On the Solution of Differential Equations by Definite Integrals

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(*Received* 12*th December* 1930. *Read* 16*th January* 1931.)

§ 1. *Introduction.*

It is well known that in many cases the solutions of a linear differential equation can be expressed as definite integrals, different solutions of the same equation being represented by integrals which have the same integrand, but different paths of integration. Thus, the various solutions of the hypergeometric differential equation

$$z(z-1)\frac{d^2 y}{dz^2} + \{-c + (a+b+1)z\}\frac{dy}{dz} + aby = 0$$

can be represented by integrals of the type

$$\int t^{a-c}(t-1)^{c-b-1}(t-z)^{-a}\,dt,$$

the path of integration being (for one particular solution) a closed circuit encircling the point $t = 0$ in the positive direction, then the point $t = 1$ in the positive direction, then the point $t = 0$ in the negative direction, and lastly the point $t = 1$ in the negative direction; or (for another particular solution) an arc in the t-plane joining the points $t = 1$ and $t = \infty$.

The object of the present paper is to communicate a general theorem regarding the solution of both ordinary and partial differential equations by means of families of definite integrals. The theorem gives only the integrand of the integral: but when in any particular case the integrand is known, there is usually not much difficulty in determining a double-circuit or other path of integration which will give any desired solution.

In § 2 the theorem is stated in its general form. The rest of the paper is devoted to working out particular cases of it, the results being believed to be in most cases new.

§ 2. *The general theorem.*

The general theorem may be stated as follows:—

Consider any contact-transformation from a set of variables $(q_1, q_2, .. q_n, p_1, p_2, .. p_n)$ *to a set of variables* $(Q_1, Q_2, .. Q_n, P_1, P_2, .. P_n)$. As is

https://doi.org/10.1017/S0013091500007768 Published online by Cambridge University Press

known,[1] we can obtain a contact-transformation by taking any function $W(q_1, q_2, \ldots q_n, Q_1, Q_2, \ldots Q_n)$ of the q's and Q's, and writing down the equations

$$P_r = \frac{\partial W}{\partial Q_r}, \qquad p_r = -\frac{\partial W}{\partial q_r} \qquad (r = 1, 2, \ldots n) \tag{1}$$

and solving them with respect to the Q's and P's. *Suppose that the variables* $(Q_1, Q_2, \ldots Q_n, P_1, P_2, \ldots P_n)$, *when expressed in terms of* $(q_1, q_2, \ldots q_n, p_1, p_2, \ldots p_n)$, *are denoted by* $Q_r(q_1, q_2, \ldots q_n, p_1, p_2 \ldots p_n)$, $P_r(q_1, q_2, \ldots q_n, p_1, p_2, \ldots p_n)$ $(r = 1, 2, \ldots n)$. *Then the set of partial differential equations*

$$\left.\begin{aligned} Q_r\left(q_1, q_2, \ldots q_n, \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \ldots \frac{\partial}{\partial q_n}\right)\chi &= t_r\,\chi \\ P_r\left(q_1, q_2, \ldots q_n, \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \ldots \frac{\partial}{\partial q_n}\right)\chi &= -\frac{\partial \chi}{\partial t_r} \end{aligned}\right\} (r = 1, 2, \ldots n) \tag{2}$$

is a compatible set of differential equations, and possesses a solution $\chi(q_1, q_2, \ldots q_n, t_1, t_2, \ldots t_n)$. It should be observed here that the equations (2) are symbolic: that is to say, $\left(\frac{\partial}{\partial q_r}\right)^2 \chi$ is to be taken to mean $\left(\frac{\partial^2 \chi}{\partial q_r^2}\right)$, etc.: and it should also be observed that certain rules are to be observed in replacing the p's by the $\frac{\partial}{\partial q}$'s in Q_r and P_r, in order to settle *e.g.* whether a term $p_1 q_1$ is to become $q_1 \frac{\partial}{\partial q_1}$ or $\frac{\partial}{\partial q_1} . q_1$: but this matter we defer for the present.

Now let it be required to solve a set of n *compatible linear partial differential equations in* n *independent variables, say*

$$F_r\left(q_1, q_2, \ldots q_n, \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \ldots \frac{\partial}{\partial q_n}\right)\psi = 0, \qquad (r = 1, 2, \ldots n) \tag{3}$$

where $\psi(q_1, q_2, \ldots q_n)$ *is the function to be determined. Suppose that when the* q'*s and* p'*s are replaced by their values in terms of the* Q'*s and* P'*s from* (1), *the function* $F_r(q_1, q_2, \ldots q_n, p_1, p_2, \ldots p_n)$ *becomes* $G_r(Q_1, Q_2, \ldots Q_n, P_1, P_2, \ldots P_n)$. *Then the solution of the partial differential equations* (3) *is furnished by a family of definite integrals of the type*

$$\psi(q_1, q_2, \ldots q_n) = \iiint \ldots \int \chi(q_1, q_2, \ldots q_n, t_1, t_2, \ldots t_n)\,\phi(t_1, t_2, \ldots t_n)\,dt_1\,dt_2 \ldots dt_n \tag{4}$$

[1] *Cf. e.g.* Whittaker, *Analytical Dynamics*, § 126.

when $\phi(t_1, t_2, \ldots t_n)$ *is the solution of the set of* n *compatible linear partial differential equations*

$$G_r\left(t_1, t_2, \ldots t_n \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots \frac{\partial}{\partial t_n}\right)\phi = 0 \qquad (r = 1, 2, \ldots n).$$

My own original way of establishing the theorem was defective as a proof, and has been superseded by a proof devised by Dr Kermack and Dr M'Crea, which is given in the paper following this. I will therefore not discuss the proof here, but confine myself to working out particular cases, chiefly with a view to obtaining new results.

§ 3. *An extension of Laplace's transformation.*

As a first example, let us suppose that the contact-transformation is specified by the equations

$$P_r = \frac{\partial W}{\partial Q_r}, \qquad p_r = -\frac{\partial W}{\partial q_r} \qquad (r = 1, 2, \ldots n)$$

when

$$W = -q_1 Q_1 - q_2 Q_2 - \ldots - q_n Q_n,$$

so that

$$P_r = -q_r, \qquad p_r = Q_r \qquad (r = 1, 2, \ldots n)$$

The equations (2) for the function χ become

$$\frac{\partial \chi}{\partial q_r} = t_r \chi, \qquad \frac{\partial \chi}{\partial t_r} = q_r \chi \qquad (r = 1, 2 \ldots n)$$

of which the solution is

$$\chi = e^{q_1 t_1 + q_2 t_2 + \ldots + q_n t_n}$$

(neglecting now, as always, an arbitrary multiplicative constant). Thus the theorem asserts that *if* $\psi(q_1, q_2, \ldots q_n)$ *is a solution of* n *compatible partial differential equations*

$$F_r\left(q_1, q_2, \ldots q_r, \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \ldots \frac{\partial}{\partial q_n}\right)\psi = 0, \qquad (r = 1, 2, \ldots n) \tag{5}$$

then a relation exists of the type

$$\psi(q_1, q_2, \ldots q_n) = \iint \ldots \int e^{q_1 t_1 + q_2 t_2 + \ldots + q_n t_n} \phi(t_1, t_2, \ldots t_n)\, dt_1\, dt_2 \ldots dt_n \tag{6}$$

where $\phi(t_1, t_2, \ldots t_n)$ *is a solution of the set of compatible partial differential equations*

$$F_r\left(-\frac{\partial}{\partial t_1}, -\frac{\partial}{\partial t_2}, \ldots -\frac{\partial}{\partial t_n}, t_1, t_2, \ldots t_n\right)\phi = 0. \tag{7}$$

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It will be seen that this is essentially the extension, to partial differential equations in n variables, of the well-known method,[1] due to Laplace, for solving ordinary linear differential equations by means of integrals of the form

$$\int e^{qt}\,\phi(t)\,dt.$$

As a particular case, suppose that the equations (5) are

$$\left.\begin{array}{l}\left(\dfrac{\partial}{\partial q_1} - a_{11}q_1 - a_{12}q_2 - \ldots\ldots - a_{1n}q_n\right)\psi = 0\\ \left(\dfrac{\partial}{\partial q_2} - a_{21}q_1 - a_{22}q_2 - \ldots\ldots - a_{2n}q_n\right)\psi = 0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \left(\dfrac{\partial}{\partial q_n} - a_{n1}q_1 - a_{n2}q_2 - \ldots\ldots - a_{nn}q_n\right)\psi = 0\end{array}\right\}$$

where the a's are constants, and $a_{ik} = a_{ki}$. These partial differential equations are compatible, the solution being evidently

$$\psi(q_1, q_2, \ldots q_n) = e^{\frac{1}{2}(a_{11}q_1^2 + a_{22}q_2^2 + \ldots + a_{nn}q_n^2 + 2a_{12}q_1q_2 + \ldots)}.$$

The corresponding equations (7) for ϕ are

$$\left.\begin{array}{l}\left(t_1 + a_{11}\dfrac{\partial}{\partial t_1} + a_{12}\dfrac{\partial}{\partial t_2} + \ldots\ldots + a_{1n}\dfrac{\partial}{\partial t_n}\right)\phi = 0\\ \left(t_2 + a_{21}\dfrac{\partial}{\partial t_1} + a_{22}\dfrac{\partial}{\partial t_2} + \ldots\ldots + a_{2n}\dfrac{\partial}{\partial t_n}\right)\phi = 0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \left(t_n + a_{n1}\dfrac{\partial}{\partial t_1} + a_{n2}\dfrac{\partial}{\partial t_2} + \ldots\ldots + a_{nn}\dfrac{\partial}{\partial t_n}\right)\phi = 0\end{array}\right\}$$

If we denote the determinant $\|a_{ik}\|$ by Δ, and denote $(1/\Delta)\times$ the co-factor of a_{ik} in Δ by A_{ik}, so that the quadratic form $\sum\limits_{ik} A_{ik}x_i x_k$ is the reciprocal of the quadratic form $\sum\limits_{ik} a_{ik}x_i x_k$, then the last set of equations may be written

$$\left.\begin{array}{l}\left(\dfrac{\partial}{\partial t_1} + A_{11}t_1 + A_{12}t_2 + \ldots\ldots + A_{1n}t_n\right)\phi = 0\\ \left(\dfrac{\partial}{\partial t_2} + A_{21}t_1 + A_{22}t_2 + \ldots\ldots + A_{2n}t_n\right)\phi = 0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \left(\dfrac{\partial}{\partial t_n} + A_{n1}t_1 + A_{n2}t_2 + \ldots\ldots + A_{nn}t_n\right)\phi = 0\end{array}\right\}$$

[1] *Cf.*, *e.g.*, A. R. Forsyth, *Treatise on Differential Equations*, Chapter VII., "Solution by Definite Integrals."

and they are compatible, their solution being

$$\phi = e^{-\frac{1}{2}(A_{11}t_1^2 + A_{22}t_2^2 + \ldots + 2A_{12}t_1t_2 + \ldots)}$$

Thus the theorem asserts that a relation exists of the form

$$\exp\left(\tfrac{1}{2}\sum_{i,k} a_{ik}q_iq_k\right) = \text{constant} \times \iint \ldots \int \exp(q_1t_1 + q_2t_2 + \ldots + q_nt_n) \exp\left(-\tfrac{1}{2}\sum_{i,k} A_{ik}t_it_k\right)dt_1\,dt_2\ldots dt_n,$$

where the paths of integration may consist of loops surrounding the singularities of the integrand, or paths to infinity. Since the integrand has no finite singularities, we infer that the paths must terminate at infinity: and in fact we readily find that *the precise equation is*

$$\exp\left(\tfrac{1}{2}\sum_{i,k} a_{ik}\,q_i\,q_k\right) = (2\pi)^{-\frac{1}{2}n}\,\Delta^{-\frac{1}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\exp\left(q_1t_1 + q_2t_2 + \ldots + q_nt_n - \tfrac{1}{2}\sum_{i,k}A_{ik}t_it_k\right)dt_1\,dt_2\ldots dt_n,$$

which represents the result of the theorem in this particular case. We can represent the expression $\sum_{i,k} A_{ik}t_it_k$ in terms of the a's in the form

$$-\frac{1}{\Delta}\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} & t_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & t_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{2n} & t_n \\ t_1 & t_2 & \ldots & t_n & 0 \end{vmatrix}.$$

§ 4. *A double-integral connected with the hypergeometic function.*

As a second example, suppose that the contact-transformation is specified by the function

$$W = -q_1\log Q_1 - q_2\log Q_2,$$

so that

$$q_1 = -Q_1P_1, \quad q_2 = -Q_2P_2, \quad p_1 = \log Q_1, \quad p_2 = \log Q_2,$$

or $\quad Q_1 = e^{p_1}, \quad Q_2 = e^{p_2}, \quad P_1 = -e^{-p_1}q_1, \quad P_2 = -e^{-p_2}q_2.$

The equations (2) for determining χ are now

$$\left\{\exp\left(\frac{\partial}{\partial q_1}\right)\right\}\chi = t_1\chi, \qquad \text{or} \quad \chi(q_1+1, q_2) = t_1\chi(q_1, q_2)$$

$$\left\{\exp\left(\frac{\partial}{\partial q_2}\right)\right\}\chi = t_2\chi, \qquad \text{or} \quad \chi(q_1, q_2+1) = t_2\chi(q_1, q_2)$$

$$-\left\{\exp\left(-\frac{\partial}{\partial q_1}\right)\right\}q_1\chi = -\frac{\partial\chi}{\partial t_1}, \quad \text{or} \quad (q_1-1)\chi(q_1-1, q_2) = \frac{\partial\chi(q_1, q_2)}{\partial t_1}$$

$$-\left\{\exp\left(-\frac{\partial}{\partial q_2}\right)\right\}q_2\chi = -\frac{\partial\chi}{\partial t_2}, \quad \text{or} \quad (q_2-1)\chi(q_1, q_2-1) = \frac{\partial\chi(q_1, q_2)}{\partial t_2}$$

and the solution of these equations is

$$\chi = t_1^{q_1-1}t_2^{q_2-1}.$$

Thus the theorem asserts that *the solution of a pair of compatible partial differential equations*

$$\begin{cases} F_1\left(q_1, q_2, \dfrac{\partial}{\partial q_1}, \dfrac{\partial}{\partial q_2}\right)\psi = 0 \\ F_2\left(q_1, q_2, \dfrac{\partial}{\partial q_1}, \dfrac{\partial}{\partial q_2}\right)\psi = 0 \end{cases}$$

may be expressed by a definite integral

$$\psi(q_1, q_2) = \iint \phi(t_1, t_2)\, t_1^{q_1 - 1} t_2^{q_2 - 1}\, dt_1\, dt_2, \tag{8}$$

where ϕ is a solution of the equations

$$\left.\begin{aligned} F_1\left(-t_1\frac{\partial}{\partial t_1}, -t_2\frac{\partial}{\partial t_2}, \log t_1, \log t_2\right)\phi = 0, \\ F_2\left(-t_1\frac{\partial}{\partial t_1}, -t_2\frac{\partial}{\partial t_2}, \log t_1, \log t_2\right)\phi = 0. \end{aligned}\right\} \tag{9}$$

As a particular case, take

$$\psi(q_1, q_2) = \Gamma(q_1)\,\Gamma(q_2)\,F(q_1, q_2, \tfrac{1}{2}, z), \tag{10}$$

where $F(a, b, c, z)$ denotes the hypergeometric function. From the recurrence-formulae for the hypergeometric function we see that ψ satisfies the two difference-equations

$$\begin{aligned} &\psi(q_1, q_2 + 1) - \psi(q_1 + 1, q_2) + (q_1 - q_2)\psi(q_1, q_2) = 0 \\ &\{\tfrac{1}{2} - 2q_1 - (q_2 - q_1)z\}\psi(q_1, q_2) + (1 - z)\psi(q_1 + 1, q_2) \\ &\qquad - (\tfrac{1}{2} - q_1)(q_1 - 1)\psi(q_1 - 1, q_2) = 0, \end{aligned}$$

which may be written

$$\begin{cases} (e^{\partial/\partial q_2} - e^{\partial/\partial q_1} + q_1 - q_2)\psi = 0 \\ \{\tfrac{1}{2} - 2q_1 - (q_2 - q_1)z + (1 - z)e^{\partial/\partial q_1} - (\tfrac{1}{2} - q_1)(q_1 - 1)e^{-\partial/\partial q_1}\}\psi = 0, \end{cases}$$

so the corresponding equations (9) for ϕ are

$$\left(t_2 - t_1 - t_1\frac{\partial}{\partial t_1} + t_2\frac{\partial}{\partial t_2}\right)\phi = 0 \tag{11}$$

$$\left\{\tfrac{1}{2} + 2t_1\frac{\partial}{\partial t_1} + zt_2\frac{\partial}{\partial t_2} - zt_1\frac{\partial}{\partial t_1} + (1 - z)t_1 + \left(\tfrac{1}{2} + t_1\frac{\partial}{\partial t_1}\right)\left(t_1\frac{\partial}{\partial t_1} + 1\right)\left(\frac{1}{t_1}\right)\right\}\phi = 0 \tag{12}$$

The solution to equation (11) is

$$\phi = e^{-t_1 - t_2} f(s), \tag{13}$$

where $s = t_1 t_2$, and f denotes an arbitrary function.

Equation (12) may be written

$$\tfrac{1}{2}\phi + 2t_1\frac{\partial\phi}{\partial t_1} + zt_2\frac{\partial\phi}{\partial t_2} - zt_1\frac{\partial\phi}{\partial t_1} + (1-z)\,t_1\,\phi + \tfrac{1}{2}\frac{\partial\phi}{\partial t_1} + t_1\frac{\partial^2\phi}{\partial t_1^2} = 0.$$

Substituting in this the value of ϕ given by (13), it becomes

$$s\,\frac{d^2 f}{ds^2} + \tfrac{1}{2}\frac{df}{ds} - zf = 0,$$

of which the solution is

$$f = \cosh\left(2\sqrt{zt_1\,t_2} + \epsilon\right),$$

where ϵ denotes a constant. The theorem (8) therefore gives

$$\Gamma(q_1)\,\Gamma(q_2)\,F(q_1, q_2, \tfrac{1}{2}, z) =$$
$$\text{Constant.}\iint e^{-t_1-t_2}\cosh\left(2\sqrt{t_1\,t_2\,z} + \epsilon\right) t_1^{q_1-1}\,t_2^{q_2-1}\,dt_1\,dt_2;$$

it is not difficult to determine the paths of integration and the constants, *the final result being*

$$\Gamma(q_1)\,\Gamma(q_2)\,F(q_1, q_2, \tfrac{1}{2}, z) = \int_0^\infty\int_0^\infty e^{-t_1-t_2}\cosh\left(2\sqrt{t_1\,t_2\,z}\right) t_1^{q_1-1}\,t_2^{q_2-1}\,dt_1\,dt_2.$$

§ 5. *A new property of the hypergeometric function.*

It is well-known[1] from the work of Pincherle and Mellin that the hypergeometric function can be represented by definite integrals in which the integrand is a product of Gamma-functions, *e.g.*

$$F(\alpha, \beta, \gamma, 1-q) = \frac{1}{2\pi i}\,\frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\beta)\,\Gamma(\gamma-\alpha)} \times$$
$$\int_C \Gamma(-t)\,\Gamma(\alpha+t)\,\Gamma(\beta+t)\,\Gamma(\gamma-\alpha-\beta-t)\,q^t\,dt, \qquad (14)$$

where the path of integration C is parallel to the imaginary axis in the t-plane and passes between the sequences of positive and negative poles of the integrand. It will now be shown that these Pincherle-Mellin integrals are particular cases of a much more general formula which expresses a new property of the hypergeometric function.

[1] *Cf. e.g.* Whittaker and Watson, *Modern Analysis*, § 14.5.

For this purpose, we consider the contact-transformation defined by the function

$$W = -Q \log(-q)$$

so that the equations

$$P = \frac{\partial W}{\partial Q}, \qquad p = -\frac{\partial W}{\partial q},$$

give

$$Q = qp, \qquad P = -\log(-q).$$

The equations (2) for χ are now

$$q\frac{\partial \chi}{\partial q} = t\chi, \qquad -\log(-q)\,\chi = -\frac{\partial \chi}{\partial t},$$

giving

$$\chi = (-q)^t.$$

Thus the theorem (4) asserts that *any linear differential equation*

$$F\left(q, \frac{d}{dq}\right)\psi = 0 \tag{15}$$

is satisfied by a family of definite integrals of the type

$$\psi(q) = \int (-q)^t \phi(t)\,dt, \tag{16}$$

where $\phi(t)$ is a solution of the differential equation

$$F\left(-e^{-\frac{d}{dt}}, \; -e^{\frac{d}{dt}}.\,t\right)\phi = 0. \tag{17}$$

In particular, let

$$\psi(q) = F(\alpha, \beta, \gamma, z+q),$$

where F now denotes the hypergeometric function, so that ψ satisfies the differential equation (corresponding to (15) above)

$$\left[\left\{(z+q)\frac{d}{dq} + \alpha\right\}\left\{(z+q)\frac{d}{dq} + \beta\right\} - \left\{(z+q)\frac{d}{dq} + \gamma\right\}\frac{d}{dq}\right]\psi = 0.$$

The equation for $\phi(t)$, corresponding to (17), is therefore

$$\{(-ze^{\frac{d}{dt}}t + t + \alpha)(-ze^{\frac{d}{dt}}t + t + \beta) + (-ze^{\frac{d}{dt}}t + t + \gamma)\,e^{\frac{d}{dt}}t\}\,\phi = 0$$

or

$$z(z-1)(t+1)(t+2)\,\phi(t+2)$$
$$+ (t+1)\{\gamma + t - (\lambda + \beta + 2t + 1)z\}\,\phi(t+1) + (\alpha + t)(\beta + t)\,\phi(t) = 0,$$

a difference-equation whose solution is found (by comparing it with the recurrence-formulae for the hypergeometric function) to be

$$\phi(t) = \frac{\Gamma(-t)\,\Gamma(\alpha+t)\,\Gamma(\beta+t)}{\Gamma(\gamma+t)} F(\alpha+t, \beta+t, \gamma+t, z).$$

Thus the equation (16) becomes

$$F(\alpha, \beta, \gamma, z+q) = \text{Constant} \times \int \frac{\Gamma(-t)\,\Gamma(\alpha+t)\,\Gamma(\beta+t)}{\Gamma(\gamma+t)} F(\alpha+t, \beta+t, \gamma+t, z)\,(-q)^t\,dt.$$

It is readily seen that the path of integration must be the same as for the Pincherle-Mellin integral (14), and in fact the complete formula is

$$F(\alpha, \beta, \gamma, z+q) = \frac{1}{2\pi i}\,\frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)} \times$$

$$\int_C \frac{\Gamma(-t)\,\Gamma(\alpha+t)\,\Gamma(\beta+t)}{\Gamma(\gamma+t)} F(\alpha+t, \beta+t, \gamma+t, z)\,(-q)^t\,dt. \qquad (18)$$

This formula expresses the general hypergeometric function of the sum of two arguments, z and q, in terms of hypergeometric functions of the argument z. The Pincherle-Mellin formula (14) is evidently the special case of it which arises when $z = 1$, so that the hypergeometric function in the integrand reduces to a product of Gamma-functions; but in this special case the most characteristic property of the more general formula (18) is lost.

§ 6. *The linear differential equation with four regular singularities.*

It is well-known[1] that the linear differential equation of the second order with three regular singularities, that is, the equation of the general hypergeometric function or Riemann P-function

$$P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$$

can be solved by means of definite integrals of the type

$$\int (t-a)^p\,(t-b)^q\,(t-c)^r\,(z-t)^\nu\,dt.$$

It is therefore natural to enquire whether the linear differential equation of the second order with four regular singularities, that is the equation of the function

$$P\left\{\begin{matrix} a & b & c & e & \\ \alpha & \beta & \gamma & \epsilon & z \\ \alpha' & \beta' & \gamma' & \epsilon' & \end{matrix}\right\}$$

can be solved by means of definite integrals of the type

$$\int (t-a)^p\,(t-b)^q\,(t-c)^r\,(t-e)^s\,(z-t)^\nu\,dt.$$

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[1] *Cf. e.g.*, Whittaker and Watson, *Modern Analysis*, § 14.6.

The answer is in the negative: in fact definite integrals of this last type do occur[1] in the solution of linear differential equations with four singularities, but *these linear differential equations are of the third order*.

To discuss this question, consider the contact-transformation specified by the function

$$W = -(n+1)\log(q-Q),$$

so that

$$p = -\frac{\partial W}{\partial q} = \frac{n+1}{q-Q} \quad , \quad P = \frac{\partial W}{\partial Q} = \frac{n+1}{q-Q},$$

and therefore

$$Q = q - \frac{n+1}{p} \quad , \quad P = p.$$

The equations (2) for χ thus become

$$\left\{q - (n+1)\left(\frac{\partial}{\partial q}\right)^{-1}\right\}\chi = t\chi \tag{19}$$

$$\frac{\partial \chi}{\partial q} = -\frac{\partial \chi}{\partial t}. \tag{20}$$

Now (19) may be written

$$\frac{\partial}{\partial q}\left\{(q-t)\chi\right\} = (n+1)\chi$$

or

$$\frac{1}{\chi}\frac{\partial \chi}{\partial q} = \frac{n}{q-t}.$$

This with equation (20) gives

$$\chi = (t-q)^n.$$

Thus the theorem (4) asserts that *a linear differential equation*

$$F\left(q, \frac{d}{dq}\right)\psi = 0$$

is satisfied by a family of integrals of the type

$$\psi(q) = \int (t-q)^n \phi(t)\, dt,$$

where $\phi(t)$ is a solution of the differential equation

$$F\left\{t + (n+1)\left(\frac{d}{dt}\right)^{-1}, \ \frac{d}{dt}\right\}\phi = 0.$$

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[1] Such integrals have been used in the solution of third-order equations by L. Schlesinger, *Math. ZS.*, 27 (1928), p. 504.

In particular, suppose that

$$\phi(t) = t^{\alpha}(t-1)^{\beta}(t-e)^{\gamma},$$

which obviously satisfies the differential equation

$$\left(\frac{d}{dt} - \frac{\alpha}{t} - \frac{\beta}{t-1} - \frac{\gamma}{t-e}\right)\phi = 0. \tag{21}$$

Now since when c is any constant and ϕ is any function of t, we have

$$\left\{\frac{d}{dt}(t-c) - 1\right\}\phi = \left\{(t-c)\frac{d}{dt}\right\}\phi,$$

therefore $$t - c = \left\{\frac{d}{dt}(t-c) - 1\right\}\left(\frac{d}{dt}\right)^{-1},$$

and therefore equation (21) may be written

$$\left[\frac{d}{dt} - \frac{\alpha}{\left\{\frac{d}{dt}t - 1\right\}\left(\frac{d}{dt}\right)^{-1}} - \frac{\beta}{\left\{\frac{d}{dt}(t-1) - 1\right\}\left(\frac{d}{dt}\right)^{-1}} - \frac{\gamma}{\left\{\frac{d}{dt}(t-e) - 1\right\}\left(\frac{d}{dt}\right)^{-1}}\right]\phi = 0$$

or, since $\dfrac{1}{HK^{-1}}\phi = KH^{-1}\phi$, whatever operators H and K may be,

$$\frac{d}{dt}\left[1 - \frac{\alpha}{\frac{d}{dt}t - 1} - \frac{\beta}{\frac{d}{dt}(t-1) - 1} - \frac{\gamma}{\frac{d}{dt}(t-e) - 1}\right]\phi = 0$$

or, operating on this with $\left(\dfrac{d}{dt}\right)^{-1}$,

$$\left[1 - \frac{\alpha}{\frac{d}{dt}t - 1} - \frac{\beta}{\frac{d}{dt}(t-1) - 1} - \frac{\gamma}{\frac{d}{dt}(t-e) - 1}\right]\phi = 0. \tag{22}$$

Now the above theorem asserts that the integral

$$\psi(q) = \int t^{\alpha}(t-1)^{\beta}(t-e)^{\gamma}(t-q)^{n}\,dt \tag{23}$$

satisfies the equation which is obtained from (22) by replacing $\dfrac{d}{dt}$ by $\dfrac{d}{dq}$ and replacing t by $\left\{q - (n+1)\left(\dfrac{d}{dq}\right)^{-1}\right\}$. Thus the integral (23) satisfies the equation

$$\left[1 - \frac{\alpha}{\frac{d}{dq}q - n - 2} - \frac{\beta}{\frac{d}{dq}(q-1) - n - 2} - \frac{\gamma}{\frac{d}{dq}(q-e) - n - 2}\right]\psi = 0.$$

Operating with $\frac{d}{dq}(q-e)-(n+2)$, we obtain

$$\left[\frac{d}{dq}(q-e)-n-2-\frac{\alpha}{\left\{\frac{d}{dq}q-n-2\right\}\left\{\frac{d}{dq}(q-e)-n-2\right\}^{-1}}\right.$$
$$\left.-\frac{\beta}{\left\{\frac{d}{dq}(q-1)-n-2\right\}\left\{\frac{d}{dq}(q-e)-n-2\right\}^{-1}}-\gamma\right]\psi=0. \qquad (24)$$

Now we have

$$\left\{\frac{d}{dq}(q-1)-n-1\right\}\left\{\frac{d}{dq}(q-e)-n-2\right\}=\left\{\frac{d}{dq}(q-e)-n-1\right\}\left\{\frac{d}{dq}(q-1)-n-2\right\}$$

and therefore

$$\left\{\frac{d}{dq}(q-e)-n-1\right\}^{-1}\left\{\frac{d}{dq}(q-1)-n-1\right\}=\left\{\frac{d}{dq}(q-1)-n-2\right\}\left\{\frac{d}{dq}(q-e)-n-2\right\}^{-1}. \qquad (25)$$

Thus (24) may be written

$$\left[\frac{d}{dq}(q-e)-n-2-\frac{\alpha}{\left\{\frac{d}{dq}(q-e)-n-1\right\}^{-1}\left\{\frac{d}{dq}q-n-1\right\}}\right.$$
$$\left.-\frac{\beta}{\left\{\frac{d}{dq}(q-e)-n-1\right\}^{-1}\left\{\frac{d}{dq}(q-1)-n-1\right\}}-\gamma\right]\psi=0$$

or

$$\left[\frac{d}{dq}(q-e)-n-2-\left\{\frac{d}{dq}q-n-1\right\}^{-1}\left\{\frac{d}{dq}(q-e)-n-1\right\}\alpha\right.$$
$$\left.-\left\{\frac{d}{dq}(q-1)-n-1\right\}^{-1}\left\{\frac{d}{dq}(q-e)-(n+1)\right\}\beta-\gamma\right]\psi=0.$$

Operating with $\frac{d}{dq}(q-1)-(n+1)$, we have

$$\left[\begin{array}{l}\left\{\frac{d}{dq}(q-1)-n-1\right\}\left\{\frac{d}{dq}(q-e)-n-2\right\}\\ \quad-\left\{\frac{d}{dq}(q-1)-n-1\right\}\left\{\frac{d}{dq}q-n-1\right\}^{-1}\left\{\frac{d}{dq}(q-e)-n-1\right\}\alpha\\ \quad-\left\{\frac{d}{dq}(q-e)-n-1\right\}\beta-\left\{\frac{d}{dq}(q-1)-(n+1)\right\}\gamma\end{array}\right]\psi=0,$$

https://doi.org/10.1017/S0013091500007768 Published online by Cambridge University Press

or, using a formula similar to (28),

$$\left[\begin{array}{l} \left\{\frac{d}{dq}(q-1)-n-1\right\}\left\{\frac{d}{dq}(q-e)-n-2\right\} \\ \quad -\left(\frac{d}{dq}q-n\right)^{-1}\left\{\frac{d}{dq}(q-1)-n\right\}\left\{\frac{d}{dq}(q-e)-n-1\right\}\alpha \\ \quad -\left\{\frac{d}{dq}(q-e)-n-1\right\}\beta-\left\{\frac{d}{dq}(q-1)-n-1\right\}\gamma \end{array}\right]\psi=0,$$

or, operating with $\left(\frac{d}{dq}q-n\right)$,

$$\left[\begin{array}{l} \left\{\frac{d}{dq}q-n\right\}\left\{\frac{d}{dq}(q-1)-n-1\right\}\left\{\frac{d}{dq}(q-e)-n-2\right\} \\ \quad -\left\{\frac{d}{dq}(q-1)-n\right\}\left\{\frac{d}{dq}(q-e)-n-1\right\}\alpha \\ \quad -\left\{\frac{d}{dq}-n\right\}\left\{\frac{d}{dq}(q-e)-n-1\right\}\beta-\left\{\frac{d}{dq}-n\right\}\left\{\frac{d}{dq}(q-1)-n-1\right\}\gamma \end{array}\right]\psi=0,$$

or, performing the operations,

$$\frac{d^3\psi}{dq^3}+\left(\frac{1-\alpha-n}{q}+\frac{1-\beta-n}{q-1}+\frac{1-\gamma-n}{q-e}\right)\frac{d^2\psi}{dq^2}$$
$$+(n-1)\left\{\frac{\alpha+\beta+n}{q(q-1)}+\frac{\alpha+\gamma+n}{q(q-e)}+\frac{\beta+\gamma+n}{(q-1)(q-e)}\right\}\frac{d\psi}{dq}$$
$$+\frac{n(1-n)(\alpha+\beta+\gamma+n+1)}{q(q-1)(q-e)}\psi=0.$$

This is the differential equation satisfied by the family of definite integrals (23). It has:—

a singularity at $q=0$, with the exponents	$0,$	$1,$	$\alpha+n+1$
,, ,, ,, $q=1$, ,, ,, ,,	$0,$	$1,$	$\beta+n+1$
,, ,, ,, $q=e$, ,, ,, ,,	$0,$	$1,$	$\gamma+n+1$
,, ,, ,, $q=\infty$, ,, ,, ,,	$-n,$	$1-n,$	$-\alpha-\beta-\gamma-n-1.$

Thus *the integral* (23) *satisfies a linear differential equation with four regular singularities, but this differential equation is of the third order.* We might indeed have guessed that it could not in general satisfy a differential equation of the second order, since the number of independent functions which can be obtained from the integral (23), by taking different double-circuit paths of integration round the various singularities, is greater than the number of independent solutions of a linear differential equation of the second order.

§ 7. *Integrals involving hypergeometric functions.*

We shall now consider the solution of linear differential equations by means of definite integrals which involve a hypergeometric function in the integrand.

By taking the function $W(q, Q)$ which determines the contact-transformation to be a function of the product qQ, we see that the equations

$$Q = p\sigma(qp), \qquad P = -q/\sigma(qp),$$

where $\sigma(qp)$ is any function of the product qp, define a contact-transformation. Hence, in particular, the equations

$$\begin{cases} Q = 1 - \dfrac{(qp+\gamma)\,p}{(qp+\alpha)(qp+\beta)} \\ P = \dfrac{q\,(qp+\alpha)(qp+\beta)}{qp+\gamma} \end{cases}$$

define a contact-transformation. The corresponding equations (2) for $\chi(q, t)$ are

$$\left\{1 - \left(q\frac{\partial}{\partial q}+\beta\right)^{-1}\left(q\frac{\partial}{\partial q}+\alpha\right)^{-1}\left(q\frac{\partial}{\partial q}+\gamma\right)\frac{\partial}{\partial q}\right\}\chi = t\chi, \tag{26}$$

$$q\left(q\frac{\partial}{\partial q}+\gamma\right)^{-1}\left(q\frac{\partial}{\partial q}+\alpha\right)\left(q\frac{\partial}{\partial q}+\beta\right)\chi = -\frac{\partial\chi}{\partial t}. \tag{27}$$

Now (26) gives

$$\left(q\frac{\partial}{\partial q}+\beta\right)^{-1}\left(q\frac{\partial}{\partial q}+\alpha\right)^{-1}\left(q\frac{\partial}{\partial q}+\gamma\right)\frac{\partial}{\partial q}\chi = (1-t)\chi$$

or, operating with $q\left(q\dfrac{\partial}{\partial q}+\gamma\right)^{-1}\left(q\dfrac{\partial}{\partial q}+\alpha\right)\left(q\dfrac{\partial}{\partial q}+\beta\right)$ on both sides,

$$\begin{aligned} q\frac{\partial\chi}{\partial q} &= (1-t)\,q\left(q\frac{\partial}{\partial q}+\gamma\right)^{-1}\left(q\frac{\partial}{\partial q}+\alpha\right)\left(q\frac{\partial}{\partial q}+\beta\right)\chi \\ &= -(1-t)\frac{\partial\chi}{\partial t}, \text{ by (27).} \end{aligned}$$

Thus χ satisfies the partial differential equation

$$q\frac{\partial\chi}{\partial q} + (1-t)\frac{\partial\chi}{\partial t} = 0$$

whose solution is $\chi = \theta(s)$, where $s = q - qt$ and θ is the arbitrary function.

Substituting this value of χ in (27), we have

$$\left(s\frac{d}{ds}+\gamma\right)^{-1}\left(s\frac{d}{ds}+\alpha\right)\left(s\frac{d}{ds}+\beta\right)\theta=\frac{d\theta}{ds}$$

or
$$\left(s\frac{d}{ds}+\alpha\right)\left(s\frac{d}{ds}+\beta\right)\theta-\left(s\frac{d}{ds}+\gamma\right)\frac{d\theta}{ds}=0,$$

which gives $\theta=F(\alpha,\beta,\gamma,s)$, where F is the hypergeometric function.

Thus $\qquad \chi=F(\alpha,\beta,\gamma,\ q-qt)$,

and the theorem (4) asserts that *if* $\phi(t)$ *is a solution of a linear differential equation*

$$G\left(t,\frac{d}{dt}\right)\phi=0 \tag{28}$$

then the linear differential equation

$$G\left\{1-\left(q\frac{d}{dq}+\beta\right)^{-1}\left(q\frac{d}{dq}+\alpha\right)^{-1}\left(q\frac{d}{dq}+\gamma\right)\frac{d}{dq},\right.$$
$$\left.q\left(q\frac{d}{dq}+\gamma\right)^{-1}\left(q\frac{d}{dq}+\alpha\right)\left(q\frac{d}{dq}+\beta\right)\right\}\psi=0 \tag{29}$$

is satisfied by a family of definite integrals of the type

$$\psi(q)=\int F(\alpha,\beta,\gamma,q-qt)\,\phi(t)\,dt. \tag{30}$$

In particular, take $\phi(t)=t^{\kappa}$, so that the equation (28) becomes

$$\left\{\frac{d}{dt}\,t-\kappa-1\right\}\phi=0$$

and therefore the equation (29), satisfied by ψ, becomes

$$\left\{q\left(q\frac{d}{dq}+\gamma\right)^{-1}\left(q\frac{d}{dq}+\alpha\right)\left(q\frac{d}{dq}+\beta\right)-q\frac{d}{dq}-\kappa-1\right\}\psi=0,$$

or
$$\left\{\left(q\frac{d}{dq}+\alpha\right)\left(q\frac{d}{dq}+\beta\right)-\left(q\frac{d}{dq}+\gamma\right)\frac{d}{dq}-(\kappa+1)\left(q\frac{d}{dq}+\gamma\right)\left(\frac{1}{q}\right)\right\}\psi=0$$

or
$$q(q-1)\frac{d^2\psi}{dq^2}+\{(\alpha+\beta+1)q-(\gamma+\kappa+1)\}\frac{d\psi}{dq}+\left\{\alpha\beta-\frac{(\kappa+1)(\gamma-1)}{q}\right\}\psi=0,$$

which is the differential equation of the function

$$\psi=q^{-\kappa-1}F(\alpha-\kappa-1,\ \beta-\kappa-1,\ \gamma-\kappa-1,\ q),$$

where F denotes the hypergeometric function.

Thus the formula (30) becomes

$$q^{-\kappa-1}F(\alpha-\kappa-1,\ \beta-\kappa-1,\ \gamma-\kappa-1,q)=\text{Constant}\times\int F(\alpha,\beta,\gamma,q-qt)\,t^{\kappa}\,dt$$

or, writing t for qt, and s for $-\kappa-1$,

$$F(\alpha+s,\ \beta+s,\ \gamma+s,\ q)=\text{Constant}\times\int F(\alpha,\ \beta,\ \gamma,\ q-t)\,t^{-s-1}\,dt.$$

It is not difficult to determine the path of integration and the constant, *the final result being*

$$F(a+s, \beta+s, \gamma+s, q) = \frac{\Gamma(a)\,\Gamma(\beta)\,\Gamma(\gamma+s)}{\Gamma(\gamma)\,\Gamma(-s)\,\Gamma(a+s)\,\Gamma(\beta+s)} \int_0^\infty F(a, \beta, \gamma, q-t)\, t^{-s-1}\, dt.$$

This somewhat remarkable property of the hypergeometric function, which is new to me, is in a certain sense the *reciprocal* of the other new property expressed by equation (18), being in fact (as Dr Copson has kindly pointed out to me) connected with it by the Riemann-Mellin Theorem.[1]

§ 8. *Results involving Bessel functions.*

If the contact-transformation is taken to be

$$Q = -p^2 - \frac{p}{q}, \quad P = \frac{q}{2p},$$

the equations to determine χ are

$$\begin{cases} \left(-\dfrac{\partial^2}{\partial q^2} - \dfrac{1}{q}\dfrac{\partial}{\partial q}\right)\chi = t\chi \\ \qquad\qquad \tfrac{1}{2}\left(\dfrac{\partial}{\partial q}\right)^{-1} q\chi = -\dfrac{\partial \chi}{\partial t}, \end{cases}$$

and these have the solution $\chi = J_0(q\sqrt{t})$ where J_0 is the Bessel function of zero order: from this it is easy to derive such formulae as

$$\exp\left(-\frac{q^2}{2\kappa^2}\right) = \kappa^2 \int_0^\infty \exp\left(-\tfrac{1}{2}\kappa^2 t\right) J_0(q\sqrt{t})\, dt.$$

Similarly the contact-transformations

$$Q = -p^2 - \frac{p}{q} + \frac{n^2}{q^2}, \quad P = -\tfrac{1}{2}pq\Big/\left(-p^2 - \frac{p}{q} + \frac{n^2}{q^2}\right).$$

gives the χ-function $\chi = J_n(q\sqrt{t})$ and leads to definite integrals having the Bessel function of order n in the integrand.

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[1] Riemann, *Berlin Monatsb.*, Nov. 1859: Mellin, *Acta Math.*, 25 (1902), p. 138: *Math. Ann.*, 68 (1909), p. 305.