22. THE INFLUENCE OF PROPERTIES OF A SET OF OBSERVATIONS ON THE WEIGHTS OF DETERMINATION OF THE ORBITAL ELEMENTS OF A ONE-APPARITION COMET

M. BIELICKI

Astronomical Observatory, Warsaw University, Warsaw, Poland

Abstract. An expression is developed for the dependence of the weights of the determination of the elements of the orbit of a one-apparition comet on the number of observations, the length of the observational interval and the apparent motion of the comet during that interval.

The determination of the definitive orbit of a celestial body consists in finding those values of the orbital elements for which the discrepancies between the observed and calculated positions are smallest. This is generally achieved by means of the method of least squares. This method satisfies the real distribution of random errors, provided that the measurements are of equal weight and free from nonaccidental errors. The determination of weights and the rejection of observations with nonaccidental errors involve only the quality of the measurements and can frequently be treated quite objectively (Bielicki, 1972).

We are interested here, however, in certain properties of the observational material as a whole and in the influence of these properties on the accuracy of the determination of the orbital elements. In the case of a one-apparition comet it is intuitively obvious that this accuracy is affected by three factors:

(1) the number M of observational equations;

(2) the interval of time T covered by the observations;

(3) a quantity K that depends on the apparent motion of the comet during the observational interval.

By application of Cauchy's theorem Jacobi obtained the following formula connecting the solutions of particular groups of observational equations with the general solution of all the observational equations by the method of least squares:

$$E_j^m - E_j^c = \frac{\sum_{(r)=1}^{\binom{M}{p}} (D_{(r)})^2 (E_j^m - E_j^c)_{(r)}}{\sum_{(r)=1}^{\binom{M}{p}} (D_{(r)})^2}, \tag{1}$$

where $E_j^m - E_j^c$ is the most probable correction to the parameter E_j^c, with $j = 1, 2, \ldots, p$; $D_{(r)}$ is the determinant of the coefficients of the combination (r) of observational equations, selected from the total of M observational equations in $\binom{M}{p}$ discrete ways; and $(E_j^m - E_j^c)_{(r)}$ is the rigorous solution of the combination (r) of observational equations. The above formula permits only a qualitative discussion on the accuracy

Chebotarev et al. (eds.), The Motion, Evolution of Orbits, and Origin of Comets, 118–122.

of the results, and such general discussions have been described by various authors (Whittaker and Robinson, 1924; Plummer, 1939; etc.).

This prompted us to find a formula that would directly relate the coefficients in the observational equations with the weights Q_{jj}^{-1} of the parameters j of the orbit. It is as follows:

$$Q_{jj}^{-1} = \frac{\sum_{(r)=1}^{\binom{M}{p}} (D_{(r)})^2}{\sum_{(r)=1}^{\binom{M}{p-1}} (D_{(r)}^{jj})^2}, \tag{2}$$

where $D_{(r)}^{jj}$ is the determinant of the coefficients of the combination (r) of observational equations, selected in $\binom{M}{p-1}$ discrete ways, and in which the column of coefficients corresponding to the unknown j has been removed. A summary of the reasoning follows.

Let $(D_{(r)})^2$ be a random variable with the same probability density as an element of the general population. Then its expected value is

$$E[(D_{(r)})^2] = \binom{M}{p}^{-1} \sum_{(r)=1}^{\binom{M}{p}} D(_{(r)})^2.$$

Taking a sample of m elements $(D'_{(r)})^2$, we have

$$E[(D'_{(r)})^2] = E[(D_{(r)})^2].$$

When we add this sample to the total population we have $M+m$ elements $(D''_{(r)})^2$ and also

$$E[(D''_{(r)})^2] = E[(D_{(r)})^2].$$

From the above there results

$$\binom{M+m}{p}^{-1} \sum_{(r)=1}^{\binom{M+m}{p}} (D''_{(r)})^2 = \binom{M}{p}^{-1} \sum_{(r)=1}^{\binom{M}{p}} (D_{(r)})^2,$$

and, with analogous reasoning for the determinants $D_{(r)}^{jj}$, we have for $M+m$ and M observations

$$(Q_{jj}^{-1})_{M+m} = \frac{\sum_{(r)=1}^{\binom{M+m}{p}} (D''_{(r)})^2}{\sum_{(r)=1}^{\binom{M+m}{p-1}} (D''^{jj}_{(r)})^2} = \frac{M+m-p+1}{M-p+1} = (Q_{jj}^{-1})_M;$$

Since the limit M^{-1} tends to zero, we have

$$Q_{jj}^{-1} \sim M. \tag{3}$$

https://doi.org/10.1017/S0074180900006379 Published online by Cambridge University Press

Let two consecutive values $c^j_{(r)i}$ and $c^j_{(r)i+1}$ of the differential coefficient $c^j_{(r)}$ for the parameter j in the combination (r) of observational equations be of the form

$$c^j_{(r)i+1} = c^j_{(r)i} + k^j_{(r)i}(w_{(r)i+1} - w_{(r)i}) + \cdots.$$

We suppose that the independent variable w is chosen so that quadratic and higher terms may be ignored. Then

$$D_{(r)} = \begin{vmatrix} c^1_{(r)1} & c^2_{(r)1} & \cdots & c^p_{(r)1} \\ c^1_{(r)2} & c^2_{(r)2} & \cdots & c^p_{(r)2} \\ \cdot & \cdot & \cdot & \cdot \\ c^1_{(r)p} & c^2_{(r)p} & \cdots & c^p_{(r)p} \end{vmatrix}$$

$$= \begin{vmatrix} c^1_{(r)1} & c^2_{(r)1} & \cdots \\ c^1_{(r)1} + k^1_{(r)1}(w_{(r)2} - w_{(r)1}) & c^2_{(r)1} + k^2_{(r)1}(w_{(r)2} - w_{(r)1}) & \cdots \\ c^1_{(r)2} + k^1_{(r)2}(w_{(r)3} - w_{(r)2}) & c^2_{(r)2} + k^2_{(r)2}(w_{(r)3} - w_{(r)2}) & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \begin{vmatrix} c^1_{(r)1} & c^2_{(r)1} & \cdots \\ k^1_{(r)1} & k^2_{(r)1} & \cdots \\ k^1_{(r)2} & k^2_{(r)2} & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix} \times \prod_{i=1}^{p-1} (w_{(r)i+1} - w_{(r)i})$$

$$= D^{c,k}_{(r)} \prod_{i=1}^{p-1} (w_{(r)i+1} - w_{(r)i}).$$

Now, $D^{c,k}_{(r)} = c^1_{(r)1} D^{k1}_{(r)} - c^2_{(r)1} D^{k2}_{(r)} + \cdots$, where $D^{k1}_{(r)}$, $D^{k2}_{(r)}$, are the minors of the determinant $D^{c,k}_{(r)}$. Then,

$$\frac{d}{dw} D^{c,k}_{(r)} = \frac{d}{dw} c^1_{(r)1} D^{k1}_{(r)} + c^1_{(r)1} \frac{d}{dw} D^{k1}_{(r)} - \cdots$$

$$= k^1_{(r)1} D^{k1}_{(r)} - k^2_{(r)1} D^{k2}_{(r)} + \cdots$$

But the determinant $D^{k1}_{(r)}$ is of the form

$$D^{k1}_{(r)} = \begin{vmatrix} k^2_1 & k^3_1 & \cdots & k^p_1 \\ k^2_2 & k^3_2 & \cdots & k^p_2 \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \text{const} \prod_{i \times 1}^{p-2} (w_{(r)i+1} - w_{(r)i}),$$

and so are $D^{k2}_{(r)}$, etc. Thus

$$\frac{d}{dw} D^{c,k}_{(r)} \sim \prod_{i=1}^{p-2} (w_{(r)i+1} - w_{(r)i}).$$

https://doi.org/10.1017/S0074180900006379 Published online by Cambridge University Press

TABLE I

Influence of the number of observations (M), the length of the observational interval (T) and the apparent motion factor (K) on the relative weights of the elements determined for the orbit of comet 1953 I

Observations included	No. of observations	Observational interval (days)	$\bar{r}$ (AU)	$\bar{\Delta}$ (AU)	M	T^4	K^4	${}_{\text{theor}}Q_{jj}^{-1}$	${}_{\text{obs}}Q_{jj}^{-1}$
Sensitivity to M									
1–83	83	236	2.1	1.7	1.00	1.00	1.00	1.00	1.00
1, 3, 5, . . ., 81, 83	42	236	2.1	1.7	0.51	1.00	1.00	0.51	0.53
1, 4, 7, . . ., 79, 82	28	236	2.1	1.7	0.34	1.00	1.00	0.34	0.34
Sensitivity to T									
23–82	60	150	1.9	1.4	0.72	0.16	1.46	0.17	0.19
33–72	40	87	1.8	1.2	0.48	0.018	2.18	0.019	0.023
43–62	20	32	1.6	1.5	0.24	0.00034	0.56	0.000045	0.000040
Sensitivity to K									
1–20	20	59	2.4	1.6	0.24	0.0039	2.17	0.0020	0.0017
21–40	20	50	1.8	1.0	0.24	0.0020	4.51	0.0022	0.0022
41–60	20	32	1.6	1.4	0.24	0.00034	0.73	0.000060	0.000053
61–80	20	57	1.8	1.9	0.24	0.0034	0.34	0.00028	0.00023

The figures given as ${}_{\text{obs}}Q_{jj}^{-1}$ are the geometric means of the determinations of the components of position and velocity of the comet.

In which case,

$$D_{(r)}^{c,k} \sim \prod_{i=1}^{p-1} (w_{(r)i+1} - w_{(r)i}),$$

$$D_{(r)} \sim \prod_{i=1}^{p-1} (w_{(r)i+1} - w_{(r)i})^2.$$

Analogously, when p is replaced by $p-1$,

$$D_{(r)}^{jj} \sim \prod_{i=1}^{p-2} (w_{(r)i+1} - w_{(r)i})^2.$$

If the observations are uniformly distributed with respect to w, then $w_{(r)i+1} - w_{(r)i} \sim W$, the total length of the observational interval. Hence $D_{(r)} \sim (W^{p-1})^2$ and $D_{(r)}^{jj} \sim (W^{p-2})^2$, and from Equation (2) we obtain

$$Q_{jj}^{-1} \sim \frac{(W^{p-1})^4}{(W^{p-2})^4} = W^4.$$

We now define the quantity K, the measure of the comet's apparent motion, to be the average value of dw/dt during the interval of observation T (which corresponds to W). Hence

$$Q_{jj}^{-1} \sim K^4T^4,$$

and combining this with Equation (3), we find the complete dependence of the weights on the properties of the observations to be

$$Q_{jj}^{-1} \sim MK^4T^4. \tag{4}$$

In practice, e.g., when we determine the components of position and velocity as the cometary orbit, K is the average value of the ratio of the heliocentric distance r and the geocentric distance Δ during the observation interval:

$$K = \langle r/\Delta \rangle. \tag{5}$$

The results in Table I show the effects of different selections of observations on the relative weights of the orbital elements of comet 1953 I, using Equations (4) and (5); see also Sitarski (1972).

References

Bielicki, M.: 1972, this Symposium, p. 112.
Plummer, H. C.: 1939, *Probability and Frequency*, London, p. 170.
Sitarski, G.: 1972, this Symposium, p. 107.
Whittaker, E. A. and Robinson, G.: 1944, *The Calculus of Observations*, 4th ed., London and Glasgow, p. 251.