BULL. AUSTRAL. MATH. SOC. 10E10
VOL. 18 (1978), 403-405.

A new extension of Minkowski's Theorem

P.R. Scott

Let K be a closed convex set in the plane containing no non-zero point of the integral lattice. We show that if the area $A(K)$ of K is equally distributed amongst the four principal quadrants of the plane, then $A(K) < 4$.

1.

Let K be a closed, convex set in the euclidean plane which has area $A(K)$. Let Λ be a lattice in the plane with determinant $d(\Lambda)$. A well known theorem of Minkowski asserts that if K is symmetric about the origin O , and K contains no non-zero point of Λ , then $A(K) < 4d(\Lambda)$. It is known that Minkowski's Theorem in the plane remains true for a large class of non-symmetric sets (for example, [1]), but the following simple result appears to have been overlooked.

Let $\mathbf{u}, \mathbf{v}$ be vectors from O which generate the lattice Λ . Then the lines determined by $\mathbf{u}, \mathbf{v}$ divide the plane into four 'quadrants' Q_i $(1 \leq i \leq 4)$. We assume that the quadrants are indexed in an anti-clockwise direction, with Q_1 the positive quadrant

$\{x\mathbf{u}+y\mathbf{v} \mid x \geq 0, y \geq 0\}$.

THEOREM. *If K is a closed, convex set in the plane for which $A(K \cap Q_i) = \frac{1}{4}A(K)$ $(1 \leq i \leq 4)$, and K contains no non-zero point of the lattice Λ , then $A(K) < 4d(\Lambda)$.*

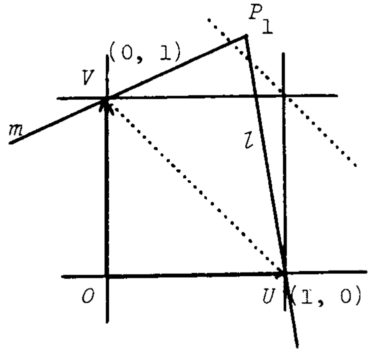
Since closure, convexity, and ratios of areas are all left invariant

Received 17 February 1978.

https://doi.org/10.1017/S0004972700008261 Published online by Cambridge University Press

under linear transformation, it is sufficient to prove the theorem when Λ is the integral lattice generated by $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 1)$; in this case $d(\Lambda) = 1$.

If K contains no non-zero point of the integral lattice, then K is bounded by lines through the points $(\pm 1, 0), (0, \pm 1)$. We may in fact assume that the line m through $V(0, 1)$ has slope less than 1 (else $A(K \cap Q_2) \le \frac{1}{2}$, and $A(K) \le 2$), and slope greater than -1 (else $A(K \cap Q_1) \le \frac{1}{2}$, and $A(K) \le 2$). Similarly, the line l through $U(1, 0)$ has slope less than -1 or greater than 1 .



Let l and m meet at the point P_1 . Since K is closed and contains no non-zero lattice points, $A(K \cap Q_1) < A(OUP_1V)$. If $P_1(x, y)$ lies in the portion of Q_1 satisfying $x + y \le 2$, then

$$A(OUP_1V) = A(\Delta OUV) + A(\Delta UVP_1) \le 1 ,$$

$A(K \cap Q_1) < 1$, and $A(K) < 4$.

Suppose then that $P_1(x, y)$ satisfies $x + y > 2$. Since the circle on UV as diameter touches the line $x + y = 2$, we deduce that $\underline{/UP_1V}$ is acute.

Using a similar argument in each quadrant Q_i we obtain: either $A(K) < 4$, or $\underline{/UP_iV}$ is acute $(1 \le i \le 4)$. Since a quadrilateral $P_1P_2P_3P_4$ cannot have four acute angles, we conclude that $A(K) < 4$. This completes the proof of the theorem.

There is an obvious generalization to n-dimensional euclidean space, E^n . The vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ generating the lattice Λ will determine $m = 2^n$ orthants, $O_1, O_2, \ldots, O_m$. Let $V(K)$ denote the n-dimensional volume of K . We can now conjecture:

If K is a closed, convex set in E^n for which $V(K \cap O_i) = \frac{1}{2^n} V(K)$ $(1 \le i \le 2^n)$, and K contains no non-zero point of the lattice Λ , then $V(K) < 2^n d(\Lambda)$.

However, even for $n = 3$ it is not immediately clear how one might establish this result.

Reference

[1] P.R. Scott, "An analogue of Minkowski's theorem in the plane", *J. London Math. Soc.* (2) 8 (1974), 647-651.

Department of Pure Mathematics,
University of Adelaide,
Adelaide,
South Australia.