Canad. Math. Bull. Vol. **37** (3), 1994 pp. 351–360

DISTORTION THEOREMS FOR DIFFEOMORPHISMS

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ABSTRACT. Generalizations of the Koebe distortion theorem to a class of diffeomorphisms are given. They are applied to univalent harmonic mappings.

1. **Introduction.** Let f be a complex-valued harmonic function defined on the unit disk U. Then f can be written in the form $f = h + \bar{g}$ where h and g belong to the linear space $H(U)$ of analytic functions on U. In order to have a unique representation, we assume that $g(0) = 0$.

Suppose now that in addition, f is also univalent on U. Without loss of generality, we may assume that f is orientation-preserving, since if not, consider the function $f(\bar{z})$. It follows then, that f is a solution of the (non-uniformly) elliptic partial differential equation

$$\overline{f_{\bar{z}}} = af_z, \quad a \in H(U), \ |a| < 1 \text{ on } U.$$

Therefore, f is a locally quasiconformal and pseudo-analytic mapping of second kind on U. (For more details see *e.g.* [4] and [1].) Observe that $f \in H(U)$ and hence is conformal if and only if $a \equiv 0$, *i.e.* $g \equiv 0$.

A well known distortion theorem due to Koebe states that

$$(1) \qquad |f(z) - f(0)| \geq \frac{|f'(0)|\,|z|}{(1+|z|)^2}, \quad z \in U,$$

for all univalent analytic functions on U. A generalization of (1) has been given by J. Clunie and T. Sheil-Small [2], which have shown that

$$(2) \qquad |f(z) - f(0)| \geq \frac{|f_z(0)|\,|z|}{4(1+|z|)^2}, \quad z \in U,$$

holds for all univalent harmonic and orientation-preserving mappings $f = h + \bar{g}$ defined on U satisfying the condition $\overline{f_{\bar{z}}(0)} = g'(0) = 0$.

In this paper we give first two distortion theorems for a large class of diffeomorphisms defined on the unit disk U. As a corollary of Theorem 1, we get a generalization of Koebe's $\frac{1}{4}$-Theorem. Our next result, Theorem 2, contains both, the sharp Koebe estimate (1) and

The first author was supported in part by a grant from the NSERC, Canada and the FCAR, Quebec.
The second author was supported in part by an Undergraduate Student Research Award from the NSERC, Canada.
Received by the editors January 4, 1993.
AMS subject classification: Primary: 30C55; secondary: 31A05.

the result (2) of Clunie and Sheil-Small. For univalent harmonic mappings satisfying the properties above, Clunie and Sheil-Small have also shown that

$$\{w\,;\,|w-f(0)|<R\}\subset f(U)$$

implies that $R \le R_0 = \frac{2\pi\sqrt{3}h'(0)}{9}$. The upper bound R_0 is best possible but there is no univalent harmonic mapping of the considered class which has the property that $\{w\,;\,|w-f(0)|<R_0\}$ belongs to $f(U)$. In Theorem 4 we give a corresponding result for univalent harmonic mappings which are defined on the exterior of the unit disk and which are of the form

$$f(z)=Az+\overline{Bz}+\sum_{n=0}^{\infty}a_n\left(\frac{1}{z}\right)^n+\overline{\sum_{n=1}^{\infty}b_n\left(\frac{1}{z}\right)^n},\qquad |B|<|A|.$$

In particular, we show that for any omitted value p, we have

$$\max\{|f(z)-p|\,;\,|z|=r\}\ge(|A|-|B|)\frac{r^2-1}{r},\qquad r>1.$$

where equality holds for the mapping

$$f(z)=Az+\overline{Bz}+p-\frac{\bar{B}}{z}-\frac{A}{\bar{z}}.$$

The proof is based on Theorem 3, where we derive a corresponding result for the class of diffeomorphisms f on U which satisfy the inequality $|f_{\bar{z}}|\le c|z|^p|f_z|$ for some $c\in[0,1]$ and some $p>0$. Finally, we give in Section 3 a univalence criterion for orientation-preserving harmonic mappings.

2. **Generalizations of the Koebe distortion theorem.** We start with the following result:

THEOREM 1. *Let f be a diffeomorphism defined on U such that for some given $p>0$ and $c\in[0,1]$*

(3) $$|f_{\bar{z}}|\le c|z|^p|f_z|,$$

for all $z\in U$. Then we have the inequality

(4) $$|f(z)-f(0)|\ge\frac{|f_z(0)|\,|z|}{4(1+c|z|^p)^{2/p}},\qquad z\in U.$$

REMARK. For $c=1$ and $p=1$, we get the inequality (2). However the case $c=0$ does not yet give the classical Koebe estimate (1).

PROOF. We modify the proof of J. Clunie and T. Sheil-Small given for the case of harmonic mappings [2, Theorem 4.4]. First, observe that the inequality (4) is satisfied if $f_z(0)=0$. Hence, we may assume that $f_z(0)\ne 0$.

Fix $r \in (0,1)$. Define

$$F(z) = \frac{f(rz) - f(0)}{r.f_z(0)} \quad \text{and} \quad \Omega = F(U). \tag{5}$$

Observe that $F(0) = 0$, $F_z(0) = 1$ and that

$$|F_{\bar{z}}(z)| \leq c.r^p|z|^p|F_z(z)| \quad z \in U. \tag{6}$$

Therefore we have also $F_{\bar{z}}(0) = 0$.

Next, choose $\varepsilon > 0$ such that $\overline{\Delta_\varepsilon} = \{w \,;\, |w| \leq \varepsilon\} \subset \Omega$ and define $\Omega_\varepsilon = \Omega \setminus \overline{\Delta_\varepsilon}$. Since F is a diffeomorphism satisfying $F(0) = F_{\bar{z}}(0) = 0$ and $F_z(0) = 1$, we have

$$\left.\begin{aligned} F(\varepsilon e^{it}) &= \varepsilon e^{it} + o(\varepsilon) \text{ and} \\ F^{-1}(\varepsilon e^{it}) &= \varepsilon e^{it} + o(\varepsilon) \end{aligned}\right\} \text{ as } \varepsilon \longrightarrow 0. \tag{7}$$

Let Γ be the set of rectifiable Jordan arcs in Ω_ε joining $\partial\Delta_\varepsilon$ to $\partial\Omega$. We say that a measurable function $\rho(w) \geq 0$ is *admissible* for Γ if

$$\int_\gamma \rho(w)\,|dw| \geq 1$$

for all $\gamma \in \Gamma$. In particular, put

$$\psi(r,z) = \frac{1 - cr^p|z|^p}{1 + cr^p|z|^p}\,\frac{1}{|z|}.$$

Then, for ε small enough,

$$\rho(w) = \begin{cases} \frac{\psi(r,z)}{|F_z| - |F_{\bar{z}}|} \Big/ \int_\varepsilon^1 \psi(r,z)\,d|z| & \text{if } \varepsilon < |z| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

is up to a uniform term of $o(1)$ admissible for Γ, where $w = F(z)$. Indeed, according to (8), we have

$$\begin{aligned} \int_\gamma \rho(w)\,|dw| &= \int_{F^{-1}(\gamma)} \frac{\psi(r,z)}{|F_z| - |F_{\bar{z}}|} |F_z\,dz + F_{\bar{z}}\,d\bar{z}| \Big/ \int_\varepsilon^1 \psi(r,z)\,d|z| \\ &\geq \int_{F^{-1}(\gamma)} \psi(r,z)\,|dz| \Big/ \int_\varepsilon^1 \psi(r,z)\,d|z| \\ &\geq 1 + o(1). \end{aligned}$$

The modulus $M(\Omega_\varepsilon)$ of the ring domain Ω_ε is defined by

$$\frac{1}{M(\Omega_\varepsilon)} = \inf \int_{\Omega_\varepsilon} \rho^2(w)\,du\,dv \quad (w = u + iv)$$

where the infimum is taken over all admissible ρ for Γ.

For the particular ρ defined in (8) we get (up to an additional term $o(1)$) from (6) and (7)

$$\begin{aligned}\frac{1}{M(\Omega_\varepsilon)} &\leq \int_{F^{-1}(\Omega_\varepsilon)} \rho^2\big(F(z)\big)(|F_z|^2 - |F_{\bar z}|^2)\,dx\,dy \quad (z = x+iy)\\ &= \int_{F^{-1}(\Omega_\varepsilon)} \psi^2(r,z)\,\frac{|F_z|+|F_{\bar z}|}{|F_z|-|F_{\bar z}|}\,dx\,dy \Big/ \Big(\int_\varepsilon^1 \psi(r,z)\,d|z|\Big)^2\\ &\leq \int_{F^{-1}(\Omega_\varepsilon)} \psi(r,z)\,d|z|\,d\theta \Big/ \Big(\int_\varepsilon^1 \psi(r,z)\,d|z|\Big)^2\\ &= \int_0^{2\pi}\int_\varepsilon^1 \psi(r,z)\,d|z|\,d\theta \Big/ \Big(\int_\varepsilon^1 \psi(r,z)\,d|z|\Big)^2 + o(1).\end{aligned}$$

as ε tends to 0.

Hence,

$$\begin{aligned}M(\Omega_\varepsilon) &\geq \frac{1}{2\pi}\int_\varepsilon^1 \psi(r,z)\,d|z| + o(1)\\ &= \frac{1}{2\pi}\Big\{-\ln[\varepsilon] - \frac{2}{p}\ln[1+cr^p] + \frac{2}{p}\ln[1+cr^p\varepsilon^p]\Big\} + o(1). \qquad (9)\end{aligned}$$

Let δ be the distance of $\partial\Omega$ from the origin. Without loss of generality, we may assume that $\delta \in \partial\Omega$. Then by Grötzsch [4], we have

$$M(\Omega_\varepsilon) \leq M(D_\varepsilon), \qquad (10)$$

where $D_\varepsilon = \mathbf{C} \setminus \big\{[\delta,\infty) \cup \{w\,;\,|w| \leq \varepsilon\}\big\}$.

For ε small enough, we conclude from [4; Section 2.3 of Chapter 2] that

$$M(D_\varepsilon) = \frac{1}{2\pi}\ln\Big[\frac{4\delta}{\varepsilon}\Big] + o(1). \qquad (11)$$

From (9), (10) and (11), we get

$$\ln[4\delta] \geq \frac{2}{p}\ln\Big[\frac{1+cr^p\varepsilon^p}{1+cr^p}\Big] + o(1).$$

By letting $\varepsilon \longrightarrow 0$, we obtain

$$|F(e^{it})| = \left|\frac{f(re^{it}) - f(0)}{r f_z(0)}\right| \geq \delta \geq \frac{1}{4(1+cr^p)^{2/p}}$$

and the theorem is proved.

As an immediate consequence, we get a generalization of the Koebe $\frac{1}{4}$-Theorem:

COROLLARY 1. *Let f be defined as in Theorem 1. Then*

$$\Big\{w\,;\,|w - f(0)| < \frac{|f_z(0)|}{4(1+c)^{2/p}}\Big\} \subset f(U). \qquad (12)$$

If c = 0 and $|z|$ is small, then the inequality (4) is not best possible. Our next result gives an improvement of Theorem 1 for the cases $p \geq 1$.

THEOREM 2. *Let f be a diffeomorphism defined on U such that for some given $p > 0$ and $c \in [0,1]$*

$$|f_{\bar{z}}| \leq c|z|^p |f_z|, \tag{13}$$

for all $z \in U$. Then we have the inequality

$$|f(z) - f(0)| \geq \frac{|f_z(0)|\,|z|}{(1+c)^{2/p}(1+|z|)^2}, \quad z \in U. \tag{14}$$

REMARKS. (1) If $c = 0$ or $p = \infty$, then f is a univalent conformal mapping and the inequality (14) reduces to the classical sharp Koebe estimate (1).

(2) Let $f = h + \bar{g}$ be a univalent harmonic and orientation-preserving mapping defined on U having the property that $g'(0) = 0$. Then, by Schwarz's lemma, the condition (13) is satisfied with c = 1 and p = 1 and the inequality (14) reduces to the form (2).

(3) For $p = 1$ and $c = 1$, we conjecture that (14) may be replaced by

$$|f(re^{it}) - f(0)| \geq |f_z(0)| r \exp\left[\frac{-4r}{1+r}\right] \tag{15}$$

where equality holds for the univalent mapping

$$f(z) = z\frac{\overline{1+z}}{1+z}\exp\left[\frac{-4z}{1+z}\right] \tag{16}$$

PROOF. Fix $r \in (0,1)$ and $t \in [0, 2\pi]$. Define

$$k_t(z) = \frac{z}{(1+e^{-it}z)^2},$$

$$\omega_t(z) = k_t^{-1}\left[\frac{4r}{(1+r)^2}k_t(z)\right]$$

$$G(z) = \frac{(1+r)^2}{4r}\left[f\big(\omega_t(z)\big) - f(0)\right] \tag{17}$$

Observe that $G(0) = 0$, $G_z(0) = f_z(0)$ and that

$$G(e^{it}) = \frac{(1+r)^2}{4r}[f(re^{it}) - f(0)] = \frac{[f(re^{it}) - f(0)]}{A(r)}. \tag{18}$$

Since $\omega_t(z)$ is a Schwarz function, *i.e.* $\omega_t(z)$ is analytic and $|\omega_t(z)| \leq |z|$ on U, we get

$$\begin{aligned}|G_{\bar{z}}(z)| &= \frac{\left|f_{\bar{z}}\big(\omega_t(z)\big)\right| |\omega_t'(z)|}{A(r)} \leq \frac{c|\omega_t(z)|^p \left|f_z\big(\omega_t(z)\big)\right| |\omega_t'(z)|}{A(r)} \leq \frac{c|z|^p \left|f_z\big(\omega_t(z)\big)\right| |\omega_t'(z)|}{A(r)} \\ &= c|z|^p |G_z(z)|\end{aligned}$$

and therefore we have

$$|G_{\bar{z}}(z)| \leq c|z|^p |G_z(z)| \quad z \in U. \tag{19}$$

Applying Corollary 1 to G we get

$$|G(e^{it})| = \left|\frac{(1+r)^2}{4r}[f(re^{it}) - f(0)]\right| \geq \frac{|f_z(0)|}{4(1+c)^{2/p}} \tag{20}$$

and Theorem 2 follows immediately.

Our next result gives a sharp estimate for the largest possible disk centered at the origin lying in the image $f(U)$.

THEOREM 3. *Let f be a diffeomorphism defined on U such that for some given $p > 0$ and $c \in [0,1]$*

$$|f_{\bar z}| \leq c|z|^p |f_z|, \tag{21}$$

for all $z \in U$. Then we have the inequality

$$\min\{|f(z) - f(0)| \,;\, |z| = r\} \leq \frac{|f_z(0)|r}{(1-cr^p)^{2/p}}, \quad z \in U. \tag{22}$$

The inequality is best possible.

PROOF. Let r, F, ε and Ω_ε be as in the proof of Theorem 1 and let Γ be the set of rectifiable Jordan arcs in Ω_ε separating $\partial\Delta_\varepsilon$ and $\partial\Omega$. Then, for ε small enough,

$$\rho(w) = \begin{cases} \frac{1}{2\pi|z|(|F_z|-|F_{\bar z}|)}, & \text{if } \varepsilon < |z| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

is up to a uniform term of o(1) admissible for Γ, where $w = F(z)$. The modulus $M(\Omega_\varepsilon)$ of the ring domain Ω_ε is determined by

$$M(\Omega_\varepsilon) = \inf \int_{\Omega_\varepsilon} \rho^2(w)\,du\,dv \quad (w = u + iv)$$

where the infimum is taken over all admissible ρ for Γ. Again let δ be the distance of $\partial\Omega$ from the origin. Since the annulus $\{w\,;\,\varepsilon < |w| < \delta\}$ lies in Ω_ε we conclude, by the superadditivity of the moduli, that

$$\begin{aligned} \frac{1}{2\pi}\ln\left(\frac{\delta}{\varepsilon}\right) \leq M(\Omega_\varepsilon) &\leq \int_{F^{-1}(\Omega_\varepsilon)} \rho^2\big(F(z)\big)(|F_z|^2 - |F_{\bar z}|^2)\,dx\,dy \quad (z = x+iy) \\ &= \int_{F^{-1}(\Omega_\varepsilon)} \frac{1}{4\pi^2|z|^2}\frac{|F_z| + |F_{\bar z}|}{|F_z| - |F_{\bar z}|}\,dx\,dy \\ &\leq \int_0^{2\pi}\int_\varepsilon^1 \frac{1}{4\pi^2|z|}\frac{1 + cr^p|z|^p}{1 - cr^p|z|^p}\,d|z|\,dt + o(1) \\ &= \frac{1}{2\pi}\left\{-\ln[\varepsilon] + \frac{2}{p}\ln\left[\frac{1 - cr^p\varepsilon^p}{1 - cr^p}\right]\right\} + o(1). \end{aligned}$$

as ε tends to 0.

Therefore, we get

$$\min\{|f(z) - f(0)| \,;\, |z| = r\} \leq \frac{|f_z(0)|r}{(1-cr^p)^{2/p}}, \quad z \in U.$$

It remains to show that the inequality is best possible. For $c = 0$ or $p = \infty$, the inequality (22) follows from the minimum modulus principle applied to $\frac{f(z)-f(0)}{z}$ and equality holds if and only if $f(z) = az + b$. Let $0 < c \leq 1$ and $p > 0$. Consider the function

$$f(z) = \frac{z}{(1 - c|z|^p)^{2/p}}.$$

Direct calculations show that the partial derivatives

$$f_{\bar{z}}(z) = \frac{c\frac{z}{\bar{z}}|z|^p}{(1 - c|z|^p)^{1+\frac{2}{p}}}$$

and

$$f_z(z) = \frac{1}{(1 - c|z|^p)^{1+\frac{2}{p}}}$$

are continuous functions on U satisfying the property

$$|f_{\bar{z}}(z)| = c|z|^p |f_z(z)|.$$

Next, we show that f is univalent on U. Since $f(0) = 0$ and $\arg f(z) \equiv \arg z$, it is sufficient to verify that $\frac{\partial |f|}{\partial |z|} > 0$ on U. We have

$$\frac{\partial\, |f(re^{it})|}{\partial r} = \frac{(1 + cr^p)}{(1 - cr^p)^{1+\frac{2}{p}}} > 0$$

and Theorem 3 is established.

Let $f = h + \bar{g}$ be a univalent harmonic and orientation-preserving mapping defined on U having the property that $g'(0) = 0$. Then, by Schwarz's lemma, the condition (13) is satisfied with $c = 1$ and $p = 1$. As we have mentioned in the introduction, Clunie and Sheil-Small [2] have shown that for such harmonic mappings the inequality

(24) $$\min\{|f(z) - f(0)|\ ;\ |z| = r\} \leq \frac{2\pi\sqrt{3}|f_z(0)|r}{9}, \quad 0 < r < 1.$$

holds. The estimate, which is best possible for $r = 1$, is much better than our inequality (22). However, we get sharp estimates for univalent harmonic mappings defined on the exterior of the unit disk.

Let $\Delta = \{w\ ;\ |w| > 1\}$ be the exterior of the closed unit disk $\bar{U}$ and let f be a univalent harmonic and orientation-preserving mapping defined on Δ which maps infinity onto itself. Then f is of the form

(25) $$f(z) = Az + \overline{Bz} + 2C\ln|z| + \sum_{n=0}^{\infty} a_n\left(\frac{1}{z}\right)^n + \overline{\sum_{n=1}^{\infty} b_n\left(\frac{1}{z}\right)^n}$$

We restrict ourself to the case where $C = 0$, *i.e.* to mappings of the form

(26) $$f(z) = Az + \overline{Bz} + \sum_{n=0}^{\infty} a_n\left(\frac{1}{z}\right)^n + \overline{\sum_{n=1}^{\infty} b_n\left(\frac{1}{z}\right)^n}$$

It follows then that $A \neq 0$ and $|B| < |A|$.

https://doi.org/10.4153/CMB-1994-052-9 Published online by Cambridge University Press

THEOREM 4. *Let f be a univalent harmonic and orientation-preserving mapping defined on Δ which is of the form (26) and let p be a point of the complement of $f(\Delta)$. Then we have*

$$\max\{|f(z)-p| \;;\; |z|=r\} \geq (|A|-|B|)\frac{r^2-1}{r}, \quad r>1. \tag{27}$$

Furthermore, equality holds for the mapping

$$f(z) = Az + \overline{Bz} + p - \frac{\bar{B}}{z} - \frac{A}{\bar{z}}. \tag{28}$$

PROOF. Define

$$f_1(z) = \frac{f(z)-p}{A},$$
$$f_2(z) = \frac{f_1(z) - \frac{\bar{B}}{A}\overline{f_1(z)}}{1-\left|\frac{B}{A}\right|^2}.$$

and

$$a(z) = \frac{(f_2)_{\bar{z}}(z)}{(f_2)_z(z)}.$$

Then

$$b(\zeta) = a\left(\frac{1}{\zeta}\right) = \alpha\zeta^2 + O(\zeta^3), \quad |\alpha| \leq 1,$$

is an analytic function on U and by Schwarz's lemma, we have $|b(\zeta)| \leq |\zeta|^2$ for all ζ in U. Define

$$g(\zeta) = \frac{1}{f_2\left(\frac{1}{\zeta}\right)}.$$

Then g is a diffeomorphism on U satisfying the inequality

$$|g_{\bar{\zeta}}(\zeta)| = |b(\zeta)|\,|g_\zeta(\zeta)| \leq |\zeta|^2|g_\zeta(\zeta)|$$

for all $\zeta \in U$. Since $g(0)=0$ and $g_\zeta(0)=1$, we conclude from Theorem 3 ($c=1$ and $p=2$) that

$$\min\{|g(z)| \;;\; |z|=r\} \leq \frac{r}{1-r^2}, \quad r<1.$$

Therefore, we have

$$\max\{|f_2(z)| \;;\; |z|=r\} \geq \frac{r^2-1}{r}, \quad r>1. \tag{29}$$

The latter inequality is best possible for the univalent harmonic mapping

$$F_2(z) = z - \frac{1}{\bar{z}}. \tag{30}$$

Since

$$f(z) - p = Af_1(z) = A\left(f_2(z) + \frac{\bar{B}}{A}\overline{f_2(z)}\right),$$

we get from (29) the estimate

$$\begin{aligned}\max\{|f(z) - p| \,;\, |z| = r\} &\geq |A|\frac{r^2 - 1}{r}\left(1 - \left|\frac{B}{A}\right|\right)\\ &= \frac{r^2 - 1}{r}(|A| - |B|), \quad r > 1.\end{aligned}$$

Since equality of (29) holds for F_2 defined in (30), and since $F_2(re^{it}) = \frac{r^2-1}{r}e^{it}$, equality in (27) holds for

(31) $$F(z) = A\left[z - \frac{1}{\bar{z}}\right] + p + \bar{B}\left[\bar{z} - \frac{1}{z}\right],$$

and the statement of Theorem 4 follows.

3. **A univalence criterion for harmonic mappings.** Let now $f = h + \bar{g}$ be a univalent harmonic mapping defined on U. Without loss of generality, we may assume that f is orientation-preserving, *i.e.* that $a = g'/h' \in H(U)$ and $|a| < 1$ on U.

Consider the conformal transformation

$$T(z) = \frac{z + z_1}{1 + \overline{z_1}z}$$

for a fixed $z_1 \in U$ and put

(32) $$G(z) = (f \circ T)(z) - \overline{(a \circ T)(0)(f \circ T)(z)}.$$

Then G is again a univalent harmonic mapping defined on U and, by Schwarz's lemma, we have

$$\left|\frac{G_{\bar{z}}}{G_z}(z)\right| = \left|\frac{(a \circ T)(z) - (a \circ T)(0)}{1 - \overline{(a \circ T)(0)}(a \circ T)(z)}\right| \leq |z|.$$

Hence, G satisfies condition (3) with $c = p = 1$ and Theorem 1 (or Theorem 2) applies. We get, according to (4) or (14),

$$|G(z) - G(0)| \geq \frac{|G_z(0)|\,|z|}{4(1 + |z|)^2}.$$

which leads us to

$$|(f \circ T)(z) - (f \circ T)(0)| \geq \frac{|z|}{4(1 + |z|)^2}|(f \circ T)_z(0)|\big(1 - |(a \circ T)(0)|\big).$$

Defining $z_2 = T(z)$, we conclude that

(33) $$|f(z_2) - f(z_1)| \geq \frac{|1 - \overline{z_1}z_2|\,|z_2 - z_1|}{(|1 - \overline{z_1}z_2| + |z_2 - z_1|)^2}(1 - |z_1|^2)|h'(z_1)|\big(1 - |a(z_1)|\big).$$

THEOREM 5. *Let $f = h + \bar{g}$ be an orientation-preserving mapping defined on U. Then f is univalent if and only if (33) holds.*

https://doi.org/10.4153/CMB-1994-052-9 Published online by Cambridge University Press

REMARK. In the case of analytic functions, there is an analogous result called the Invariant Koebe Distortion theorem [3].

PROOF. The necessity of condition (33) for univalence has been already shown. Hence, suppose that $f(\hat{z}) = f(\hat{\zeta})$ for a couple $(\hat{z}, \hat{\zeta}) \in U \times U$, $\hat{z} \neq \hat{\zeta}$. By (33), it follows that $h'(\hat{z}) = 0$ and since f is orientation-preserving, we get also $g'(\hat{z}) = 0$. Since f is a harmonic mapping, it follows that f is at least two-valent in any neighborhood of $\hat{z}$. Such a result does not hold in general for quasi-regular mappings as the example $z|z|^2$ shows. It follows then that there exist two sequences z_n, ζ_n in U, $n \in \mathbf{N}$, such that $z_n \longrightarrow \hat{z}$, $\zeta_n \longrightarrow \hat{z}$ and $f(z_n) = f(\zeta_n)$. Applying again (33), we get $h'(z_n) = g'(z_n) = 0$ and, by the identity principle, we conclude that $f = h + \bar{g}$ is a constant, which contradicts our assumption that f is an open and orientation-preserving mapping.

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