds to $\mathrm{GL}_{m_i+m_{r-1}}$ and its Levi $\mathrm{GL}_{m_i} \times \mathrm{GL}_{m_{r-1}}$ with the embedding*

$$(g_i, g_{r-1}) \mapsto \begin{pmatrix} g_i & \\ & {}^{\tau}g_{r-1}^{-1} \end{pmatrix}.$$

For $i = r-1$ and $m_{r-1} > 1$, β corresponds to $G_{m_{r-1}}$ and its Levi $\mathrm{GL}_{m_{r-1}}$ with the embedding

$$g_{r-1} \mapsto \begin{pmatrix} g_{r-1} & \\ & {}^{\tau}g_{r-1}^{-1} \end{pmatrix}.$$

Lemma 3.5 *Let $G_n = \mathrm{SO}(2n)$ and $(\beta_{r-1}, \beta_r) = (\alpha_{n-1}, \alpha_n)$.*

(1) *If $\beta = \beta_{i,r-2,r}$ and $1 \leq i \leq r-2$, then β corresponds to $\mathrm{GL}_{m_i+m_r}$ and its Levi $\mathrm{GL}_{m_i} \times \mathrm{GL}_{m_r}$ (here $m_r = 1$) with the embedding*

$$(g_i, g_r) \mapsto \begin{pmatrix} g_i & \\ & {}^{\tau}g_r^{-1} \end{pmatrix}.$$

(2) *If $\beta = \beta'_{i,j,r-1,r}$ and $1 \leq i < j \leq r-2$, then β corresponds to $\mathrm{GL}_{m_i+m_j}$ and its Levi $\mathrm{GL}_{m_i} \times \mathrm{GL}_{m_j}$ with the embedding*

$$(g_i, g_j) \mapsto \begin{pmatrix} g_i & \\ & {}^{\tau}g_j^{-1} \end{pmatrix}.$$

(3) *If $\beta = \beta''_i$ and $m_i > 1$, $1 \leq i \leq r-2$, then β corresponds to G_{m_i} and its Levi GL_{m_i} with the embedding*

$$g_i \mapsto \begin{pmatrix} g_i & \\ & {}^{\tau}g_i^{-1} \end{pmatrix}.$$

Lemma 3.6 *Let $G_n = \mathrm{SO}(2n)$ and $\beta_r = \alpha_n$ and $\beta_{r-1} \neq \alpha_{n-1}$.*

(1) *If* $\beta = \beta'_{i,j,r}$, $1 \le i < j \le r-1$, *then* β *corresponds to* $\mathrm{GL}_{m_i+m_j}$ *and its Levi* $\mathrm{GL}_{m_i} \times \mathrm{GL}_{m_j}$ *with the embedding*

$$(g_i, g_j) \mapsto \begin{pmatrix} g_i & \\ & {}^{\tau}g_j^{-1} \end{pmatrix}.$$

(2) *If* $\beta = \beta'_i$ *and* $m_i > 1$, $1 \le i \le r-1$, *then* β *corresponds to* G_{m_i} *and its Levi* GL_{m_i} *with the embedding*

$$g_i \mapsto \begin{pmatrix} g_i & \\ & {}^{\tau}g_i^{-1} \end{pmatrix}.$$

Suppose that

$$\tau = \rho_1 \otimes \cdots \otimes \rho_r \otimes \sigma$$

where each ρ_i is a unitary supercuspidal representation of $H_{m_i}(F)$ and σ is a unitary supercuspidal representation of $G_{m_0}(F)$. By the lemmas above and the known result on discrete series representations of general linear groups, we have the following propositions.

Proposition 3.1

(1) *If* $\beta = \beta_i$ *and* $1 \le i \le r-1$, *then* β *is special if and only if* $\rho_i \simeq \rho_{i+1}$.
(2) *If* $\beta = \beta_{i,j}$ *and* $1 \le i < j \le r-1$, *then* β *is special if and only if* $\rho_i \simeq \rho_{j+1}$.
(3) *If* $\beta = \beta'_{i,j}$ *and* $1 \le i < j \le r-1$, *then* β *is special if and only if* $\rho_i \simeq \tilde{\rho}_j$.

Proposition 3.2 *Let* $\beta = \beta_r$, *or* $\beta_{i,r}$.

(1) *Suppose that* $G_n \ne \mathrm{SO}(2n)$ *and* $\beta_r \ne \alpha_n$, *or* $G_n = \mathrm{SO}(2n)$ *and* $\beta_r \ne \alpha_n$ *and* $\beta_{r-1} \ne \beta_{n-1}$. *Then* β_r *(resp.* $\beta_{i,r}$*) is special if and only if* $\rho_r \otimes \sigma$ *(resp.* $\rho_i \otimes \sigma$*) is special with respect to* $G_{m_r+m_0}$ *(resp.* $G_{m_i+m_0}$ *).*
(2) *Suppose that* $G_n \ne \mathrm{SO}(2n)$ *and* $\beta_r = \alpha_n$, *or* $G_n = \mathrm{SO}(2n)$ *and* $\beta_r = \alpha_n$ *and* $\beta_{r-1} \ne \alpha_{n-1}$. *Then* β_r *is special if and only if* ρ_r *is special with respect to* G_{m_r}; $\beta_{i,r}$ *is special if and only if* $\rho_i \simeq \tilde{\rho}_r$ *when* $G_n = \mathrm{Sp}(2n)$ *or* $\mathrm{SO}(2n)$; *and* $\beta_{i,r}$ *is special if and only if* ρ_i *is special with respect to* G_{m_i} *when* $G_n = \mathrm{SO}(2n+1)$.

Proposition 3.3 *Suppose that* $G_n = \mathrm{Sp}(2n)$ *and* $\beta_r = \alpha_n$ *or* $G_n = \mathrm{SO}(2n)$ *and* $\beta_r = \alpha_n$ *and* $\beta_{r-1} \ne \alpha_{n-1}$. *Then*

(1) $\beta'_{i,j,r}$, $1 \le i < j \le r-1$, *is special if and only if* $\rho_i \simeq \tilde{\rho}_j$,
(2) β'_i *(if* $G_n = \mathrm{SO}(2n)$ *then* $m_i > 1$*)*, $1 \le i \le n-1$, *is special if and only if* ρ_i *is special with respect to* G_{m_i}.

Proposition 3.4 *Let* $G_n = \mathrm{SO}(2n)$ *and* $(\beta_{r-1}, \beta_r) = (\alpha_{n-1}, \alpha_n)$. *(In this case,* $m_r = 1$.*)*

(1) $\beta_{i,r-2,r}$, $1 \le i \le r-2$, *is special if and only if* $\rho_i \simeq \tilde{\rho}_r$.
(2) $\beta'_{i,j,r-1,r}$, $1 \le i < j \le r-2$, *is special if and only if* $\rho_i \simeq \tilde{\rho}_j$.
(3) β''_i *with* $m_i > 1$, $1 \le i \le r-2$ *is special if and only if* ρ_i *is special with respect to* G_{m_i}.
(4) $\beta_{i,r}$, $1 \le i \le r-2$, *is special if and only if* $\rho_i \simeq \tilde{\rho}_{r-1}$.
(5) $\beta_{r-1,r}$ *with* $m_{r-1} > 1$ *is special if and only if* ρ_{r-1} *is special with respect to* $G_{m_{r-1}}$.

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4 The Proof of Theorem when $m_0 \neq 0$

In this section, we assume

$$M \simeq H_{m_1} \times \cdots \times H_{m_r} \times G_{m_0}, \quad \sum_{i=1}^{r} m_i + m_0 = n,$$

with $m_0 \neq 0$. This is equivalent to that $\beta_r \neq \alpha_n$ for $G_n \neq \mathrm{SO}(2n)$, and $\beta_r \neq \alpha_n, \alpha_{n-1}$ for $G_n = \mathrm{SO}(2n)$.

We first study two simple cases when $m_i = m_j = m$ for any $1 \leq i, j \leq r$. The general situation will be reduced to these cases. We need an easy lemma on the ramification of a supercuspidal representation of a maximal Levi of $G_n(F)$ with respect to $G_n(F)$. Let τ be an irreducible admissible representation of $M(F)$, then we say that τ is ramified with respect to G_n, if there is a non-trivial element w in the Weyl group $W(G, A)$ such that w fixes τ.

Lemma 4.1 *Let $M \simeq H_m \times G_{m_0}$ with $n = m + m_0$, and $\tau = \rho \otimes \sigma$ be a unitary supercuspidal representation of $M(F)$. Then τ is ramified with respect to G_n, if and only if*

(1) $\tilde{\rho} \simeq \rho$, when $G_n \neq \mathrm{SO}(2n)$, or $G_n = \mathrm{SO}(2n)$ and m is even.
(2) $\tilde{\rho} \simeq \rho$ and $s(\sigma) \simeq \sigma$, when $G_n = \mathrm{SO}(2n)$ and m is odd. Here s is the automorphism of $\mathrm{SO}(2m)$ determined by the element in $O(2n)(F)$

$$\begin{pmatrix} I_{m-1} & & & \\ & & 1 & \\ & 1 & & \\ & & & I_{m-1} \end{pmatrix}.$$

Proof Let w_0 be the longest element of the Weyl group $W(G, A_o)$ modulo the Weyl group $W(M, A)$. When $G_n \neq \mathrm{SO}(2n)$ or $G_n = \mathrm{SO}(2n)$ and m even, $w_0(\rho \otimes \sigma) = \tilde{\rho} \otimes \sigma$. So τ is ramified if and only if $\tilde{\rho} \simeq \rho$. When $G_n = \mathrm{SO}(2n)$ and m odd, $w_0(\rho \otimes \sigma) = \tilde{\rho} \otimes s(\sigma)$, where s is as in the Lemma. So τ is ramified with respect to G_n if and only if $\tilde{\rho} \simeq \rho$ and $s(\sigma) \simeq \sigma$. ■

Remark Let τ be as in the last lemma. Then the lemma implies that for all cases, if τ is special with respect to G_n, then $\tilde{\rho} \simeq \rho$.

Proposition 4.1 *Let*

$$\tau = \overbrace{\rho \otimes \cdots \otimes \rho}^{r} \otimes \sigma$$

be an irreducible unitary supercuspidal representation of $M(F)$. Then the Theorem is true for τ.

Proof When $r = 1$, M is a maximal Levi of G_n. So by the remark above, we see $\tilde{\rho} \simeq \rho$. Hence Theorem is true for τ.

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Suppose $r > 1$. Suppose that τ is special. If $\tilde{\rho} \not\simeq \rho$, then by Propositions 3.1 and 3.2 we see

$$\Sigma^{''}(\tau) = \{\beta_i; 1 \leq i \leq r-1\} \cup \{\beta_{i,j}; 1 \leq i < j \leq r-1\}.$$

So the root system $\Sigma^{''}(\tau)$ is of the type A_{r-1}, hence the maximal number of linear independent elements of $\Sigma^{''}(\tau)$ is $r-1$, not r. This contradicts with the assumption that τ is special. So $\tilde{\rho} \simeq \rho$. Now we suppose $\tilde{\rho} \simeq \rho$. From Proposition 3.1, we always have

$$\{\beta_i; 1 \leq i \leq r-1\} \cup \{\beta_{i,j}; 1 \leq i < j \leq r-1\} \subset \Sigma^{''}(\tau).$$

Since $r > 1$, $\beta'_{r-1,r} \in \Sigma_r(P, A)$. By Proposition 3.2, $\tilde{\rho} \simeq \rho$ implies $\beta'_{r-1,r} \in \Sigma^{''}(\tau)$. Therefore

$$\{\beta_i; 1 \leq i \leq r-1\} \cup \{\beta'_{r-1,r}\}$$

is a set of r linear independent elements in $\Sigma^{''}(\tau)$. Note r is the rank of $\Sigma_r(P, A)$. So τ is special with respect to G_n. ■

Remark Let τ be as in Proposition 4.1. Suppose τ is special. From the proof of Proposition 4.1, we see that

$$\Sigma^{''}(\tau) = \Sigma_r(P, A)$$

if $\rho \otimes \sigma$ is special with respect to $G_{m_1+m_0}$, and

$$\Sigma^{''}(\tau) = \Sigma_r(P, A) - \{\beta_r, \beta_{i,r}; 1 \leq i \leq r-1\},$$

otherwise. When τ is not special, $\Sigma^{''}(\tau)$ is of the type A_{r-1}.

Proposition 4.2 *Let*

$$\tau = \overbrace{\rho \otimes \cdots \otimes \rho}^{k} \otimes \overbrace{\tilde{\rho} \otimes \cdots \otimes \tilde{\rho}}^{k'} \otimes \sigma$$

be an irreducible unitary supercuspidal representation of $M(F)$, $k + k' = r$. Then the Theorem is true for τ and G_n.

Proof When $r = 1$, it is clear.

Assume $r > 1$. Let

$$\tilde{\beta}_k = \beta'_{k,r}.$$

We always have

$$\{\beta_i; 1 \leq i \leq r-1, i \neq k\} \cup \{\tilde{\beta}_k\} \subset \Sigma''(\tau).$$

Suppose that τ is special. If $\tilde{\rho} \not\simeq \rho$, then neither $\rho \otimes \sigma$ is special with respect to G_{m+m_0} and nor is $\rho \otimes \tilde{\rho}$ special with respect to GL_{2m}. So by Propositions 3.1, 3.2, we see

$$\begin{aligned}\Sigma^{''}(\tau) = \{\beta_i; 1 \leq i \leq r-1, i \neq k\} \\ \cup \{\beta_{i,j}; 1 \leq i < j \leq k-1 \text{ or } k+1 \leq i < j \leq r-1\} \\ \cup \{\beta'_{i,j}; 1 \leq i \leq k, k+1 \leq j \leq r\}.\end{aligned}$$

We claim that every element $\beta \in \Sigma^{''}(\tau)$ is a linear combination of the $r-1$ elements of the subset S of $\Sigma^{''}(\tau)$, where

$$S = \{\beta_i; 1 \leq i \leq r-1, i \neq k\} \cup \{\tilde{\beta}_k\}.$$

Indeed, if

$$\beta \in \{\beta_{i,j}; 1 \leq i < j \leq k-1 \text{ or } k+1 \leq i < j \leq r-1\},$$

β is a linear combination of the elements of

$$\{\beta_i; 1 \leq i \leq r-1, i \neq k\}.$$

If

$$\beta = \beta'_{i,j} \in \{\beta'_{i,j}; 1 \leq i \leq k, k+1 \leq j \leq r\},$$

then

$$\beta = \beta_i + \cdots + \beta_{k-1} + \tilde{\beta}_k + \beta_j + \cdots + \beta_{r-1}.$$

Therefore the maximal number of linear independent elements of $\Sigma^{''}(\tau)$ is $r-1$. This is not possible, since we assumed τ is special. So we must have $\tilde{\rho} \simeq \rho$. The sufficiency of the proposition was proved in Proposition 4.1. ∎

We now state the following result for our further study of a unitary supercuspidal representation of $M(F)$ to be special with respect to G_n.

Proposition 4.3 *For $w \in W(G, A_o)$ such that $w(P)$ is still a standard parabolic subgroup of G, then τ is special with respect to G if and only if $w(\tau)$ is special with respect to G.*

Proof For $\lambda \in \mathfrak{a}^*$, since the induced representations $I(w\lambda, w\tau)$ and $I(\lambda, \tau)$ have the same composition factors (see [3]), the proposition is true. ∎

From this proposition, when G_n is not $\mathrm{SO}(2n)$ we may assume that

$$\tau = \overbrace{\rho_1 \otimes \cdots \otimes \rho_1}^{k_1} \otimes \cdots \otimes \overbrace{\rho_t \otimes \cdots \otimes \rho_t}^{k_t} \otimes \sigma,$$

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with $\rho_i \neq \rho_j, \tilde{\rho}_j$ for $i \neq j$. For $G = \mathrm{SO}(2n)$, since a single sign change is not in $W(G_n)$, we can not always assume τ has the form above. In this situation we can assume

$$\tau = \overbrace{\rho_1 \otimes \cdots \otimes \rho_1}^{k_1} \otimes \overbrace{\tilde{\rho}_1 \otimes \cdots \otimes \tilde{\rho}_1}^{k_1'} \otimes \cdots \otimes \overbrace{\rho_t \otimes \cdots \otimes \rho_t}^{k_t} \otimes \overbrace{\tilde{\rho}_t \otimes \cdots \otimes \tilde{\rho}_t}^{k_t'} \otimes \sigma,$$

with $\rho_i \neq \rho_j, \tilde{\rho}_j$ for $i \neq j$. Set

$$\tau_i = \overbrace{\rho_i \otimes \cdots \otimes \rho_i}^{k_i} \otimes \sigma,$$

$$\tau_i' = \overbrace{\rho_i \otimes \cdots \otimes \rho_i}^{k_i} \otimes \overbrace{\tilde{\rho}_i \otimes \cdots \otimes \tilde{\rho}_i}^{k_i'} \otimes \sigma$$

for $1 \leq i \leq t$.

For each τ_i, we let $\sum_r(P_i, A_i)$ be the reduced roots of $G_{m_i k_i + m_0}$ with respect to A_i. Here P_i is the standard parabolic subgroup of $G_{m_i k_i + m_0}$ with

$$\overbrace{\mathrm{GL}_{m_i} \times \cdots \times \mathrm{GL}_{m_i}}^{k_i} \times G_{m_0}$$

as its Levi factor which is the centralizer of a torus A_i. Then we can embed $\sum_r(P_i, A_i)$ into $\sum_r(P, A)$ such that for each $\beta \in \sum_r(P_i, A_i)$ is special with respect to $G_{m_i k_i + m_0}$ and τ_i if and only if the image of β is special with respect to G_n and τ. We use $\sum''(\tau_i)$ for the set of special roots of τ_i with respect to $G_{m_i k_i + m_0}$, also for the image of $\sum''(\tau_i)$ in $\sum''(\tau)$. Similar notations and explanations are applied to each τ_i', for $1 \leq i \leq t$.

Proposition 4.4 *Suppose*

$$\tau = \overbrace{\rho_1 \otimes \cdots \otimes \rho_1}^{k_1} \otimes \cdots \otimes \overbrace{\rho_t \otimes \cdots \otimes \rho_t}^{k_t} \otimes \sigma$$

is a unitary supercuspidal representation of $M(F)$ with $\rho_i \not\simeq \rho_j, \tilde{\rho}_j$ for $i \neq j$, $1 \leq i, j \leq r$. Then τ is special with respect to G_n if and only if every τ_i is special with respect to $G_{m_i k_i + m_0}$, for $1 \leq i \leq t$.

Proof Since $\rho_i \not\simeq \rho_j, \tilde{\rho}_j$ for $i \neq j$, it follows from Propositions 3.1 and 3.2 that

$$\Sigma''(\tau) = \Sigma''(\tau_1) \cup \cdots \cup \Sigma''(\tau_t).$$

Note that every $\Sigma''(\tau_i)$ has at most k_i linear independent elements, and that $k_1 + \cdots + k_t = r$. So we see that τ is special with respect to G_n implies each τ_i is special with respect to $G_{m_i k_i + m_0}$ for $1 \leq i \leq t$.

Conversely, suppose that every τ_i is special with respect to $G_{m_i k_i + m_0}$ for $1 \leq i \leq t$. Let

$$\tilde{\beta}_{k_1 + \cdots + k_i} = \begin{cases} \beta_{k_1 + \cdots + k_i, r} & \text{if } k_i = 1 \\ \beta'_{k_1 + \cdots + k_i - 1, k_1 + \cdots + k_i} & \text{if } k_i > 1. \end{cases}$$

for $1 \le i \le t$. From the remark of Proposition 4.1 we see that for each i, $\tilde{\beta}_{k_1+\cdots+k_i} \in \Sigma''(\tau_i)$. And it is clear that

$$\{\beta_i; 1 \le i \le r\} - \{\beta_{k_1+\cdots+k_i}; 1 \le i \le t\} \subset \Sigma''(\tau).$$

For every i, $1 \le i \le t$, we put

$$S_i = \{\beta_{k_1+\cdots+k_{i-1}+1}, \beta_{k_1+\cdots+k_{i-1}+2}, \dots, \beta_{k_1+\cdots+k_i-1}, \tilde{\beta}_{k_1+\cdots+k_i}\}.$$
$$S = S_1 \cup \cdots \cup S_t.$$

It is easy to see that S is a subset of $\Sigma''(\tau)$ consisting of r linear independent elements. Hence τ is special with respect to G_n. ■

Proposition 4.5 *Let*

$$\tau = \overbrace{\rho_1 \otimes \cdots \otimes \rho_1}^{k_1} \otimes \overbrace{\tilde{\rho}_1 \otimes \cdots \otimes \tilde{\rho}_1}^{k_1'} \otimes \cdots \otimes \overbrace{\rho_t \otimes \cdots \otimes \rho_t}^{k_t} \otimes \overbrace{\tilde{\rho}_t \otimes \cdots \otimes \tilde{\rho}_t}^{k_t'} \otimes \sigma$$

be a unitary supercuspidal representation of $M(F)$ with $\rho_i \not\simeq \rho_j, \tilde{\rho}_j$ for $i \ne j$, $1 \le i, j \le t$. Then τ is special with respect to G_n if and only if each τ_i' is special with respect to $G_{m_i(k_i+k_i')+m_0}$.

Proof The proof of the necessary conclusion is similar with that of the last proposition.

Suppose that every τ_i' is special with respect to $G_{m_i(k_i+k_i')+m_0}$. Then $\tilde{\rho}_i \simeq \rho_i$ for $1 \le i \le t$, from Proposition 4.2. Proposition 4.4 gives the specialty of τ with respect to G_n. ■

Now we have proved the Theorem when $m_0 \ne 0$, by combining all the propositions in this section together.

5 The Proof of Theorem, when $m_0 = 0$

In this section we assume $m_0 = 0$, *i.e.*,

$$M \simeq \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_r}.$$

When $G_n \ne \mathrm{SO}(2n)$, this assumption is equivalent to $\beta_r = \alpha_n$. When $G_n = \mathrm{SO}(2n)$, it is equivalent to $\beta_r = \alpha_n$ or α_{n-1}.

As we did in the last section, we also need a lemma on the ramification of a supercuspidal representation of the Levi of a standard maximal parabolic subgroup of G_n.

Lemma 5.1 *Let ρ be a supercuspidal representation of $M(F) = \mathrm{GL}_n(F)$. Then when $G_n \ne \mathrm{SO}(2n)$, ρ is ramified with respect to G_n if and only if $\tilde{\rho} \simeq \rho$; when $G_n = \mathrm{SO}(2n)$, ρ is ramified with respect to G_n if and only if n is even and $\tilde{\rho} \simeq \rho$.*

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Proof When $G_n \neq \mathrm{SO}(2n)$ or $G_n = \mathrm{SO}(2n)$ with n even, the Weyl group $W(G,A) = \{1, w\}$ and $w(\rho) \simeq \tilde{\rho}$. So in this case, ρ is ramified with respect to G_n if and only if $\tilde{\rho} \simeq \rho$. When $G_n = \mathrm{SO}(2n)$ with n odd, $W(G,A) = 1$. So in this case, ρ is unramified with respect to G_n. ■

Remark Let ρ be as in the lemma above. Then ρ is special with respect to G_n implies $\tilde{\rho} \simeq \rho$.

We first consider the situation that $G_n \neq \mathrm{SO}(2n)$, or $G_n = \mathrm{SO}(2n)$ and $\beta_r = \alpha_n$. When $G_n = \mathrm{SO}(2n)$ and $\beta_r = \alpha_n$, there are two cases to be treated separately. One is that $\beta_{r-1} \neq \alpha_{n-1}$, the other one is that $\beta_{r-1} = \alpha_{n-1}$.

Now we suppose that $G_n \neq \mathrm{SO}(2n)$, or $G_n = \mathrm{SO}(2n)$ and $\beta_r = \alpha_n$ and $\beta_{r-1} \neq \alpha_{n-1}$.

Proposition 5.1 *Assume $\beta_r = \alpha_n$, and additionally $\beta_{r-1} \neq \alpha_{n-1}$ when $G_n = \mathrm{SO}(2n)$. Let*

$$\tau \simeq \overbrace{\rho \otimes \cdots \otimes \rho}^{r}$$

be a unitary supercuspidal representation of $M(F)$. (Now $m_i = m$, for $1 \leq i \leq r$.) Then the Theorem is true for τ and G_n.

Proof When $r = 1$, it is obvious from the remark of Lemma 5.1.

Assume $r > 1$. As we see before,

$$\{\beta_i; 1 \leq i \leq r-1\} \subset \Sigma^{''}(\tau).$$

It is obvious that $\tilde{\rho} \not\simeq \rho$ implies that ρ is not special with respect to G_m, from the remark of Lemma 5.1. If $\rho \not\simeq \tilde{\rho}$, then by Propositions 3.1, 3.2, 3.3, we have

$$\Sigma^{''}(\tau) = \{\beta_i; 1 \leq i \leq r-1\} \cup \{\beta_{i,j}; 1 \leq i < j \leq r-1\}.$$

Hence the rank of $\Sigma^{''}(\tau)$ is $r-1$, not r. So τ is not special with respect to G_n.

Suppose $\rho \simeq \tilde{\rho}$. If ρ is special with respect to G_m, then $\Sigma^{''}(\tau) = \Sigma_r(P,A)$, hence τ is special with respect to G_n. If ρ is not special with respect to G_m, then

$$\Sigma^{''}(\tau) = \Sigma_r(P,A) - \{\beta_r, \beta_i'; 1 \leq i \leq r-1\}$$

for $G_n \neq \mathrm{SO}(2n+1)$, and

$$\Sigma^{''}(\tau) = \Sigma_r(P,A) - \{\beta_r, \beta_{i,r}; 1 \leq i \leq r-1\}$$

for $G_n = \mathrm{SO}(2n+1)$. Therefore

$$\beta_1, \ldots, \beta_{r-1}, \beta_{r-1,r}$$

are r linear independent elements in $\Sigma^{''}(\tau)$ for $G_n \neq \mathrm{SO}(2n+1)$, and

$$\beta_1, \ldots, \beta_{r-1}, \beta_{r-1,r}'$$

are r linear independent elements in $\Sigma^{''}(\tau)$ for $G_n = \mathrm{SO}(2n+1)$. So τ is special with respect to G_n. ■

Proposition 5.2 *Let $G_n = \mathrm{SO}(2n)$ and $\beta_r = \alpha_n$, $\beta_{r-1} \neq \alpha_{n-1}$. Suppose*

$$\tau = \overbrace{\rho \otimes \cdots \otimes \rho}^{k} \otimes \overbrace{\tilde{\rho} \otimes \cdots \otimes \tilde{\rho}}^{k'},$$

where $k + k' = r$, is a unitary supercuspidal representation of $M(F)$. Then the Theorem is true for τ and G_n.

Proof If $\tilde{\rho} \simeq \rho$, then the conclusion is clear from the last proposition. Suppose $\tilde{\rho} \not\simeq \rho$. Then

$$\begin{aligned}\Sigma''(\tau) = \{&\beta_i; 1 \leq i \leq r-1, i \neq k\}\\ &\cup \{\beta_{i,j}; 1 \leq i < j \leq k-1 \text{ or } k+1 \leq i < j \leq r-1\}\\ &\cup \{\beta_{i,r}; 1 \leq i \leq k\}\\ &\cup \{\beta'_{i,j,r}; 1 \leq i \leq k, k+1 \leq j \leq r-1\}.\end{aligned}$$

Let

$$\tilde{\beta}_k = \beta_{k,r}.$$

Then

$$\beta_1, \ldots, \beta_{k-1}, \tilde{\beta}_k, \beta_{k+1}, \ldots, \beta_{r-1}$$

are $r-1$ elements in $\Sigma''(\tau)$ such that every element in $\Sigma''(\tau)$ is a linear combination of these $r-1$ elements. Indeed,

$$\beta_{i,r} = \beta_i + \cdots + \beta_{k-1} + \tilde{\beta}_k, \quad 1 \leq i \leq k$$
$$\beta'_{i,j,r} = \beta_i + \cdots + \beta_{k-1} + \tilde{\beta}_k + \beta_j + \cdots + \beta_{r-1}, \quad 1 \leq i \leq k, k+1 \leq j \leq r-1.$$

Therefore τ is not special. ■

Now suppose

$$\tau = \overbrace{\rho_1 \otimes \cdots \otimes \rho_1}^{k_1} \otimes \cdots \otimes \overbrace{\rho_t \otimes \cdots \otimes \rho_t}^{k_t},$$

or

$$\tau = \overbrace{\rho_1 \otimes \cdots \otimes \rho_1}^{k_1} \otimes \overbrace{\tilde{\rho}_1 \otimes \cdots \otimes \tilde{\rho}_1}^{k'_1} \otimes \cdots \otimes \overbrace{\rho_t \otimes \cdots \otimes \rho_t}^{k_t} \otimes \overbrace{\tilde{\rho}_t \otimes \cdots \otimes \tilde{\rho}_t}^{k'_t}.$$

We define τ_i and τ'_i similarly as we did in the last section.

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Proposition 5.3 *Let $G_n \neq \mathrm{SO}(2n)$ and $\beta_r = \alpha_n$, or $G_n = \mathrm{SO}(2n)$ with $\beta_r = \alpha_n$ and $\beta_{r-1} = \alpha_{n-1}$.*

$$\tau = \overbrace{\rho_1 \otimes \cdots \otimes \rho_1}^{k_1} \otimes \cdots \otimes \overbrace{\rho_t \otimes \cdots \otimes \rho_t}^{k_t}$$

with $\rho \not\simeq \rho_j, \tilde{\rho}_j$ for $i \neq j$. Then τ is special with respect to G_n if and only if every τ_i is special with respect to $G_{m_i k_i}$, for $1 \leq i \leq t$.

Proof For $G_n = \mathrm{SO}(2n+1)$, the proof is the same as that of Proposition 4.4. For $G_n = \mathrm{Sp}(2n)$ or $\mathrm{SO}(2n)$, the proof is the same as that of Proposition 4.4 after we make the following changes in the proof of Proposition 4.4. For i with $1 \leq i \leq t-1$, let

$$\tilde{\beta}_{k_1+\cdots+k_i} = \begin{cases} \beta'_{k_1+\cdots+k_i} & \text{if } k_i = 1 \\ \beta'_{k_1+\cdots+k_i-1, k_1+\cdots+k_i, r} & \text{if } k_i > 1. \end{cases}$$

For $i = t$, $k_1 + \cdots + k_t = r$. We let

$$\tilde{\beta}_r = \begin{cases} \beta_r, & \text{if } k_t = 1 \\ \beta_{r-1,r} & \text{if } k_t > 1. \end{cases}$$

∎

Proposition 5.4 *Let $G_n = \mathrm{SO}(2n)$, and $\beta_r = \alpha_n$, $\beta_{r-1} \neq \alpha_{n-1}$. Suppose*

$$\tau = \overbrace{\rho_1 \otimes \cdots \otimes \rho_1}^{k_1} \otimes \overbrace{\tilde{\rho}_1 \otimes \cdots \otimes \tilde{\rho}_1}^{k'_1} \otimes \cdots \otimes \overbrace{\rho_t \otimes \cdots \otimes \rho_t}^{k_t} \otimes \overbrace{\tilde{\rho}_t \otimes \cdots \otimes \tilde{\rho}_t}^{k'_t}.$$

Then τ is special with respect to G_n if and only if each τ'_i is special with respect to $G_{m_i(k_i+k'_i)}$, for $1 \leq i \leq t$.

Proof Since $\rho_i \not\simeq \rho_j, \tilde{\rho}_j$ for $i \neq j$, we see the necessary conclusion as we did before.

Suppose that every τ'_i is special with respect to $G_{m_i(k_i+k'_i)}$, then $\tilde{\rho}_i \simeq \rho_i$ for $1 \leq i \leq t$ from Proposition 5.2. Therefore by Proposition 5.3, τ is special with respect to G_n. ∎

Putting all the propositions above in this section together, we have proved the Theorem for the case that $G_n \neq \mathrm{SO}(2n)$ and $\beta_r = \alpha_n$, or $G_n = \mathrm{SO}(2n)$ and $\beta_r = \alpha_n$ and $\beta_{r-1} \neq \alpha_{n-1}$.

Now we consider the situation that $G_n = \mathrm{SO}(2n)$ and $(\beta_{r-1}, \beta_r) = (\alpha_{n-1}, \alpha_n)$. For

$$M \simeq \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_r}$$

corresponding to $\{\beta_1, \ldots, \beta_{r-1}, \beta_r\}$, if there is some i, $1 \leq i \leq r$ such that $m_i > 1$, then our study can be reduced to the situation that $\beta_r = \alpha_n$ and $\beta_{r-1} \neq \alpha_{n-1}$, by Proposition 4.3. Therefore what we have to consider here is the situation that $P = MN$ is the fixed Borel subgroup of $\mathrm{SO}(2n)$. So we assume that we are in such a situation:

$$M \simeq \overbrace{\mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1}^{n},$$

$$\tau \simeq \rho_1 \otimes \cdots \otimes \rho_n,$$

with each ρ_i a unitary character of $\mathrm{GL}_1(F)$, for $1 \le i \le n$. Now

$$\begin{aligned}\Sigma_r(P,A) = \{\beta_i; 1 \le i \le n\} \\ \cup \{\beta_{i,j}; 1 \le i < j \le n, (i,j) \ne (r-1,r)\} \\ \cup \{\beta_{i,n-2,n}; 1 \le i < j \le n-2\} \\ \cup \{\beta'_{i,j,n-1,n}; 1 \le i < j \le n-2\}.\end{aligned}$$

Proposition 5.5 *Let $G_n = \mathrm{SO}(2n)$, $n > 1$, P be the Borel subgroup of G_n and*

$$\tau \simeq \overbrace{\rho \otimes \cdots \otimes \rho}^{n}$$

be a unitary character of $M(F)$. Then the Theorem is true for τ and G_n.

Proof It is obvious that

$$\{\beta_i; 1 \le i \le n-1\} \cup \{\beta_{i,j}; 1 \le i < j \le n-1\} \subset \Sigma^{''}(\tau).$$

And $\beta_n \in \Sigma^{''}(\tau)$ if and only if $\rho^2 = 1$, if and only if $\sum^{''}(\tau) = \sum_r(P,A)$. ∎

Proposition 5.6 *Let $G_n = \mathrm{SO}(2n)$, $n > 1$, P be the Borel subgroup of G_n and*

$$\tau \simeq \overbrace{\rho \otimes \cdots \otimes \rho}^{k} \otimes \overbrace{\rho^{-1} \otimes \cdots \otimes \rho^{-1}}^{k'},$$

with $k + k' = n$, be a unitary character of $M(F)$. Then the Theorem is true for τ and G_n.

Proof If $\rho^2 = 1$, the proposition was proved by Proposition 5.5.

Suppose that τ is special with respect to G_n, and $\rho^2 \not\simeq 1$. Then

$$\begin{aligned}\Sigma^{''}(\tau) = \{\beta_i; 1 \le i \le n-1, i \ne k\} \\ \cup \{\beta_{i,j}; 1 \le i < j \le k-1, \text{ or } k+1 \le i < j \le n-1\} \\ \cup \{\beta_{i,n-2,n}; 1 \le i \le k\} \cup \{\beta_{i,n}; 1 \le i \le k\} \\ \cup \{\beta'_{i,j,n-1,n}; 1 \le i \le k \text{ and } k+1 \le j \le n-3\}.\end{aligned}$$

Let $\tilde{\beta}_k = \beta_{k,n-2,n}$. Remember

$$\beta_{k,n-2,n} = \beta_k + \cdots + \beta_{n-2} + \beta_n.$$

Then the set

$$S = \{\beta_i; 1 \le i \le n-1, i \ne k\} \cup \{\tilde{\beta}_k\}$$

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is a subset of $\Sigma''(\tau)$ whose cardinality is $n-1$. We claim that every element $\beta \in \Sigma''(\tau)$ is a linear combination of the elements of S. Indeed, if

$$\beta \in \{\beta_i; 1 \le i \le n-1, i \ne k\} \\ \cup \{\beta_{i,j}; 1 \le i < j \le k-1\} \cup \{\beta_{i,j}; k+1 \le i < j \le r-1\},$$

the claim is obvious. If

$$\beta = \beta_{i,n-2,n} \in \{\beta_{i,n-2,n}; 1 \le i \le k\},$$

then

$$\beta = \beta_i + \cdots + \beta_{k-1} + \tilde{\beta}_k.$$

If

$$\beta = \beta_{i,n} \in \{\beta_{i,n}; 1 \le i \le k\},$$

then

$$\beta = \beta_i + \cdots + \beta_{k-1} + \tilde{\beta}_k + \beta_{n-1}.$$

If

$$\beta = \beta_{i,j,n-1,n} \in \{\beta_{i,j,n-1,n}; 1 \le i \le k \text{ and } k+1 \le j \le n-3\},$$

then

$$\beta = \beta_i + \cdots + \beta_{k-1} + \tilde{\beta}_k + \beta_j + \cdots + \beta_{n-2} + \beta_{n-1}.$$

Therefore, the maximal number of linear independent elements in $\Sigma''(\tau)$ is $n-1$, not n. So τ is not special with respect to G_n. ■

Proposition 5.7 *Let $G_n = \mathrm{SO}(2n)$, and $\beta_r = \alpha_n$, $\beta_{r-1} = \alpha_{n-1}$. Suppose*

$$\tau = \overbrace{\rho_1 \otimes \cdots \otimes \rho_1}^{k_1} \otimes \overbrace{\tilde{\rho}_1 \otimes \cdots \otimes \tilde{\rho}_1}^{k_1'} \otimes \cdots \otimes \overbrace{\rho_t \otimes \cdots \otimes \rho_t}^{k_t} \otimes \overbrace{\tilde{\rho}_t \otimes \cdots \otimes \tilde{\rho}_t}^{k_t'}$$

is a unitary supercuspidal representation of $M(F)$ with $\rho_i \not\simeq \rho_j, \tilde{\rho}_j$ for $i \ne j$. Then τ is special with respect to G_n if and only if τ_i is special with respect to $G_{k_i+k_i'}$ for $1 \le i \le t$.

Proof The necessary conclusion is clear. Suppose that each τ_i' is special with respect to $G_{k_i+k_i'}$, for $1 \le i \le t$. Then $\rho_i^2 \simeq 1$ and $k_i + k_i' > 1$. So by Proposition 5.3, τ is special with respect to G_n. ■

Putting Propositions for $G_n = \mathrm{SO}(2n)$ and P the fixed Borel subgroup of G_n together, we have proved the Theorem for $G_n = \mathrm{SO}(2n)$ and P is the fixed Borel subgroup of G_n. Therefore we have proved the Theorem for $G_n = \mathrm{SO}(2n)$ and $(\beta_r, \beta_{r-1}) = (\alpha_n, \alpha_{n-1})$.

Finally we consider the situation that $G_n = \mathrm{SO}(2n)$ and $\beta_r = \alpha_{n-1}$. Let M be the Levi of G_n corresponding to

$$\{\beta_1, \dots, \beta_{r-1}, \beta_r\},$$

and M' be the Levi of G_n corresponding to

$$\{\gamma_1, \dots, \gamma_{r-1}, \gamma_r\},$$

where $\gamma_i = \beta_i$ for $1 \leq i \leq r-1$, and $\gamma_r = \alpha_n$. Let c_n be the n-th sign change: $e_n \mapsto -e_n$, where $e_n \in X(T)$, such that $e_n(x) = x_n$ for

$$x = \mathrm{diag}(x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}) \in T(F).$$

c_n induces an outer automorphism of G_n which is F-rational. Also c_n induces a bijection of $\{\beta_1, \dots, \beta_{r-1}, \beta_r\}$ and $\{\gamma_1, \dots, \gamma_{r-1}, \gamma_r\}$ such that $\beta_i \mapsto \gamma_i$ for $1 \leq i \leq r$. Suppose that $P' = M'N'$ is the standard parabolic subgroup of G_n corresponding to $\{\gamma_1, \dots, \gamma_{r-1}, \gamma_r\}$. Let A' be the maximal torus in the center of M'. Then there is an one to one correspondence between $\Phi(P, A)$ and $\Phi(P', A')$ induced by the map $\beta_i \mapsto \gamma_i$ and $c_n\big(M(F)\big) = M'(F)$. So,

$$M(F) \simeq M'(F) \simeq \mathrm{GL}_{m_1}(F) \times \cdots \times \mathrm{GL}_{m_{r-1}}(F) \times \mathrm{GL}_{m_r}(F),$$

given by

$$g \mapsto c_n(g) = \mathrm{diag}(g_1, \dots, g_r, {}^{\tau}g_r^{-1}, \dots, {}^{\tau}g_1^{-1}) \mapsto (g_1, \dots, g_r).$$

Suppose that

$$\tau = \rho_1 \otimes \cdots \otimes \rho_{m_r}$$

is a unitary supercuspidal representation of

$$\mathrm{GL}_{m_1}(F) \times \cdots \times \mathrm{GL}_{m_r}(F).$$

Under the isomorphisms above, τ induces a unitary supercuspidal representation of τ' of $M(F)$ and τ'' of $M'(F)$, respectively. It is obvious that $\tau'' = c_n(\tau')$. Since for rank one situation, the Plancherel measure of a unitary supercuspidal representation vanishes at zero if and only if the unitary supercuspidal representation is ramified and the induced representation is irreducible, we see that for each $\beta \in \Phi(P, A)$, $\mu_\beta(0, \tau') = 0$ if and only if $\mu_\gamma(0, \tau'') = 0$. Therefore β is special with respect to τ' and G_n, if and only if β' is special with respect to τ'' and G_n. It is obvious that the map c_n preserves linear relations of the roots. Hence τ' is special with respect to M and G_n if and only if τ'' is special with respect to M' and G_n. We already proved Theorem for τ'' with respect to M' and G_n. Therefore Theorem is true for τ' with respect to M and G_n. Thus, we have completed the proof of Theorem.

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Centre de recherches mathématiques
Case postale 6128
Succursale centre-ville
Montréal, QC
H3C 3J7
email: yz@math.purdue.edu