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Abstract

In [M. Kontsevich and Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants*, Preprint (2011), arXiv:1006.2706v2[math.AG]], the authors, in particular, associate to each finite quiver Q with a set of vertices I the so-called cohomological Hall algebra $\mathcal{H}$, which is $\mathbb{Z}^I_{\geqslant 0}$-graded. Its graded component $\mathcal{H}_\gamma$ is defined as cohomology of the Artin moduli stack of representations with dimension vector γ. The product comes from natural correspondences which parameterize extensions of representations. In the case of a symmetric quiver, one can refine the grading to $\mathbb{Z}^I_{\geqslant 0}\times\mathbb{Z}$, and modify the product by a sign to get a super-commutative algebra $(\mathcal{H},\star)$ (with parity induced by the $\mathbb{Z}$-grading). It is conjectured in [M. Kontsevich and Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants*, Preprint (2011), arXiv:1006.2706v2[math.AG]] that in this case the algebra $(\mathcal{H}\otimes\mathbb{Q},\star)$ is free super-commutative generated by a $\mathbb{Z}^I_{\geqslant 0}\times\mathbb{Z}$-graded vector space of the form $V=V^{\mathrm{prim}}\otimes\mathbb{Q}[x]$, where x is a variable of bidegree $(0,2)\in\mathbb{Z}^I_{\geqslant 0}\times\mathbb{Z}$, and all the spaces $\bigoplus_{k\in\mathbb{Z}}V^{\mathrm{prim}}_{\gamma,k}$, $\gamma\in\mathbb{Z}^I_{\geqslant 0}$. are finite-dimensional. In this paper we prove this conjecture (Theorem 1.1). We also prove some explicit bounds on pairs (γ,k) for which $V^{\mathrm{prim}}_{\gamma,k}\neq 0$ (Theorem 1.2). Passing to generating functions, we obtain the positivity result for quantum Donaldson–Thomas invariants, which was used by Mozgovoy to prove Kac's conjecture for quivers with sufficiently many loops [S. Mozgovoy, *Motivic Donaldson–Thomas invariants and Kac conjecture*, Preprint (2011), arXiv:1103.2100v2[math.AG]]. Finally, we mention a connection with the paper of Reineke [M. Reineke, *Degenerate cohomological Hall algebra and quantized Donaldson–Thomas invariants for m-loop quivers*, Preprint (2011), arXiv:1102.3978v1[math.RT]].

1. Introduction

In this paper we study the cohomological Hall algebra (COHA) introduced by Kontsevich and Soibelman [KS11], in the case of a symmetric quiver without potential. Our main result is the proof of the Kontsevich–Soibelman conjecture on the freeness of the COHA of a symmetric quiver.

Consider a finite quiver Q with a set of vertices I and with a_{ij} edges from $i\in I$ to $j\in I$, so that $a_{ij}\in\mathbb{Z}_{\geqslant 0}$. One can choose trivial stability conditions on the category of complex finite-dimensional representations, so that stable representations are precisely the simple ones, and

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they all have the same slope. In particular, each representation is semi-stable with the same slope. Then, for each dimension vector

$$\gamma = \{\gamma^i\}_{i \in I} \in \mathbb{Z}^I_{\geqslant 0},$$

the moduli space of representations of Q is an Artin quotient stack M_γ/G_γ, where M_γ is an affine space of all representations in coordinate vector spaces $\mathbb{C}^{\gamma^i}$, $G_\gamma = \prod_{i \in I} \mathrm{GL}(\gamma^i, \mathbb{C})$, and the action is by conjugation (see § 2.1). One then defines a $\mathbb{Z}^I_{\geqslant 0}$-graded $\mathbb{Q}$-vector space $\mathcal{H}$ by the formula

$$\mathcal{H} = \bigoplus_{\gamma \in \mathbb{Z}^I_{\geqslant 0}} \mathcal{H}_\gamma, \quad \mathcal{H}_\gamma := H^\bullet_{G_\gamma}(M_\gamma, \mathbb{Q}).$$

Note that originally in [KS11], one takes cohomology with integer coefficients, but we will deal only with the result of tensoring by $\mathbb{Q}$.

Now, for every choice of two vectors $\gamma_1, \gamma_2 \in \mathbb{Z}^I_{\geqslant 0}$, one has a natural correspondence $M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2}$ between the stacks $M_{\gamma_1}/G_{\gamma_1}$ and $M_{\gamma_2}/G_{\gamma_2}$, which parameterizes all extensions (§ 2.1). We get natural maps of stacks

$$(M_{\gamma_1}/G_{\gamma_1}) \times (M_{\gamma_2}/G_{\gamma_2}) \leftarrow M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2} \to M_{\gamma_1+\gamma_2}/G_{\gamma_1+\gamma_2},$$

which allow one to define a multiplication

$$H^\bullet_{G_{\gamma_1}}(M_{\gamma_1}) \otimes H^\bullet_{G_{\gamma_2}}(M_{\gamma_2}) \to H^{\bullet - 2\chi_Q(\gamma_1,\gamma_2)}_{G_{\gamma_1+\gamma_2}}(M_{\gamma_1+\gamma_2}), \tag{1.1}$$

where $\chi_Q(\gamma_1, \gamma_2)$ is the Euler form

$$\chi_Q(\gamma_1, \gamma_2) = \sum_{i \in I} \gamma_1^i \gamma_2^i - \sum_{i,j \in I} a_{ij} \gamma_1^i \gamma_2^j.$$

It is proved in [KS11, Theorem 1] that the resulting product on $\mathcal{H}$ is associative, so this makes $\mathcal{H}$ into a $\mathbb{Z}^I_{\geqslant 0}$-graded algebra, which is called the (rational) cohomological Hall algebra of a quiver Q.

Now we restrict to the case of a symmetric quiver Q, i.e. to the case $a_{ij} = a_{ji}$. In this case the Euler form $\chi_Q(\gamma_1, \gamma_2)$ is symmetric as well. One defines a $(\mathbb{Z}^I_{\geqslant 0} \times \mathbb{Z})$-graded algebra structure on $\mathcal{H}$, by assigning to a subspace $H^k_{G_\gamma}(M_\gamma)$ a bigrading $(\gamma, k + \chi_Q(\gamma, \gamma))$. It follows from (1.1) that the product is compatible with this grading. We also define a parity on $\mathcal{H}$ to be induced by the $\mathbb{Z}$-grading (see § 2.3).

In general, the algebra $\mathcal{H}$ for symmetric quiver is not super-commutative, but it becomes so after twisting the product by a sign (§ 2.3). Denote by $\star$ the resulting super-commutative product. Our main result is the following theorem which was conjectured in [KS11, Conjecture 1].

Theorem 1.1. *For any finite symmetric quiver Q, the $(\mathbb{Z}^I_{\geqslant 0} \times \mathbb{Z})$-graded algebra $(\mathcal{H}, \star)$ is a free super-commutative algebra generated by a $(\mathbb{Z}^I_{\geqslant 0} \times \mathbb{Z})$-graded vector space V of the form $V = V^{\mathrm{prim}} \otimes \mathbb{Q}[x]$, where x is a variable of degree $(0, 2) \in \mathbb{Z}^I_{\geqslant 0} \times \mathbb{Z}$, and for any $\gamma \in \mathbb{Z}^I_{\geqslant 0}$ the space $V^{\mathrm{prim}}_{\gamma,k}$ is non-zero (and finite-dimensional) only for finitely many $k \in \mathbb{Z}$.*

The second result in this paper gives explicit bounds on pairs (γ, k) for which $V^{\mathrm{prim}}_{\gamma,k} \neq 0$. For a given symmetric quiver Q, and $\gamma \in \mathbb{Z}^I_{\geqslant 0} \backslash \{0\}$, we put

$$N_\gamma(Q) := \frac{1}{2}\Bigg(\sum_{\substack{i,j \in I,\\ i \neq j}} a_{ij}\gamma^i\gamma^j + \sum_{i \in I} \max(a_{ii} - 1, 0)\gamma^i(\gamma^i - 1)\Bigg) - \sum_{i \in I} \gamma^i + 2.$$

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Theorem 1.2. *In the notation of Theorem 1.1, if* $V^{\mathrm{prim}}_{\gamma,k} \neq 0$, *then* $\gamma \neq 0$,

$$k \equiv \chi_Q(\gamma,\gamma) \mod 2 \quad \textit{and} \quad \chi_Q(\gamma,\gamma) \leqslant k < \chi_Q(\gamma,\gamma) + 2N_\gamma(Q).$$

The only non-trivial statement in Theorem 1.2 is the upper bound on k. In the proofs of both theorems, we use explicit formulas for the product in $\mathcal{H}$ from [KS11, Theorem 2]. Namely, since the affine space M_γ is G_γ-equivariantly contractible, we have

$$\mathcal{H}_\gamma \cong H^\bullet(\mathrm{B}\, G_\gamma),$$

and the right-hand side is isomorphic to the algebra of polynomials in $x_{i,\alpha}$, where $i \in I$, $1 \leqslant \alpha \leqslant \gamma^i$, which are invariant with respect to the product of symmetric groups S_{γ^i}. Then, given two polynomials $f_1 \in \mathcal{H}_{\gamma_1}$, $f_2 \in \mathcal{H}_{\gamma_2}$, their product $f_1 \cdot f_2 \in \mathcal{H}_\gamma$, $\gamma = \gamma_1 + \gamma_2$, equals the sum over all shuffles (for any $i \in I$) of the following rational function in variables $(x'_{i,\alpha})_{i\in I,\alpha\in\{1,\dots,\gamma_1^i\}}$, $(x''_{i,\alpha})_{i\in I,\alpha\in\{1,\dots,\gamma_2^i\}}$:

$$f_1((x'_{i,\alpha}))f_2((x''_{i,\alpha}))\frac{\prod_{i,j\in I}\prod_{\alpha_1=1}^{\gamma_1^i}\prod_{\alpha_2=1}^{\gamma_2^j}(x''_{j,\alpha_2}-x'_{i,\alpha_1})^{a_{ij}}}{\prod_{i\in I}\prod_{\alpha_1=1}^{\gamma_1^i}\prod_{\alpha_2=1}^{\gamma_2^i}(x''_{i,\alpha_2}-x'_{i,\alpha_1})}.$$

Theorems 1.1 and 1.2 imply the corresponding results for the generating functions for cohomological Hall algebras, and, in particular, positivity for quantum Donaldson–Thomas invariants. The positivity result was used by Mozgovoy to prove Kac's conjecture for quivers with at least one loop at each vertex [Moz11].

The paper is organized as follows.

Section 2 is devoted to some preliminaries on cohomological Hall algebras for quivers. We follow [KS11, § 2]. In § 2.1 we give a definition of the rational cohomological Hall algebra for an arbitrary finite quiver. Section 2.2 is devoted to explicit formulas for the product in cohomological Hall algebras. In § 2.3 we define an additional $\mathbb{Z}$-grading on the COHA of a symmetric quiver, so that we get a $(\mathbb{Z}^I_{\geqslant 0} \times \mathbb{Z})$-graded algebra. Then, we show how to modify the product on $\mathcal{H}$ by a sign to get a super-commutative algebra $(\mathcal{H}, \star)$, with parity induced by the $\mathbb{Z}$-grading.

Section 3 is devoted to the proofs of Theorem 1.1 (Theorem 3.1) and Theorem 1.2 (Theorem 3.10).

In § 4 we discuss applications of our results to the generating function of COHA, or, in other words, to quantized Donaldson–Thomas invariants.

2. Preliminaries on cohomological Hall algebras

In this section we recall some definitions and results from [KS11, § 2].

2.1 COHA of a quiver

Let Q be a finite quiver. Denote its set of vertices by I, and let $a_{ij} \in \mathbb{Z}_{\geqslant 0}$ be the number of arrows from i to j, where $i, j \in I$. Fix a dimension vector $\gamma = (\gamma^i)_{i\in I} \in \mathbb{Z}^I_{\geqslant 0}$. We have an affine variety of representations of Q in complex coordinate vector spaces $\mathbb{C}^{\gamma^i}$:

$$M_\gamma = \prod_{i,j\in I} \mathbb{C}^{a_{ij}\gamma^i\gamma^j}.$$

The variety M_γ is acted on via conjugation by the complex algebraic group $G_\gamma = \prod_{i\in I} \mathrm{GL}(\gamma^i, \mathbb{C})$.

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Recall that the infinite-dimensional Grassmannian

$$\mathrm{Gr}(d,\infty)=\varinjlim \mathrm{Gr}(d,\mathbb{C}^n),\quad n\to+\infty,$$

is a model for the classifying space of $\mathrm{GL}(d,\mathbb{C})$. Put

$$\mathrm{B}\,G_\gamma:=\prod_{i\in I}\mathrm{B}\,\mathrm{GL}(\gamma^i,\mathbb{C})=\prod_{i\in I}\mathrm{Gr}(\gamma^i,\infty).$$

We have a standard universal G_γ-bundle $\mathrm{E}\,G_\gamma\to\mathrm{B}\,G_\gamma$, and the Artin stack M_γ/G_γ gives a universal family over $\mathrm{B}\,G_\gamma$:

$$M_\gamma^{\mathrm{univ}}:=(\mathrm{E}\,G_\gamma\times M_\gamma)/G_\gamma\to\mathrm{E}\,G_\gamma/G_\gamma=\mathrm{B}\,G_\gamma.$$

Define a $\mathbb{Z}^I_{\geqslant 0}$-graded $\mathbb{Q}$-vector space

$$\mathcal{H}=\bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}}\mathcal{H}_\gamma,$$

putting

$$\mathcal{H}_\gamma:=H^\bullet_{G_\gamma}(M_\gamma,\mathbb{Q})=\bigoplus_{n\geqslant 0}H^n(M_\gamma^{\mathrm{univ}},\mathbb{Q}).$$

Now we define a multiplication on $\mathcal{H}$ which makes it into an associative unital $\mathbb{Z}^I_{\geqslant 0}$-graded algebra over $\mathbb{Q}$. Take two vectors $\gamma_1,\gamma_2\in\mathbb{Z}^I_{\geqslant 0}$, and put $\gamma:=\gamma_1+\gamma_2$. Consider the affine subspace $M_{\gamma_1,\gamma_2}\subset M_\gamma$, which consists of representations for which the standard subspaces $\mathbb{C}^{\gamma_1^i}\subset\mathbb{C}^{\gamma^i}$ form a subrepresentation. The subspace M_{γ_1,γ_2} is preserved by the action of the subgroup $G_{\gamma_1,\gamma_2}\subset G_\gamma$ which consists of transformations preserving the subspaces $\mathbb{C}^{\gamma_1^i}\subset\mathbb{C}^{\gamma^i}$. We use a model for $\mathrm{B}\,G_{\gamma_1,\gamma_2}$ which is the total space of a bundle over $\mathrm{B}\,G_\gamma$ with fiber $G_\gamma/G_{\gamma_1,\gamma_2}$ (i.e. a product of infinite-dimensional partial flag varieties $\mathrm{Fl}(\gamma_1^i,\gamma_i,\infty)$). We have a natural projection $\mathrm{E}\,G_\gamma\to\mathrm{B}\,G_{\gamma_1,\gamma_2}$ which is a universal G_{γ_1,γ_2}-bundle.

Now define the morphism

$$m_{\gamma_1,\gamma_2}:\mathcal{H}_{\gamma_1}\otimes\mathcal{H}_{\gamma_2}\to\mathcal{H}_\gamma$$

as the composition of the Künneth isomorphism

$$\otimes: H^\bullet_{G_{\gamma_1}}(M_{\gamma_1,\mathbb{Q}})\otimes H^\bullet_{G_{\gamma_2}}(M_2,\mathbb{Q})\xrightarrow{\cong}H^\bullet_{G_{\gamma_1}\times G_{\gamma_2}}(M_{\gamma_1}\times M_{\gamma_2},\mathbb{Q})$$

and the following morphisms:

$$H^\bullet_{G_{\gamma_1}\times G_{\gamma_2}}(M_{\gamma_1}\times M_{\gamma_2},\mathbb{Q})\xrightarrow{\cong}H^\bullet_{G_{\gamma_1,G_{\gamma_2}}}(M_{\gamma_1,\gamma_2},\mathbb{Q})\to H^{\bullet+2c_1}_{G_{\gamma_1,\gamma_2}}(M_\gamma,\mathbb{Q})\to H^{\bullet+2c_1+2c_2}_{G_\gamma}(M_\gamma).$$

Here the first map is induced by natural surjective homotopy equivalences

$$M_{\gamma_1,\gamma_2}\xrightarrow{\sim}M_{\gamma_1}\times M_{\gamma_2},\quad G_{\gamma_1,\gamma_2}\to G_{\gamma_1}\times G_{\gamma_2}.$$

The other two maps are natural pushforward morphisms, with

$$c_1=\dim_{\mathbb{C}}M_\gamma-\dim_{\mathbb{C}}M_{\gamma_1,\gamma_2},\quad c_2=\dim_{\mathbb{C}}G_{\gamma_1,\gamma_2}-\dim_{\mathbb{C}}G_\gamma.$$

Theorem 2.1 [KS11, Theorem 1]. *The constructed product m on $\mathcal{H}$ is associative.*

Note that

$$c_1+c_2=-\chi_Q(\gamma_1,\gamma_2),\tag{2.1}$$

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where

$$\chi_Q(\gamma_1,\gamma_2)=\sum_{i\in I}\gamma_1^i\gamma_2^i-\sum_{i,j\in I}a_{ij}\gamma_1^i\gamma_2^j$$

is the Euler form of the quiver Q. That is, given two representations R_1, R_2 (over any field) of the quiver Q, with dimension vectors γ_1,γ_2 respectively, one has

$$\sum_i(-1)^i\dim\operatorname{Ext}^i(R_1,R_2)=\dim\operatorname{Hom}(R_1,R_2)-\dim\operatorname{Ext}^1(R_1,R_2)=\chi_Q(\gamma_1,\gamma_2).$$

2.2 Explicit description of the COHA of a quiver

Since the affine spaces M_γ are G_γ-equivariantly contractible, we have natural isomorphisms

$$\mathcal{H}_\gamma\cong H^\bullet(\mathrm{B}\,G_\gamma,\mathbb{Q})=\bigotimes_{i\in I}H^\bullet(\mathrm{B}\,\mathrm{GL}(\gamma^i,\mathbb{C}),\mathbb{Q}).$$

Recall that

$$H^\bullet(\mathrm{B}\,\mathrm{GL}(d,\mathbb{C}),\mathbb{Q})\cong\mathbb{Q}[x_1,\dots,x_d]^{S_d}.$$

For a vector $\gamma\in\mathbb{Z}^I_{\geqslant 0}$, introduce variables $x_{i,\alpha}$, where $i\in I$, $\alpha\in\{1,\dots,\gamma^i\}$. Then, we get natural isomorphisms

$$\mathcal{H}_\gamma\cong\mathbb{Q}[\{x_{i,\alpha}\}_{i\in I,\alpha\in\{1,\dots,\gamma^i\}}]^{\prod_{i\in I}S_{\gamma^i}}.$$

From this moment, we identify the elements of $\mathcal{H}_\gamma$ with the corresponding polynomials.

Definition 2.2. For non-negative integers p,q, we define a (p,q)-shuffle to be a permutation $\sigma\in S_{p+q}$ such that

$$\sigma(1)<\dots<\sigma(p),\quad \sigma(p+1)<\dots<\sigma(p+q).$$

Further, take a pair of dimension vectors $\gamma_1,\gamma_2\in\mathbb{Z}^I_{\geqslant 0}$, and put $\gamma:=\gamma_1+\gamma_2$. We define a (γ_1,γ_2)-shuffle to be an element $\sigma\in P_\gamma:=\prod_{i\in I}S_{\gamma^i}$ such that for each $i\in I$ the component $\sigma_i\in S_{\gamma^i}$ is a (γ_1^i,γ_2^i)-shuffle.

Theorem 2.3 [KS11, Theorem 2]. *Given two polynomials $f_1\in\mathcal{H}_{\gamma_1}$, $f_2\in\mathcal{H}_{\gamma_2}$, their product $f_1\cdot f_2\in\mathcal{H}_\gamma$, $\gamma=\gamma_1+\gamma_2$, equals the sum over all (γ_1,γ_2)-shuffles of the following rational function in variables $(x'_{i,\alpha})_{i\in I,\alpha\in\{1,\dots,\gamma_1^i\}}$, $(x''_{i,\alpha})_{i\in I,\alpha\in\{1,\dots,\gamma_2^i\}}$:*

$$f_1((x'_{i,\alpha}))f_2((x''_{i,\alpha}))\frac{\prod_{i,j\in I}\prod_{\alpha_1=1}^{\gamma_1^i}\prod_{\alpha_2=1}^{\gamma_2^j}(x''_{j,\alpha_2}-x'_{i,\alpha_1})^{a_{ij}}}{\prod_{i\in I}\prod_{\alpha_1=1}^{\gamma_1^i}\prod_{\alpha_2=1}^{\gamma_2^i}(x''_{i,\alpha_2}-x'_{i,\alpha_1})}.$$

2.3 Additional grading in the symmetric case

Now assume that the quiver Q is symmetric, i.e. $a_{ij}=a_{ji}$, $i,j\in I$. Then the Euler form

$$\chi_Q(\gamma_1,\gamma_2)=\sum_{i\in I}\gamma_1^i\gamma_2^i-\sum_{i,j\in I}a_{ij}\gamma_1^i\gamma_2^i$$

is symmetric as well.

We make $\mathcal{H}$ into a $(\mathbb{Z}^I_{\geqslant 0}\times\mathbb{Z})$-graded algebra as follows. For a polynomial $f\in\mathcal{H}_\gamma$ of degree k we define its bigrading to be $(\gamma,2k+\chi_Q(\gamma,\gamma))$. It follows from either (2.1) or Theorem 2.3 that the product on $\mathcal{H}$ is compatible with this bigrading. Define the super-structure on $\mathcal{H}$ to be induced by the $\mathbb{Z}$-grading.

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For two elements $a_{\gamma,k} \in \mathcal{H}_{\gamma,k}$, $a_{\gamma',k'} \in \mathcal{H}_{\gamma',k'}$, we have

$$a_{\gamma,k}a_{\gamma',k'} = (-1)^{\chi_Q(\gamma,\gamma')}a_{\gamma',k'}a_{\gamma,k}.$$

In general, this does not mean that $\mathcal{H}$ is super-commutative. However, it is easy to twist the product by a sign, so that $\mathcal{H}$ becomes super-commutative. This can be done as follows.

Define the homomorphism of abelian groups $\epsilon : \mathbb{Z}^I \to \mathbb{Z}/2\mathbb{Z}$ by the formula

$$\epsilon(\gamma) = \chi_Q(\gamma,\gamma) \mod 2.$$

Note that the parity of the element $a_{\gamma,k}$ equals $\epsilon(\gamma)$ (by the definition). We have a bilinear form

$$\mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}/2\mathbb{Z}, \quad (\gamma_1,\gamma_2) \mapsto (\chi_Q(\gamma_1,\gamma_2) + \epsilon(\gamma_1)\epsilon(\gamma_2)) \mod 2,$$

which induces a symmetric form β on the space $(\mathbb{Z}/2\mathbb{Z})^I$, such that $\beta(\gamma,\gamma) = 0$ for all $\gamma \in (\mathbb{Z}/2\mathbb{Z})^I$. Hence, there exists a bilinear form ψ on $(\mathbb{Z}/2\mathbb{Z})^I$ such that

$$\psi(\gamma_1,\gamma_2) + \psi(\gamma_2,\gamma_1) = \beta(\gamma_1,\gamma_2).$$

Then the twisted product on $\mathcal{H}$ is defined by the formula

$$a_{\gamma,k} \star a_{\gamma',k'} = (-1)^{\psi(\gamma,\gamma')}a_{\gamma,k} \cdot a_{\gamma',k'}.$$

It follows from the definition that the product $\star$ is associative, and the algebra $(\mathcal{H},\star)$ is super-commutative. From now on, we fix the choice of bilinear form ψ, and the corresponding product $\star$ on $\mathcal{H}$.

3. Freeness of the COHA of a symmetric quiver

Theorem 3.1. *For any finite symmetric quiver Q, the $(\mathbb{Z}^I_{\geqslant 0} \times \mathbb{Z})$-graded algebra $(\mathcal{H},\star)$ is a free super-commutative algebra generated by a $(\mathbb{Z}^I_{\geqslant 0} \times \mathbb{Z})$-graded vector space V of the form $V = V^{\mathrm{prim}} \otimes \mathbb{Q}[x]$, where x is a variable of bidegree $(0,2) \in \mathbb{Z}^I_{\geqslant 0} \times \mathbb{Z}$, and for any $\gamma \in \mathbb{Z}^I_{\geqslant 0}$ the space $V^{\mathrm{prim}}_{\gamma,k}$ is non-zero (and finite-dimensional) only for finitely many $k \in \mathbb{Z}$.*

Before giving a proof of this theorem, we illustrate it in some examples.

Let Q_d be a quiver with one vertex and d loops, $d \geqslant 0$. Then $\mathcal{H}_{n,k}$ is the space of symmetric polynomials in n variables of degree $(k-(1-d)n^2)/2$. In this case we do not need to modify the product by a sign.

Example 3.2. For $d=0$, the super-commutative algebra $\mathcal{H}$ is freely generated by odd elements $\psi_{2k+1} := x_1^k \in \mathcal{H}_{1,2k+1}$, $k \in \mathbb{Z}_{\geqslant 0}$, Thus $V = \mathcal{H}_1 \subset \mathcal{H}$, and the space $V^{\mathrm{prim}} = \mathcal{H}_{1,1} = \mathbb{Q}\cdot\psi_1$ is one-dimensional.

Example 3.3. For $d=1$, the super-commutative algebra $\mathcal{H}$ is freely generated by even elements $\phi_{2k} := x_1^k \in \mathcal{H}_{1,2k}$, $k \in \mathbb{Z}_{\geqslant 0}$, Thus again $V = \mathcal{H}_1 \subset \mathcal{H}$, and the space $V^{\mathrm{prim}} = \mathcal{H}_{1,0} = \mathbb{Q}\cdot\phi_0$ is one-dimensional.

These two cases were considered in [KS11, § 2.5]. However, for $d \geqslant 2$ the picture becomes much more complicated.

Example 3.4. Consider the case $d=2$. It is not hard to see that all the spaces V_n, $n \geqslant 1$, have to be non-zero and contain $1 \in \mathcal{H}_{n,-n^2}$. We write down here V_n and V_n^{prim} for $n \leqslant 3$.

We have to take the component $V_1 = \bigoplus_k V_{1,k}$ to be equal to $\mathcal{H}_1 = \bigoplus_{k\geqslant 0}\mathcal{H}_{1,2k-1}$, and hence $V_1^{\mathrm{prim}} = V_{1,-1}^{\mathrm{prim}} = \mathbb{Q}\cdot x_1^0$. Further, the subspace of $\mathcal{H}_2$ generated by $\mathcal{H}_1$ consists of symmetric

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polynomials divisible by $(x_1-x_2)^2$. Hence, we have to take $V_2\subset\mathcal{H}_2$ to be some complementary subspace, for example, $V_2=\mathbb{Q}[x_1+x_2]\subset\mathcal{H}_2$. Then $V_2^{\mathrm{prim}}=V_{2,-4}^{\mathrm{prim}}=\mathbb{Q}\cdot(x_1+x_2)^0$. One can show that subspace of $\mathcal{H}_3$ generated by $V_1\oplus V_2$ consists of symmetric polynomials which vanish on the line $\{x_1=x_2=x_3\}$. Hence, we can choose $V_3=\mathbb{Q}[x_1+x_2+x_3]$, and $V_3^{\mathrm{prim}}=V_{3,-9}^{\mathrm{prim}}=\mathbb{Q}\cdot(x_1+x_2+x_3)^0$.

Proof. Our first step is to construct the space V. It will be convenient to treat $\mathcal{H}_\gamma$ itself as a $\mathbb{Z}$-graded algebra (with the usual multiplication of polynomials, and the standard even grading). To distinguish between the product in $\mathcal{H}_\gamma$ and the product in $\mathcal{H}$, we will always denote the latter product by '$\star$'.

For convenience, we put $A_\gamma:=\mathbb{Q}[\{x_{i,\alpha}\}_{i\in I,1\leqslant\alpha\leqslant\gamma^i}]$ (considered as a $\mathbb{Z}$-graded algebra) and $P_\gamma:=\prod_{i\in I}S_{\gamma^i}$. Then we have that $\mathcal{H}_\gamma=A_\gamma^{P_\gamma}$. Further, put

$$A_\gamma^{\mathrm{prim}}:=\mathbb{Q}[(x_{j,\alpha_2}-x_{i,\alpha_1})_{i,j\in I,1\leqslant\alpha_1\leqslant\gamma^i,1\leqslant\alpha_2\leqslant\gamma^j}],\quad \sigma_\gamma:=\sum_{\substack{i\in I,\\ 1\leqslant\alpha\leqslant\gamma^i}}x_{i,\alpha}\in A_\gamma.$$

Then $A_\gamma=A_\gamma^{\mathrm{prim}}\otimes\mathbb{Q}[\sigma_\gamma]$. Further, we have

$$\mathcal{H}_\gamma=\mathcal{H}_\gamma^{\mathrm{prim}}\otimes\mathbb{Q}[\sigma_\gamma],\quad \mathcal{H}_\gamma^{\mathrm{prim}}:=(A_\gamma^{\mathrm{prim}})^{P_\gamma}.$$

Now, for each $\gamma\in\mathbb{Z}_{\geqslant0}^I$, denote by J_γ the smallest P_γ-stable A_γ^{prim}-submodule of the localization $A_\gamma^{\mathrm{prim}}[(x_{i,\alpha_2}-x_{i,\alpha_1})^{-1}_{i\in I,1\leqslant\alpha_1<\alpha_2\leqslant\gamma^i}]$, such that for all decompositions $\gamma=\gamma_1+\gamma_2$, $\gamma_1,\gamma_2\in\mathbb{Z}_{\geqslant0}^I\backslash\{0\}$, we have that

$$\frac{\prod_{i,j\in I}\prod_{\alpha_1=1}^{\gamma_1^i}\prod_{\alpha_2=\gamma_1^j+1}^{\gamma^j}(x_{j,\alpha_2}-x_{i,\alpha_1})^{a_{ij}}}{\prod_{i\in I}\prod_{\alpha_1=1}^{\gamma_1^i}\prod_{\alpha_2=\gamma_1^i+1}^{\gamma^i}(x_{i,\alpha_2}-x_{i,\alpha_1})}\in J_\gamma.$$

Remark 3.5. Some arguments below become simpler in the case when the quiver Q has at least one loop at each vertex, i.e. $a_{ii}\geqslant1$, $i\in I$. The reason is that in this case $J_\gamma\subset A_\gamma^{\mathrm{prim}}$, and we do not need to take the localization.

It is not hard to see that $J_\gamma^{P_\gamma}\subset\mathcal{H}_\gamma^{\mathrm{prim}}$. Namely, we have that

$$J_\gamma\subset A_\gamma^{\mathrm{prim}}\cdot M^{-1},\quad M=\prod_{i\in I}\prod_{1\leqslant\alpha<\beta\leqslant\gamma^i}(x_{i,\beta}-x_{i,\alpha}),$$

and

$$(A_\gamma^{\mathrm{prim}}\cdot M^{-1})^{P_\gamma}\subset(A_\gamma\cdot M^{-1})^{P_\gamma}=\mathcal{H}_\gamma.$$

Define $V_\gamma^{\mathrm{prim}}\subset\mathcal{H}_\gamma^{\mathrm{prim}}$ to be a graded subspace such that

$$\mathcal{H}_\gamma^{\mathrm{prim}}=V_\gamma^{\mathrm{prim}}\oplus J_\gamma^{P_\gamma}.$$

Further, put

$$V_\gamma:=V_\gamma^{\mathrm{prim}}\otimes\mathbb{Q}[\sigma_\gamma]\subset\mathcal{H}_\gamma,\quad V:=\bigoplus_{\gamma\in\mathbb{Z}_{\geqslant0}^I}V_\gamma.$$

We will prove that V freely generates $\mathcal{H}$, and that all the spaces V_γ^{prim} are finite-dimensional (this would imply the theorem).

Lemma 3.6. *The subspace $V \subset \mathcal{H}$ generates $\mathcal{H}$ as an algebra.*

Proof. Note that for each $\gamma \in \mathbb{Z}_{\geqslant 0}^I$, the image of the multiplication map

$$\bigoplus_{\substack{\gamma_1+\gamma_2=\gamma,\\ \gamma_1,\gamma_2\in\mathbb{Z}_{\geqslant 0}^I\backslash\{0\}}} \mathcal{H}_{\gamma_1}\otimes\mathcal{H}_{\gamma_2}\to\mathcal{H}_\gamma$$

is precisely $J_\gamma^{P_\gamma}\otimes\mathbb{Q}[\sigma_\gamma]$. Indeed, this image clearly is contained in $(J_\gamma\otimes\mathbb{Q}[\sigma_\gamma])^{P_\gamma}=J_\gamma^{P_\gamma}\otimes\mathbb{Q}[\sigma_\gamma]$. On the other hand, the latter space is linearly spanned by P_γ-symmetrizations of expressions of the form

$$f_1(x_{i,\alpha},1\leqslant\alpha\leqslant\gamma_1^i)f_2(x_{i,\beta+\gamma_1^i},1\leqslant\beta\leqslant\gamma_2^i)\cdot\frac{\prod_{i,j\in I}\prod_{\alpha_1=1}^{\gamma_1^i}\prod_{\alpha_2=\gamma_1^j+1}^{\gamma^j}(x_{j,\alpha_2}-x_{i,\alpha_1})^{a_{ij}}}{\prod_{i\in I}\prod_{\alpha_1=1}^{\gamma_1^i}\prod_{\alpha_2=\gamma_1^i+1}^{\gamma^i}(x_{i,\alpha_2}-x_{i,\alpha_1})},\tag{3.1}$$

where $\gamma_1+\gamma_2=\gamma$. Taking first symmetrization with respect to $P_{\gamma_1}\times P_{\gamma_2}\subset P_\gamma$, we may consider only expressions (3.1) with $f_1\in\mathcal{H}_{\gamma_1}$, $f_2\in\mathcal{H}_{\gamma_2}$. The P_γ-symmetrization of such an expression is, up to a constant, just a product $f_1\star f_2$.

Hence, it follows by induction on $\sum_{i\in I}\gamma^i$ that the subspace $\mathcal{H}_\gamma$ is contained in the subalgebra generated by V. This proves the lemma. □

Remark 3.7. The proof of the above lemma shows that, for any possible choice of a free generating subspace V, we have that $V_\gamma\oplus(J_\gamma^{P_\gamma}\otimes\mathbb{Q}[\sigma_\gamma])=\mathcal{H}_\gamma$. Our choice just reflects the fact that $V\cong V^{\mathrm{prim}}\otimes\mathbb{Q}[x]$ as a graded vector space, with $\deg x=(0,2)$.

Now we will show that the spaces V_γ^{prim} are finite-dimensional.

Lemma 3.8. *For each $\gamma\in\mathbb{Z}_{\geqslant 0}^I$, the space V_γ^{prim} is finite-dimensional.*

Proof. In other words, we need to show that the ideal $J_\gamma^{P_\gamma}\subset\mathcal{H}_\gamma^{\mathrm{prim}}$ has finite codimension. First note that if we replace a_{ii} by $a_{ii}+1$, then the fractional ideal J_γ would become smaller or equal. Hence, we may and will assume that $a_{ii}>0$ for $i\in I$, and so $J_\gamma\subset A_\gamma^{\mathrm{prim}}$.

Since we have natural injective morphisms

$$\mathcal{H}_\gamma^{\mathrm{prim}}/J_\gamma^{P_\gamma}\hookrightarrow A_\gamma^{\mathrm{prim}}/J_\gamma,$$

it suffices to show that the ideal $J_\gamma\subset A_\gamma^{\mathrm{prim}}$ has finite codimension. It will be convenient to treat the algebra A_γ^{prim} as the algebra of functions on the hyperplane $W\subset\mathbb{A}_{\mathbb{Q}}^{\sum_{i\in I}\gamma^i}$, given by equation $\sigma_\gamma(x)=0$.

It suffices to show that

$$\mathrm{Supp}(A_\gamma^{\mathrm{prim}}/J_\gamma)=\{0\}\subset W.$$

Assume the converse is true. Then there exists a point $y\in W_{\bar{\mathbb{Q}}}$, $y\neq 0$, such that all the functions from J_γ vanish at y. Since $\sigma_\gamma(y)=0$, we have that not all of the coordinates $y_{i,\alpha}$ are equal to each other. Since the ideal J_γ is P_γ-stable, we may assume that there exists a decomposition $\gamma=\gamma_1+\gamma_2$, $\gamma_1,\gamma_2\in\mathbb{Z}_{\geqslant 0}^I\backslash\{0\}$, such that

$$y_{i,\alpha_1}\neq y_{j,\alpha_2}\quad\text{for } 1\leqslant\alpha_1\leqslant\gamma_1^i,\ \gamma_1^j+1\leqslant\alpha_2\leqslant\gamma^j.$$

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Then, however, the function

$$\frac{\prod_{i,j\in I}\prod_{\alpha_1=1}^{\gamma_1^i}\prod_{\alpha_2=\gamma_1^j+1}^{\gamma^j}(x_{j,\alpha_2}-x_{i,\alpha_1})^{a_{ij}}}{\prod_{i\in I}\prod_{\alpha_1=1}^{\gamma_1^i}\prod_{\alpha_2=\gamma_1^i+1}^{\gamma^i}(x_{i,\alpha_2}-x_{i,\alpha_1})}\in J_\gamma$$

does not vanish at y, a contradiction.

The lemma is proved. □

It remains to prove the freeness.

Lemma 3.9. *The subspace $V\subset\mathcal{H}$ freely generates $\mathcal{H}$.*

Proof. We have already shown the generation. So we need to show freeness.

Choose an order on I, and fix the corresponding lexicographical order on $\mathbb{Z}^I_{\geqslant 0}$ (denoted by $\gamma\succeq\gamma'$). Further, denote by $e_{\gamma,\beta}$, $1\leqslant\beta\leqslant\dim V_\gamma^{\mathrm{prim}}$, a homogeneous basis of V_γ^{prim}. We have the lexicographical order on all of the elements $e_{\gamma,\beta}$ (for all γ and β). Further, the elements $e_{\gamma,\beta}\sigma_\gamma^m$ (for all γ,β,m) form a basis of V, and again we have a lexicographical order on them, which we denote by $\succeq$.

Fix some $\gamma\in\mathbb{Z}^I_{\geqslant 0}$. Consider the set Seq_γ of all *non-increasing* sequences $(e_{\gamma_1,\beta_1}\sigma_{\gamma_1}^{m_1},\dots,e_{\gamma_d,\beta_d}\sigma_{\gamma_d}^{m_d})$ such that:

(1) $\gamma_1+\cdots+\gamma_d=\gamma$;

(2) an equality $(\gamma_i,\beta_i,m_i)=(\gamma_{i+1},\beta_{i+1},m_{i+1})$ implies $\epsilon(\gamma_i)=0$.

Clearly, we have a natural lexicographical order on Seq_γ (which we again denote by $\succeq$). For a sequence $t\in\mathrm{Seq}_\gamma$, we denote by $M_t\in\mathcal{H}_\gamma$ the corresponding product.

What we need to show is non-vanishing of each non-trivial linear combination:

$$T=\sum_{i=1}^n\lambda_i M_{t_i}\neq 0,\quad t_1,\dots,t_n\in\mathrm{Seq}_\gamma,\quad t_1\succ\cdots\succ t_n,\quad \lambda_1\dots\lambda_n\neq 0. \tag{3.2}$$

Fix some $t_1,\dots,t_n$ and $\lambda_1,\dots,\lambda_n$ as in (3.2). Denote by $(\gamma_1,\dots,\gamma_k)$ the underlying sequence of elements in $\mathbb{Z}^I_{\geqslant 0}$ for the sequence $t_1\in\mathrm{Seq}_\gamma$. Then $\gamma_1+\cdots+\gamma_k=\gamma$, and $\gamma_i\neq 0$, $1\leqslant i\leqslant k$. We have a natural isomorphism

$$A_\gamma\cong A_{\gamma_1}\otimes\cdots\otimes A_{\gamma_k}=:\widetilde{A_\gamma},$$

which induces an inclusion

$$\iota:\mathcal{H}_\gamma\hookrightarrow\mathcal{H}_{\gamma_1}\otimes\cdots\otimes\mathcal{H}_{\gamma_k}=:\widetilde{\mathcal{H}_\gamma}.$$

Put $\widetilde{P_\gamma}:=P_{\gamma_1}\times\cdots\times P_{\gamma_k}$. Then we have $\widetilde{\mathcal{H}_\gamma}=\widetilde{A_\gamma}^{\widetilde{P_\gamma}}$. Further, take the ideal

$$(J_{\gamma_1}\cap A_{\gamma_1}^{\mathrm{prim}})\widetilde{A_\gamma}+\cdots+(J_{\gamma_k}\cap A_{\gamma_k}^{\mathrm{prim}})\widetilde{A_\gamma}=:\widetilde{J_\gamma}\subset\widetilde{A_\gamma}.$$

We will write $x_{i,\alpha}^{(p)}\in\widetilde{A_\gamma}$ for variables from the pth factor $A_{\gamma_p}\subset\widetilde{A_\gamma}$.

Claim. *The elements $(x_{j,\alpha_2}^{(q)}-x_{i,\alpha_1}^{(p)})\in\widetilde{A_\gamma}$, $1\leqslant p<q\leqslant k$, are not zero divisors in the quotient ring*

$$\widetilde{A_\gamma}/\widetilde{J_\gamma}.$$

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Proof. For convenience, we may assume that the sequence $\gamma_1, \ldots, \gamma_k$ is not necessarily non-increasing, and $q = k$. Any element $g \in \widetilde{A_\gamma}$ can be written (in a unique way) as a sum

$$g = \sum_{\nu=0}^{N} g_\nu \sigma_{\gamma_k}^{\nu}, \quad g_\nu \in A_{\gamma_1} \otimes \cdots \otimes A_{\gamma_{k-1}} \otimes A_{\gamma_k}^{\mathrm{prim}}.$$

The following are obviously equivalent:

(i) $g \notin \widetilde{J_\gamma}$;

(ii) for some $\nu \in \{0, \ldots, N\}$, $g_\nu \notin \widetilde{J_\gamma}$.

Now suppose that $g \notin \widetilde{J_\gamma}$. We need to show that

$$(x_{j,\alpha_2}^{(k)} - x_{i,\alpha_1}^{(p)}) g \notin \widetilde{J_\gamma}. \tag{3.3}$$

We may assume that $g_N \notin \widetilde{J_\gamma}$. Put

$$x_{av}^{(k)} := \frac{1}{\sum_{i \in I} \gamma_k^i} \sum_{i,\alpha} x_{i,\alpha}^{(k)} = \frac{1}{\sum_{i \in I} \gamma_k^i} \sigma_{\gamma_k}.$$

Then $x_{j,\alpha_2}^{(k)} - x_{av}^{(k)} \in A_{\gamma_k}^{\mathrm{prim}}$, and we have

$$(x_{j,\alpha_2}^{(k)} - x_{i,\alpha_1}^{(p)}) g = (x_{j,\alpha_2}^{(k)} - x_{av}^{(k)} - x_{i,\alpha_1}^{(p)}) g + x_{av}^{(k)} g = \frac{1}{\sum_{i \in I} \gamma_k^i} g_N \sigma_{\gamma_k}^{N+1} + \sum_{\nu=0}^{N} g'_\nu \sigma_{\gamma_k}^{\nu}$$

for some $g'_\nu \in A_{\gamma_1} \otimes \cdots \otimes A_{\gamma_{k-1}} \otimes A_{\gamma_k}^{\mathrm{prim}}$. Since $(1/\sum_{i \in I} \gamma_k^i) g_N \notin \widetilde{J_\gamma}$ by our assumption, this implies (3.3). The claim is proved. □

We put

$$\widetilde{A_\gamma}' := \widetilde{A_\gamma}[(x_{j,\alpha_2}^{(q)} - x_{i,\alpha_1}^{(p)})^{-1}_{1 \leqslant p < q \leqslant k}], \quad \widetilde{\mathcal{H}_\gamma}' := (\widetilde{A_\gamma}')^{\widetilde{P_\gamma}}.$$

We denote by the same letter L the localization maps $L : \widetilde{A_\gamma} \to \widetilde{A_\gamma}'$, $L : \widetilde{\mathcal{H}_\gamma} \to \widetilde{\mathcal{H}_\gamma}'$. Also put $\widetilde{J_\gamma}' := \widetilde{A_\gamma}' L(\widetilde{J_\gamma})$. It follows directly from the claim that the induced maps

$$L : \widetilde{A_\gamma}/\widetilde{J_\gamma} \to \widetilde{A_\gamma}'/\widetilde{J_\gamma}', \quad L : \widetilde{\mathcal{H}_\gamma}/(\widetilde{J_\gamma})^{\widetilde{P_\gamma}} \to \widetilde{\mathcal{H}_\gamma}'/(\widetilde{J_\gamma}')^{P} \tag{3.4}$$

are injective.

Now, let $r \in \{1, \ldots, n\}$ be the maximal number such that the underlying sequence of elements in $\mathbb{Z}_{\geqslant 0}^I$ for t_r coincides with $(\gamma_1, \ldots, \gamma_k)$. Then it is straightforward to check that

$$L\iota(M_{t_l}) \in (\widetilde{J_\gamma}')^{\widetilde{P_\gamma}} \quad \text{for } r+1 \leqslant l \leqslant n.$$

Thus, it suffices to show that

$$L\iota\left(\sum_{i=1}^{r} \lambda_i M_{t_i}\right) \notin (\widetilde{J_\gamma}')^{\widetilde{P_\gamma}}. \tag{3.5}$$

For all relevant β_i, m_i we have the following comparison:

$$\begin{aligned} &L\iota(e_{\gamma_1,\beta_1}\sigma_{\gamma_1}^{m_1} \star \cdots \star e_{\gamma_k,\beta_k}\sigma_{\gamma_k}^{m_k}) \\ &\quad \equiv F_{\gamma_1,\ldots,\gamma_k} \cdot \sum_{\tau} s(\tau) e_{\gamma_1,\beta_{\tau(1)}} \sigma_{\gamma_1}^{m_{\tau(1)}} \otimes \cdots \otimes e_{\gamma_k,\beta_{\tau(k)}} \sigma_{\gamma_k}^{m_{\tau(k)}} \bmod (\widetilde{J_\gamma}')^{\widetilde{P_\gamma}}, \end{aligned} \tag{3.6}$$

where the sum is taken over all permutations $\tau \in S_k$ such that $\gamma_p = \gamma_{\tau(p)}$ for all $p \in \{1, \ldots, k\}$, and $s(\tau)$ is the Koszul sign (recall that the parity of $e_{\gamma,\beta}\sigma_\gamma^k$ equals $\epsilon(\gamma)$), and $F_{\gamma_1,\ldots,\gamma_k} \in \widetilde{\mathcal{H}_\gamma}'$ is (up to sign) the product of some powers (positive and (-1)st) of the differences

$$(x_{j,\alpha_2}^{(q)} - x_{i,\alpha_1}^{(p)}) \in \widetilde{A_\gamma}, \quad 1 \leqslant p < q \leqslant k.$$

Thus, $F_{\gamma_1,\ldots,\gamma_k}$ is invertible, and, according to (3.6) and injectivity of the maps (3.4), we are left to check that

$$\sum_\tau s(\tau) e_{\gamma_1,\beta_{\tau(1)}} \sigma_{\gamma_1}^{m_{\tau(1)}} \otimes \cdots \otimes e_{\gamma_k,\beta_{\tau(k)}} \sigma_{\gamma_k}^{m_{\tau(k)}} \notin \widetilde{J_\gamma}^{\widetilde{P_\gamma}}.$$

However, this follows from the condition (2) in the above definition of the set of sequences Seq_γ, and from the definition of $e_{\gamma_i,\beta}$. This proves (3.3), hence the desired linear independence (3.2), and hence free generation. The lemma is proved. □

The theorem is proved. □

It is clear that if $V_{\gamma,k}^{\mathrm{prim}} \neq 0$ in the notation of the above theorem, then $k \equiv \chi_Q(\gamma,\gamma) \bmod 2$ and $k \geqslant \chi_Q(\gamma,\gamma)$. Our next result is an upper bound on k (depending on γ) for which $V_{\gamma,k} \neq 0$.

For a given symmetric quiver Q and $\gamma \in \mathbb{Z}_{\geqslant 0}^I \backslash \{0\}$, we put

$$N_\gamma(Q) := \frac{1}{2}\Bigg(\sum_{\substack{i,j \in I,\\ i \neq j}} a_{ij}\gamma^i\gamma^j + \sum_{i \in I} \max(a_{ii}-1,0)\gamma^i(\gamma^i-1)\Bigg) - \sum_{i \in I}\gamma^i + 2.$$

Theorem 3.10. *In the notation of Theorem 3.1, if $V_{\gamma,k}^{\mathrm{prim}} \neq 0$, then $\gamma \neq 0$,*

$$k \equiv \chi_Q(\gamma,\gamma) \bmod 2 \quad \textit{and} \quad \chi_Q(\gamma,\gamma) \leqslant k < \chi_Q(\gamma,\gamma) + 2N_\gamma(Q).$$

Proof. According to the proof of Theorem 3.1, we have

$$\dim V_{\gamma,k}^{\mathrm{prim}} = \dim(\mathcal{H}_\gamma^{\mathrm{prim}}/J_\gamma^{P_\gamma})^{k-\chi_Q(\gamma,\gamma)}. \tag{3.7}$$

Recall that $P_\gamma = \prod_{i \in I} S_{\gamma^i}$,

$$A_\gamma^{\mathrm{prim}} := \mathbb{Q}[(x_{j,\alpha_2} - x_{i,\alpha_1})_{i,j \in I, 1 \leqslant \alpha_1 \leqslant \gamma^i, 1 \leqslant \alpha_2 \leqslant \gamma^j}], \quad \mathcal{H}_\gamma^{\mathrm{prim}} := (A_\gamma^{\mathrm{prim}})^{P_\gamma},$$

and J_γ is the smallest P_γ-stable A_γ^{prim}-submodule of the localization

$$A_\gamma^{\mathrm{prim}}[(x_{i,\alpha_2} - x_{i,\alpha_1})^{-1}_{i \in I, 1 \leqslant \alpha_1 < \alpha_2 \leqslant \gamma^i}],$$

such that for all decompositions $\gamma = \gamma_1 + \gamma_2$, $\gamma_1, \gamma_2 \in \mathbb{Z}_{\geqslant 0}^I \backslash \{0\}$, we have that

$$\frac{\prod_{i,j \in I} \prod_{\alpha_1=1}^{\gamma_1^i} \prod_{\alpha_2=\gamma_1^j+1}^{\gamma^j} (x_{j,\alpha_2} - x_{i,\alpha_1})^{a_{ij}}}{\prod_{i \in I} \prod_{\alpha_1=1}^{\gamma_1^i} \prod_{\alpha_2=\gamma_1^i+1}^{\gamma^i} (x_{i,\alpha_2} - x_{i,\alpha_1})} \in J_\gamma. \tag{3.8}$$

Recall that we take the standard even grading on A_γ^{prim} with $\deg(x_{j,\alpha_2} - x_{i,\alpha_1}) = 2$, and the induced grading on $\mathcal{H}_\gamma^{\mathrm{prim}}$.

According to (3.7), it suffices to prove inclusions

$$(A_\gamma^{\mathrm{prim}})^d \subset J_\gamma \quad \text{for } d \geqslant 2N(Q). \tag{3.9}$$

For any $i, j \in I$, put

$$a'_{ij} := \begin{cases} a_{ij} & \text{if } i \neq j, \\ \max(1, a_{ii}) & \text{if } i = j. \end{cases}$$

Take the quiver $Q' := (I, a'_{ij})$. Note that $N_\gamma(Q) = N_\gamma(Q')$, and if we replace Q by Q' then the new fractional J_γ will be contained in the initial one. Hence, in order to prove inclusions (3.9), we may and will assume that $a_{ii} \geqslant 1$ for $i \in I$, and so $J_\gamma \subset A_\gamma^{\mathrm{prim}}$. We will deduce (3.9) from the following more general result.

Lemma 3.11. *Let* k *be an arbitrary field, and consider the graded algebra of polynomials* $B = \mathrm{k}[z_1, \ldots, z_n]$, $n \geqslant 1$, *with grading* $\deg(z_i) = 1$. *Suppose that* $l_1, \ldots, l_s \in B^1$ *are pairwise linearly independent non-zero linear forms in* z_i. *Take some non-empty set of polynomials* $\{P_1, \ldots, P_r\} \subset B$ *of the form*

$$P_i = l_1^{d_{i1}} \cdots l_s^{d_{is}},$$

where $d_{ij} \in \mathbb{Z}_{\geqslant 0}$. *Put* $d_j := \max_{1 \leqslant i \leqslant r} d_{ij}$, $1 \leqslant j \leqslant s$. *Then the following are equivalent.*

(i) $B^d \subset (P_1, \ldots, P_r)$ *for* $d \geqslant d_1 + \cdots + d_s - n + 1$.

(ii) *The ideal* $(P_1, \ldots, P_r) \subset B$ *has finite codimension.*

(iii) *For any sequence* $p_1, \ldots, p_r$ *of numbers in* $\{1, \ldots, s\}$, *such that* $d_{i,p_i} > 0$ *for* $1 \leqslant i \leqslant r$, *the linear forms* $l_{p_1}, \ldots, l_{p_r}$ *generate the space* B^1.

Proof. Both implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are evident. So we are left to prove implication (iii) $\Rightarrow$ (i).

Put $D := d_1 + \cdots + d_s - n + 1$. If $D \leqslant 0$, then one of the polynomials P_i is constant, and there is nothing to prove. So, we assume that $D > 0$.

We proceed by induction on $D + n$. If $D + n = 2$, then $n = s = d_1 = D = 1$, hence $(P_1, \ldots, P_r) \supset (z_1)$, and the statement is proved.

Assume that the implication holds for $D + n < k_0 > 2$. We will prove that it holds for $D + n = k_0$. Consider the following cases.

Case 0. One of P_i is constant. Then, there is nothing to prove.

Case 1. We have $P_i = l_j$ for some i, j. Then it suffices to show that the images of $P_{i'}$ with $d_{i'j} = 0$ in $B/(l_j)$ generate $(B/(l_j))^d$ for $d \geqslant D$. If $n = 1$ then this is clear, and if $n > 1$ then this follows from the induction hypothesis.

Case 2. All P_i have degree at least 2. Take $d \geqslant D$, and $f \in B^d$. Choose some sequence $p_1, \ldots, p_r$ of numbers in $\{1, \ldots, s\}$, such that $d_{i,p_i} > 0$ for $1 \leqslant i \leqslant r$. Then by statement (iii) we can write

$$f = \sum_{i=1}^{r} l_{p_i} g_i, \quad g_i \in B^{d-1}.$$

It suffices to show that for each $1 \leqslant i \leqslant r$, the polynomial g_i belongs to an ideal generated by $P_{i'}$ with $l_{p_i} \nmid P_{i'}$, and $P_{i''}/l_{p_i}$ with $l_{p_i} \mid P_{i''}$. However, this follows from the induction hypothesis.

In each case, we have proved the desired implication. The induction statement is proved. The lemma is proved. □

Now, consider the cases. If $\sum_i \gamma^i = 1$, then $N_\gamma(Q) = 1$, and $A_\gamma^{\mathrm{prim}} = \mathbb{Q}$, and hence inclusions (3.9) hold. Further, if $\sum_i \gamma^i \geqslant 2$, then we apply Lemma 3.11 to $B = A_\gamma^{\mathrm{prim}}$, the linear forms $(x_{j,\alpha_2} - x_{i,\alpha_1})$ (defined up to sign), and polynomials which are in the P_γ-orbit of the expressions (3.8). They generate precisely the ideal $J_\gamma \subset A_\gamma^{\mathrm{prim}}$. We have already shown in the

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proof of Theorem 3.1 that the ideal $J_\gamma \subset A_\gamma^{\mathrm{prim}}$ has finite codimension. Therefore, the implication (ii) $\Rightarrow$ (i) from Lemma 3.11 gives the desired inclusions (3.9). Indeed, we have that

$$d_1 + \cdots + d_s = \frac{1}{2}\Bigg(\sum_{\substack{i,j\in I,\\ i\neq j}} a_{ij}\gamma^i\gamma^j + \sum_{i\in I}(a_{ii}-1)\gamma^i(\gamma^i-1)\Bigg), \quad n = \sum_{i\in I}\gamma^i - 1,$$

and hence $N_\gamma(Q) = d_1 + \cdots + d_s - n + 1$. The inclusions (3.9) and the theorem are proved. $\square$

4. Applications to quantum DT invariants

Define the generating function for the COHA $\mathcal{H}$ of a symmetric quiver Q by the following formula:

$$H_Q(\{t_i\}_{i\in I}, q) := \sum_{\gamma\in\mathbb{Z}^I_{\geqslant 0}, k\in\mathbb{Z}} (-1)^k \dim(\mathcal{H}_{\gamma,k}) t^\gamma q^{k/2} \in \mathbb{Z}((q^{\frac{1}{2}}))[[\{t_i\}_{i\in I}]],$$

where $t^\gamma := \prod_{i\in I} t_i^{\gamma^i}$. Note that we have an equality

$$H_Q = \sum_{\gamma\in\mathbb{Z}^I_{\geqslant 0}} \frac{(-q^{\frac{1}{2}})^{\chi_Q(\gamma,\gamma)}}{\prod_{i\in I}(1-q)(1-q^2)\ldots(1-q^{\gamma^i})} t^\gamma. \tag{4.1}$$

Recall the notation

$$(z;q)_\infty := \prod_{n\in\mathbb{Z}_{\geqslant 0}} (1-q^n z)$$

(the so-called q-Pochhammer symbol).

COROLLARY 4.1. *Let Q be a symmetric quiver. Then we have a decomposition*

$$H_Q(\{t_i\}_{i\in I}, q) = \prod_{\gamma\in\mathbb{Z}^I_{\geqslant 0}, k\in\mathbb{Z}} (q^{k/2}x^\gamma; q)_\infty^{(-1)^{k-1}c_{\gamma,k}},$$

where $c_{\gamma,k}$ are non-negative integer numbers. Moreover, if $c_{\gamma,k}\neq 0$, then $\gamma\neq 0$,

$$k \equiv \chi_Q(\gamma,\gamma) \bmod 2 \quad \text{and} \quad \chi_Q(\gamma,\gamma) \leqslant k < \chi_Q(\gamma,\gamma) + 2N_\gamma(Q).$$

In particular, for a fixed γ only finitely many of $c_{\gamma,k}$ are non-zero.

Proof. The corollary follows immediately from Theorems 3.1 and 3.10 if we put $c_{\gamma,k} = \dim V^{\mathrm{prim}}_{\gamma,k}$. Indeed, the generating function of the free super-commutative subalgebra generated by one element of bidegree (γ,k) equals

$$(1-q^{k/2}t^\gamma)^{(-1)^{k-1}}.$$

The resulting decomposition follows from free generation of $\mathcal{H}$ by V, and from Theorem 3.10. $\square$

In the notation of Corollary 4.1 and the terminology of [KS11], the polynomials

$$\Omega(\gamma)(q) := \sum_{k\in\mathbb{Z}} c_{\gamma,k} q^{k/2} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$$

are quantum Donaldson–Thomas invariants of the quiver Q with trivial potential, trivial stability, and the dimension vector γ. It follows from Corollary 4.1 that for $\gamma\neq 0$ we have

$$\Omega(\gamma)(q) = q^{\frac{1}{2}\chi_Q(\gamma,\gamma)}\tilde{\Omega}(\gamma)(q),$$

where $\tilde{\Omega}(\gamma)(q)$ is a polynomial with non-negative coefficients, $\tilde{\Omega}(\gamma)(0) = 1$, and $\deg(\tilde{\Omega}(\gamma)(q)) < N_\gamma(Q)$.

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We would like to mention a connection with the paper of Reineke [Rei11]. In that paper, for each integer $m \geqslant 1$, the following q-hypergeometric series is considered:

$$H(q,t) = H_m(q,t) := \sum_{n \geqslant 0} \frac{q^{(m-1)\binom{n}{2}}}{(1-q^{-1})(1-q^{-2})\cdots(1-q^{-n})} t^n \in \mathbb{Z}(q)[[t]].$$

Denote by Q_m the m-loop quiver (a quiver with one vertex and m loops). Since $\chi_{Q_m}(n_1,n_2) = (1-m)n_1n_2$, the formula (4.1) implies

$$H_m(q,t) = H_{Q_m}((-1)^{m-1}tq^{(1-m)/2}, q^{-1}).$$

Also, we have $N_n(Q_m) = (m-1)\binom{n}{2} - n + 2$. Therefore, Corollary 4.1 implies the following corollary.

Corollary 4.2.

$$H_m(q,(-1)^{m-1}t) = \prod_{n \geqslant 1, k \in \mathbb{Z}} (q^k t^n; q^{-1})^{-(-1)^{(m-1)n} d_{n,k}},$$

where $d_{n,k}$ are non-negative integers, and the inequality $d_{n,k} > 0$ implies

$$n-1 \leqslant k \leqslant (m-1)\binom{n}{2}.$$

In particular, for a fixed n only finitely many of $d_{n,k}$ are non-zero.

This corollary is stronger than Conjecture 3.3 in [Rei11]. According to the notation of [Rei11], the quantized Donaldson–Thomas type invariant $DT_n^{(m)}(q)$ equals $\sum_{k \in \mathbb{Z}} d_{n,k} q^k$. Thus, Corollary 4.2 implies that $DT_n^{(m)}(q)$ is a monic polynomial of degree $(m-1)\binom{n}{2}$, divisible by q^{n-1}, with non-negative coefficients.

With the above said, the numbers $d_{n,k}$ are the dimensions of graded components of the finite-dimensional graded algebras $\mathcal{H}_n^{\mathrm{prim}}/J_n^{S_n}$. It would be interesting to compare this interpretation with the explicit formulas for $DT_n^{(m)}(q)$ in [Rei11, Theorem 6.8].

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