STABILITY OF WEIGHTED DARMA FILTERS

K. J. HARRISON, J. A. WARD AND L-J. EATON

ABSTRACT. We study the stability of linear filters associated with certain types of linear difference equations with variable coefficients. We show that stability is determined by the locations of the poles of a rational transfer function relative to the spectrum of an associated weighted shift operator. The known theory for filters associated with constant-coefficient difference equations is a special case.

1. **Introduction.** An adaptive DARMA (deterministic, autoregressive, moving average) filter is a linear operator which associates each sequence x in its domain with a sequence y which is related to x according to the linear difference equation

$$(1) \qquad \sum_{i=0}^{r} a_{n,i} y_{n-i} = \sum_{j=0}^{s} b_{n,j} x_{n-j} \quad \text{for each } n \in \mathbf{Z},$$

where the coefficients $a_{n,i}$ and $b_{n,j}$, for $0 \le i \le r$, $0 \le j \le s$, and $n \in \mathbf{Z}$, are complex numbers. We study certain types of stability of such filters which arise naturally when the filters are regarded as mappings between the various ℓ^p spaces. Two notions, which we call (p, q)-*stability* and (p, q)-*boundedness*, are defined precisely below.

Let S denote the vector space of doubly-infinite complex-valued sequences, and for $1 \le p \le \infty$ let ℓ^p denote the subspace of S consisting of p-summable sequences. Thus $x = (x_n)_{n \in \mathbf{Z}} \in \ell^p$ if $\|x\|_p = (\sum_{n \in \mathbf{Z}} |x_n|^p)^{1/p} < \infty$ for $p < \infty$, and $x \in \ell^\infty$ if $\|x\|_\infty = \sup_{n \in \mathbf{Z}} |x_n| < \infty$. A *sequence space operator* is a linear operator T whose domain $D(T)$ and range $R(T)$ are subspaces of S. A sequence space operator T is a *filter* for (1) if x and y satisfy (1) whenever $x \in D(T)$ and $y = Tx$.

DEFINITION 1. Suppose that T is a sequence space operator, $1 \le p, q \le \infty$, and that X is a subspace of S. We say that T is (p, q)-stable on X if $\ell^p \cap X \subseteq D(T)$, and $Tx \in \ell^q$ for each $x \in \ell^p \cap X$, and that T is (p, q)-bounded on X if there exists $\kappa \ge 0$ such that $\|Tx\|_q \le \kappa \|x\|_p$ for each $x \in \ell^p \cap X$.

A (p, q)-bounded operator is necessarily (p, q)-stable, but the converse does not hold in general. However, as we shall see in Section 3, there are certain types of sequence space operators which, if (p, q)-stable, are automatically (p, q)-bounded.

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If, for each i and j, the numbers $a_{n,i}$ and $b_{n,j}$ are independent of n, (1) becomes a *constant-coefficient* difference equation

$$(2) \qquad \sum_{i=0}^{r} a_i y_{n-i} = \sum_{j=0}^{s} b_j x_{n-j} \quad \text{for each } n \in \mathbf{Z},$$

where the a_i, $0 \le i \le r$, and b_j, $0 \le j \le s$ are fixed complex numbers. A standard method of dealing with (2) is to introduce the shift operator S, which is defined by $(Sx)_{n+1} = x_n$ for each $n \in \mathbf{Z}$ and for each sequence x, as in [2]. Then (2) becomes

$$(3) \qquad a(S)y = b(S)x,$$

where a and b are the polynomials defined by

$$a(\zeta) = a_0 + a_1\zeta + \cdots + a_r\zeta^r \quad \text{and} \quad b(\zeta) = b_0 + b_1\zeta + \cdots + b_s\zeta^s,$$

and the stability of filters associated with (3) is essentially determined by the location of the poles of the rational transfer function

$$(4) \qquad r(\zeta) = \frac{b(\zeta)}{a(\zeta)}.$$

There is no satisfactory general theory for the stability of adaptive filters, that is, filters associated with the variable-coefficient difference equation (1). However Ramsey [4] gives conditions, in terms of the norms of products of associated companion matrices, for the (∞, ∞)-stability of adaptive filters on causal sequences (as defined below). These results may be used to investigate, for example, the stability of adaptive filters where the coefficients are periodic or piecewise-constant [5]. In this paper we examine adaptive filters for difference equations which can be written in the form

$$(5) \qquad a(W)y = b(W)x,$$

where W is the weighted shift defined on $\mathcal{S}$ by

$$(Wx)_{n+1} = w_n x_n, \quad \text{for each } n \in \mathbf{Z} \text{ and for each } x \in \mathcal{S},$$

for a fixed weight sequence $w = (w_n)$ of positive numbers.

It is not difficult to show that (1) reduces to (5) if and only if the coefficients $a_{n,i}$ and $b_{n,j}$ satisfy relations of the form

$$a_{n,i} = a_i \beta_n \beta_{n-i}^{-1} \quad \text{and} \quad b_{n,j} = b_j \beta_n \beta_{n-j}^{-1},$$

where $\beta = (\beta_n)$ is a sequence of positive numbers with $\beta_0 = 1$, and where $a_0, a_1, \dots, a_r$ and $b_0, b_1, \dots, b_s$ are fixed complex numbers. The weight sequence w is then related to β according to the formula

$$\beta_{n+1} = w_n \beta_n \quad \text{for each } n \in \mathbf{Z}.$$

We shall assume that a_r and b_s are non-zero, and that there are positive numbers ρ_1 and ρ_2 such that

(6) $$0 < \rho_1 \leq w_n \leq \rho_2 < \infty \quad \text{for all } n \in \mathbf{Z}.$$

We shall say that T is a *weighted DARMA filter* if

$$a(W)Tx = b(W)x \quad \text{for each } x \in D(T),$$

and in this case we say that T is a filter for (5). We shall see that the stability and boundedness of such a T are determined by the locations of the non-zero poles of the transfer function $r(\zeta)$ defined by (4), relative to certain annular subsets of the complex plane associated with the weighted shift W. If $w_n = 1$ for all n, then (5) reduces to (3), and so the stability theory which we shall develop for weighted filters includes the constant-coefficient case (see [3], for example).

The block diagram in Figure 1 gives a physical realization of (5) in terms of adders, gains and unit delays. (For simplicity we have assumed here that $r = s$ and that $a_0 \neq 0$.) It is based on the standard realization of equation (3), but includes at each node an additional common, time-varying, gain of w_n at time n.
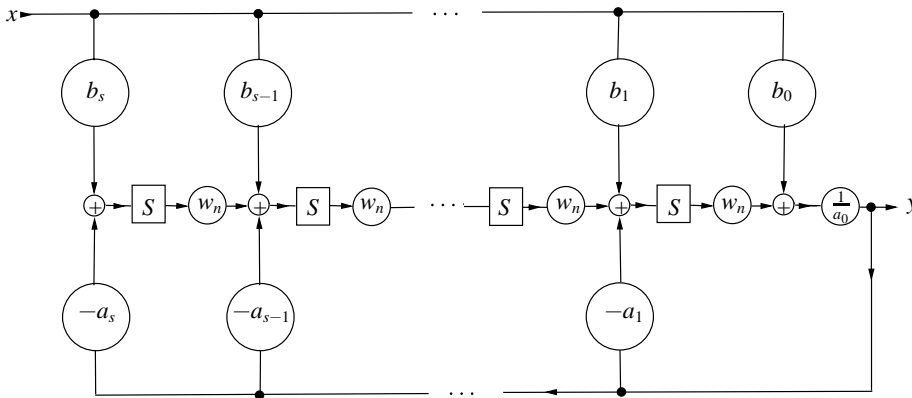

FIGURE 1

1.1. *Weighted shifts.* We include some basic facts about weighted shifts. For a more detailed discussion see the survey paper [6]. The norm of W, as an operator from ℓ^p into ℓ^p, is the supremum of the weights w_n. It follows from (6) that this supremum is bounded. It is also independent of p, and so we write

$$\|W\| = \sup_{k \in \mathbf{Z}} w_k = \sup_{k \in \mathbf{Z}} \beta_{k+1}\beta_k^{-1}.$$

For each positive integer n, the operator norm of W^n is the supremum of the sliding products $w_k w_{k+1} \cdots w_{k+n-1}$, for $k \in \mathbf{Z}$, that is,

$$\|W^n\| = \sup_{k \in \mathbf{Z}} w_k w_{k+1} \cdots w_{k+n-1} = \sup_{k \in \mathbf{Z}} \beta_{k+n}\beta_k^{-1}.$$

The spectrum of W, as an operator from ℓ^p into ℓ^p, is an annulus centred at 0 in the complex plane. The outer radius R is equal to $\rho(W)$ the spectral radius of W, and so

$$R = \rho(W) = \lim_{n\to\infty} \|W^n\|^{1/n} = \lim_{n\to\infty} \left(\sup_{k\in\mathbf{Z}} \beta_{k+n}\beta_k^{-1}\right)^{1/n}.$$

The inner radius L of the annular spectrum of W is given by

$$L = \lim_{n\to\infty} \left(\inf_{k\in\mathbf{Z}} \beta_{k+n}\beta_k^{-1}\right)^{1/n}.$$

Since the weights w_k are bounded away from 0, W has a bounded inverse as an operator on ℓ^p. In fact,

$$(W^{-1}x)_n = w_n^{-1}x_{n+1}, \quad \text{for each } n \in \mathbf{Z} \text{ and for each } x \in S,$$

and $\|W^{-1}\| = (\inf_{k\in\mathbf{Z}} w_k)^{-1}$. Furthermore, L^{-1} is the spectral radius of W^{-1}, and so

$$L = \rho(W^{-1})^{-1} = \lim_{n\to\infty} \|W^{-n}\|^{-1/n}.$$

According to the spectral mapping formula, $\rho(W^n) = \rho(W)^n$ for any $n \geq 1$. Since $\|W^n\| \geq \rho(W^n)$, it follows that

$$(7) \qquad\qquad \sup_{k\in\mathbf{Z}} \beta_{n+k}\beta_k^{-1} = \|W^n\| \geq R^n \quad \text{for each } n \geq 0.$$

We are also interested in the behaviour of W when restricted to certain subspaces of S. Recall that a sequence $x = (x_n)$ is *causal* if it is supported on $[0, \infty)$, that is, $x_n = 0$ for all $n < 0$. The set of all causal sequences is denoted by S_+. More generally, for each subset A of $\mathbf{Z}$, S_A denotes the set of sequences supported on A. The set of sequences with finite support is denoted by S_{00}. We say a sequence x has *finite past* if $x \in S_{[k,\infty)}$ for some $k \in \mathbf{Z}$, and we denote the set of all finite past sequences by S_{fp}. Thus $S_+ = S_{[0,\infty)}$, and $S_{fp} = \bigcup_{k\in\mathbf{Z}} S_{[k,\infty)}$. We say that a subset X of S is *W-invariant* if $Wx \in X$ for each $x \in X$. Each of the subspaces $S_{[k,\infty)}$, S_{fp} and S_{00} is W-invariant. The restriction of W to S_+ is called a *unilateral weighted shift*, and is denoted by W_+. The norm of W_+ is given by

$$\|W_+\| = \sup_{k\geq 0} w_k = \sup_{k\geq 0} \beta_{k+1}\beta_k^{-1},$$

and its spectral radius $\rho(W_+) = R_+$ is given by

$$R_+ = \lim_{n\to\infty} \|W_+^n\|^{1/n} = \lim_{n\to\infty} (\sup_{k\geq 0} \beta_{k+n}\beta_k^{-1})^{1/n}.$$

As an operator on $\ell_+^p = \ell^p \cap S_+$, W_+ is not invertible, and its spectrum is the closed disc centred at the origin and with radius R_+. The analogue of (7) is the formula

$$(8) \qquad\qquad \sup_{k\geq 0} \beta_{n+k}\beta_k^{-1} = \|W_+^n\| \geq R_+^n \quad \text{for each } n \geq 0.$$

For the unweighted shift S, $R = R_+ = 1$. The numbers R and R_+ are important to the analysis of stability and continuity of weighted filters.

An operator T is *power bounded* if the numbers $\|T^n\|$, $n = 1, 2, 3, \ldots$, are bounded above. We shall say that T is *power dominated* if $T/\rho(T)$ is power bounded, where $\rho(T)$ is the spectral radius of T. Thus the weighted shift W is power dominated if the numbers $\|W^n\|R^{-n}$, $n = 1, 2, 3, \ldots$, are bounded above. Clearly the unweighted shift is power dominated.

1.2. *Some simplifications.* The following observations will simplify the analysis of weighted filters. The first concerns the indices p and q.

LEMMA 1. *If there is a filter for (5) which is (p, q)-stable on S_+ then $p \leq q$.*

PROOF. Suppose that T is a filter for (5) which is (p, q)-stable on S_+. Choose $t > s$ and $x \in \ell_+^p$. Write $x = \sum_{n \geq 0} x_n \delta^{(n)}$, where $\delta^{(n)}$ is the sequence with 1 in the $n'th$ place and 0 elsewhere, and let $x^\# = \sum_{n \geq 0} x_n \delta^{(tn)}$. Then $x^\# \in \ell_+^p$, and so

$$\|b(W)x^\#\|_q = \|a(W)Tx^\#\|_q \leq \|a(W)\| \, \|Tx^\#\|_q < \infty.$$

On the other hand, $b(W)x^\# = \sum_{j=0}^s b_j W^j x^\#$, and since the supports of $W^j x^\#$ for $j = 0, 1, 2 \ldots, s$ are disjoint,

$$\|b_j W^j x^\#\|_q \leq \|b(W)x^\#\|_q < \infty \quad \text{for each } j.$$

Since $b_s \neq 0$ and w is bounded below, it follows that

$$(9) \qquad \|x\|_q = \|x^\#\|_q \leq \|W\|^{-s}\|W^s x^\#\|_q \leq |b_s|^{-1}\|W\|^{-s}\|b_s W^s x^\#\|_q < \infty.$$

Since (9) holds for each $x \in \ell_+^p$, $p \leq q$.

The second lemma will allow us to restrict attention to cases in which the polynomials $a(\zeta)$ and $b(\zeta)$ are powers of a common linear polynomial.

LEMMA 2. *Suppose that X is a W-invariant linear subspace of S. Then there is a filter for (5) which is (p, q)-stable (bounded) on X if and only if, for each non-zero pole λ^{-1} of $r(\zeta)$, there is a filter for the equation*

$$(10) \qquad\qquad (1 - \lambda W)^{\mu+\nu} y = (1 - \lambda W)^\nu x$$

which is (p, q)-stable (bounded) on X, where μ is the multiplicity of the pole λ^{-1}, and where $\nu \geq 0$ is the multiplicity of λ^{-1} as a zero of $b(\zeta)$.

PROOF. Let $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_k^{-1}$ be the non-zero poles of $r(\zeta)$ with multiplicities $\mu_1, \mu_2, \ldots, \mu_k$ respectively. Each λ_i^{-1} is a zero of $a(\zeta)$ with multiplicity $\mu_i + \nu_i$ for some $\nu_i \geq 0$, and λ_i^{-1} is a zero of $b(\zeta)$ with multiplicity ν_i if $\nu_i > 0$.

Suppose that T is a filter for (5) which is (p, q)-stable on X. Choose any one of the poles λ_i^{-1} of $r(\zeta)$ and polynomials $u_i(\zeta)$ and $v_i(\zeta)$ such that

$$u_i(\zeta)(1 - \lambda_i\zeta)^{\mu_i+\nu_i} + v_i(\zeta)b(\zeta) = (1 - \lambda_i\zeta)^{\nu_i}.$$

Let $a_i(\zeta) = a(\zeta)(1 - \lambda_i\zeta)^{-\mu_i-\nu_i}$, and let

$$(11) \qquad\qquad\qquad T_i = u_i(W) + a_i(W)v_i(W)T.$$

Then, since X is W-invariant and T is (p, q)-stable on X, for each $x \in X \cap \ell^p$

$$(1 - \lambda_iW)^{\mu_i+\nu_i}T_ix = \left((1 - \lambda_iW)^{\mu_i+\nu_i}u_i(W) + v_i(W)a(W)T\right)x = (1 - \lambda_iW)^{\nu_i}x,$$

and so T_i is a filter for (10) corresponding to the pole λ_i^{-1}. Furthermore, if T is (p, q)-bounded on X then so too is T_i.

Now suppose that, for $i = 1, \ldots, k$, T_i is a filter for (10) corresponding to the pole λ_i^{-1} which is (p, q)-stable on X. Let $\tilde{a}_i(\zeta) = a(\zeta)(1 - \lambda_i\zeta)^{-\mu_i}$ for each i. Since the greatest common divisor of the polynomials $\tilde{a}_i(\zeta)$, $i = 1, 2, \ldots, k$, is also a divisor of $\zeta^\tau b(\zeta)$, for some $\tau \geq 0$, there are polynomials $c_1(\zeta), c_2(\zeta), \ldots, c_k(\zeta)$ such that

$$\zeta^\tau b(\zeta) = \sum_{i=1}^{k} c_i(\zeta)\tilde{a}_i(\zeta).$$

Let

$$(12) \qquad\qquad\qquad T = W^{-\tau}\sum_{i=1}^{k} c_i(W)T_i.$$

Then

$$a(W)Tx = W^{-\tau}\sum_{i=1}^{k} c_i(W)a(W)T_ix = W^{-\tau}\sum_{i=1}^{k} c_i(W)\tilde{a}_i(W)x = b(W)x$$

for each $x \in X \cap \ell^p$. So T is a filter for (5) which is (p, q)-stable on X. Furthermore, if each T_i is (p, q)-bounded on X then so too is T.

2. Finite past filters. There is a unique filter for (5) which leaves invariant the subspace S_{fp}. It can be defined in terms of a Laurent series of the transfer function $r(\zeta)$. Since $r(\zeta)$ is meromorphic, it has an expansion $\sum_{j=\rho}^{\infty} h_j\zeta^j$ which converges in a punctured neighbourhood of $\zeta = 0$. The operator $r(W)_+$, which is called the *finite past filter* for (5), is defined on S_{fp} by

$$(13) \qquad\qquad r(W)_+x = \sum_{j=\rho}^{\infty} h_jW^jx \quad \text{for each } x \in S_{fp}.$$

This sum converges pointwise for each $x \in S_{fp}$. In fact, if $x \in S_{[k,\infty)}$ then $r(W)_+x \in S_{[k+\rho,\infty)}$ and

$$(r(W)_+x)_n = \beta_n\sum_{j=\rho}^{n-k} h_j\beta_{n-j}^{-1}x_{n-j} \quad \text{for each } n \geq k + \rho.$$

The reason why $r(W)_+$ is a filter for (5) is best explained in terms of convolution products. For any two sequences u and v, the (weighted) convolution product $u * v$ is defined by

$$(14) \qquad\qquad (u * v)_n = \beta_n\sum_{i+j=n} u_iv_j\beta_i^{-1}\beta_j^{-1} \quad \text{for each } n \in \mathbf{Z},$$

provided that each of the sums in (14) converges absolutely.

The convolution product $u * v$ is not defined for all pairs of sequences u and v, but when it is, it is commutative and distributive over pointwise addition. It is not, in general, associative. However, the subspace S_{fp} is a field under the operations of convolution and pointwise addition, and it is not difficult to check that

$$a_{fp}^{-1} * b = h,$$

where $b = (b_n)$ is the sequence of coefficients of $b(\zeta)$, $h = (h_n)$ is the sequence of coefficients in the Laurent series for $r(\zeta)$, and where a_{fp}^{-1} is the unique finite past inverse of $a = (a_n)$, the coefficient sequence of $a(\zeta)$. (We define $a_n = 0$ if $n \notin [0, r]$, $b_n = 0$ if $n \notin [0, s]$, and $h_n = 0$ if $n < \rho$.) Therefore

$$r(W)_+ x = h * x \quad \text{for each } x \in S_{fp}.$$

It follows that $h = r(W)_+ \delta^{(0)}$, and for this reason h is known as the *impulse response* of the filter $r(W)_+$. Since (5) can be expressed as the convolution equation $a * y = b * x$, we have, for each $x \in S_{fp}$,

$$a * r(W)_+ x = a * h * x = a * a_{fp}^{-1} * b * x = b * x.$$

The uniqueness of $r(W)_+$ follows from the fact that if T is another filter for (5) with domain S_{fp}, then $\left(T - r(W)_+\right)x \in \ker a(W)$, the kernel of $a(W)$. It is easy to check that $\ker a(W) \cap S_{fp} = \{0\}$, and so if S_{fp} is invariant under T then $Tx = r(W)_+ x$ for all $x \in S_{fp}$.

2.1. *Stability on causal sequences.* In this section we study the (p, q)-stability and (p, q)-boundedness of $r(W)_+$ on S_+. It turns out that these are equivalent because of the special nature of $r(W)_+$. We say a sequence space operator T is *causal* if, for each integer k and each $x \in S_{[k,\infty)} \cap D(T)$, $Tx \in S_{[k,\infty)}$. The following automatic continuity result for causal sequence space operators is well known [1].

LEMMA 3.   *If C is a sequence space operator which is causal and (p, q)-stable on S_+, then C is (p, q)-bounded on S_+.*

COROLLARY 1.   *The finite past filter $r(W)_+$ for (5) is (p, q)-bounded on S_+ if and only if it is (p, q)-stable on S_+.*

PROOF.   Clearly W is causal, and since $W^{-\rho} r(W)_+$ is a linear combination of non-negative powers of W, it too is causal. So by Lemma 3, $W^{-\rho} r(W)_+$ is (p, q)-bounded on S_+. The same is true of $r(W)_+$, since $r(W)_+ = W^\rho W^{-\rho} r(W)_+$, and W^ρ is (q, q)-bounded.

We now obtain a necessary and sufficient condition for the (p, q)-boundedness on S_+ of the operator $r(W)_+$. First we assume that the rational function $r(\zeta)$ has a single pole, in which case $r(W)_+ = (1 - \lambda W)_+^{-\mu}$ for some $\mu > 0$.

LEMMA 4.   *The finite past operator $(1 - \lambda W)_+^{-\mu}$ is (p, q)-bounded on S_+ if and only if $1 \leq p \leq q \leq \infty$ and $|\lambda| R_+ < 1$, or $p = 1$, $q = \infty$, $\mu = 1$, $|\lambda| R_+ = 1$ and W_+ is power dominated.*

PROOF. First suppose that $(1 - \lambda W)_+^{-\mu}$ is (p, q)-bounded on $\mathcal{S}_+$. Since $(1 - \lambda W)_+^{-\mu}$ is a filter for the equation $(1 - \lambda W)^\mu y = x$, Lemma 1 implies that $p \leq q$.

Let τ denote the (p, q) norm of $(1 - \lambda W)_+^{-\mu}$ restricted to $\mathcal{S}_+$. Then for any $x \in \ell_+^p$,

$$(15) \qquad \|(1 - \lambda W)_+^{-\mu} x\|_q \leq \tau \|x\|_p.$$

Now $(1 - \lambda \zeta)^{-\mu} = \sum_{n=0}^{\infty} \binom{\mu+n-1}{n} \lambda^n \zeta^n$, where $\binom{n}{j}$ is the generalised binomial coefficient, defined for integers n and $j \geq 0$, by

$$\binom{n}{j} = \frac{n+1}{1} \times \frac{n+2}{2} \times \cdots \times \frac{n+j}{j} \quad \text{for } j > 0, \text{ and } \binom{n}{0} = 1.$$

So for each $x \in \mathcal{S}_{fp}$, $(1 - \lambda W)_+^{-\mu} x = \sum_{n=0}^{\infty} \binom{\mu+n-1}{n} \lambda^n W^n x$, and in particular

$$(16) \qquad (1 - \lambda W)_+^{-\mu} \delta^{(k)} = \sum_{n=0}^{\infty} \binom{\mu + n - 1}{n} \lambda^n \beta_{n+k} \beta_k^{-1} \delta^{(n+k)}$$

for each $k \geq 0$. So by (15), $\binom{\mu+n-1}{n} |\lambda|^n \beta_{n+k} \beta_k^{-1} \leq \tau$ for each $n \geq 0$ and $k \geq 0$. Therefore

$$(17) \qquad \binom{\mu+n-1}{n} |\lambda|^n R_+^n \leq \binom{\mu+n-1}{n} |\lambda|^n \|W_+^n\| \leq \tau$$

for each $n \geq 0$ by (8), and hence $|\lambda| R_+ \leq 1$.

Now suppose that $|\lambda| R_+ = 1$. Then (17) implies that $\mu = 1$, and hence

$$(18) \qquad |\lambda|^n \beta_{n+k} \beta_k^{-1} \leq |\lambda|^n \|W_+^n\| = \|W_+^n\| R_+^{-n} \leq \tau$$

for each $k \geq 0$ and each $n \geq 0$. So W_+ is power dominated. Furthermore, (16) becomes $(1 - \lambda W)_+^{-1} \delta^{(k)} = \sum_{n=0}^{\infty} \lambda^n \beta_{n+k} \beta_k^{-1} \delta^{(n+k)}$, and so by (15)

$$(19) \qquad \tau \geq \|(1 - \lambda W)_+^{-1} \delta^{(k)}\|_q \geq \left\| \sum_{n=0}^{m} \lambda^n \beta_{n+k} \beta_k^{-1} \delta^{(n+k)} \right\|_q$$

for each $m \geq 0$ and each $k \geq 0$. Now (18) implies that $|\lambda|^n \beta_{n+k} \beta_k^{-1} \geq \tau^{-1} |\lambda|^m \beta_{m+k} \beta_k^{-1}$ for $0 \leq n \leq m$. So by (19)

$$\tau \geq \tau^{-1} |\lambda|^m \beta_{m+k} \beta_k^{-1} \left\| \sum_{n=0}^{m} \delta^{(n+k)} \right\|_q = \tau^{-1} |\lambda|^m \beta_{m+k} \beta_k^{-1} (m+1)^{1/q}$$

for each $m \geq 0$ and each $k \geq 0$ (where $1/\infty = 0$). Therefore, since $|\lambda| R_+ = 1$, it follows from (8) that $\tau^2 \geq (m+1)^{1/q}$. Also since $(m+1)^{1/q} \to \infty$ as $m \to \infty$ if $q < \infty$, $q = \infty$.

Now choose $k \geq 0$, $m \geq 1$ and let

$$x = \frac{1}{m} \sum_{n=0}^{m-1} \lambda^n W^n \delta^{(k)} = \frac{1}{m} (1 - \lambda W)_+^{-1} (1 - \lambda^m W^m) \delta^{(k)}.$$

Then

$$(1 - \lambda W)_+^{-1} x = \sum_{n=0}^{m-1} \frac{n+1}{m} \lambda^n \beta_{n+k} \beta_k^{-1} \delta^{(n+k)} + \sum_{n=m}^{\infty} \lambda^n \beta_{n+k} \beta_k^{-1} \delta^{(n+k)},$$

and so $\|(1 - \lambda W)_+^{-1} x\|_\infty \geq |\lambda|^m \beta_{m+k} \beta_k^{-1}$. Since $|\lambda| R_+ = 1$, it follows from (8) that $\|(1 - \lambda W)_+^{-1} x\|_\infty \geq 1$. On the other hand,

$$\|x\|_p = \left\| \sum_{n=0}^{m-1} \frac{1}{m} \lambda^n \beta_{n+k} \beta_k^{-1} \delta^{(n+k)} \right\|_p \leq \tau \left\| \sum_{n=0}^{m-1} \frac{1}{m} \delta^{(m+k)} \right\|_p = \tau m^{-1+1/p}$$

by (18). So for each $m \geq 1$,

$$1 \leq \|(1 - \lambda W)_+^{-1} x\|_\infty \leq \tau \|x\|_p \leq \tau^2 m^{-1+1/p},$$

and since $m^{-1+1/p} \to 0$ as $m \to \infty$ if $p > 1$, it follows that $p = 1$.

To prove the converse, first suppose that $1 \leq p \leq q \leq \infty$ and that $|\lambda| R_+ < 1$. Then by (8)

$$\|(1 - \lambda W)_+^{-\mu}\|_p \leq \sum_{n=0}^{\infty} \binom{\mu + n - 1}{n} |\lambda|^n \|W_+^n\| < \infty.$$

So $(1 - \lambda W)_+^{-\mu}$ is (p, p)-bounded, and hence (p, q)-bounded, on S_+.

Now suppose that $|\lambda| R_+ = 1$ and that W_+ is power dominated. Let $\tau = \sup_{n \geq 0} \|W_+^n R_+^{-n}\|$. Then $\|(1 - \lambda W)_+^{-1} \delta^{(k)}\|_\infty = \sup_{n \geq 0} |\lambda|^n \beta_{n+k} \beta_k^{-1} \leq \sup_{n \geq 0} |\lambda|^n \|W_+^n\| = \tau$ for each $k \geq 0$. Hence for each $x = (x_n) \in \ell_+^1$,

$$\|(1 - \lambda W)_+^{-1} x\|_\infty \leq \sum_{n=0}^{\infty} |x_n| \|(1 - \lambda W)_+^{-1} \delta^{(n)}\|_\infty \leq \tau \|x\|_1.$$

So $(1 - \lambda W)_+^{-1}$ is $(1, \infty)$-bounded on S_+.

We turn now to the general case, where $r(\zeta)$ has possibly more than one non-zero pole. The following result is a modification of Lemma 2.

LEMMA 5. *Suppose that X is a W-invariant linear subspace of S. Then $r(W)_+$ is (p, q)-bounded on X if and only if, for each non-zero pole λ^{-1} of $r(\zeta)$ of multiplicity μ, $(1 - \lambda W)_+^{-\mu}$ is (p, q)-bounded on X.*

PROOF. First suppose that $r(W)_+$ is (p, q)-bounded on X, and that λ^{-1} is a pole of $r(\zeta)$ of multiplicity μ. Then the operator $T = v(W)a(W)r(W)_+ + u(W)$, as given by (11) in the proof of Lemma 2 is a filter for (10) which is (p, q)-bounded on X. But S_{fp} is invariant under T, and so by the uniqueness of finite past filters, $T = (1 - \lambda W)_+^{-\mu}$.

Conversely, if $(1 - \lambda W)_+^{-\mu}$ is (p, q)-bounded on X for each non-zero pole λ^{-1} of $r(\zeta)$ (with multiplicity μ), then the operator $T = W^{-\tau} \sum_{i=1}^{k} c_i(W)(1 - \lambda_i W)_+^{-\mu_i}$, as given by (12) in the proof of Lemma 2, is a filter for (5) which is (p, q)-bounded on X. Therefore, since T leaves S_{fp} invariant, $T = r(W)_+$.

Lemma 4 and Lemma 5 provide a necessary and sufficient condition for the (p, q)-boundedness of $r(W)_+$ on S_+.

THEOREM 1. *The finite past filter $r(W)_+$ is (p, q)-bounded on S_+ if and only if $1 \leq p \leq q \leq \infty$, the non-zero poles of $r(\zeta)$ lie on or outside the circle $|\zeta| = R_+$, and, if $r(\zeta)$ has poles on the circle $|\zeta| = R_+$, then each such pole is simple, $p = 1$, $q = \infty$, and W_+ is power dominated.*

PROOF. Suppose that $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_k^{-1}$ are the non-zero poles of $r(\zeta)$ with multiplicities $\mu_1, \mu_2, \ldots, \mu_k$ respectively. By Lemma 4, the conditions in the theorem are the conditions under which each of the single-pole factors $(1 - \lambda W)_+^{-\mu}$ of $r(W)_+$ is (p, q)-bounded on S_+, and by Lemma 5 this is both necessary and sufficient for the (p, q)-boundedness on S_+ of $r(W)_+$.

Since $R_+ = 1$ for the unweighted shift S_+, and S_+ is power dominated, it is easy to recover the known conditions [3] for stability of unweighted finite past filter from Theorem 1.

COROLLARY 2. *The unweighted finite past filter $r(S)_+$ is (p, q)-bounded on S_+ if and only if $1 \leq p \leq q \leq \infty$, the non-zero poles of $r(\zeta)$ lie on or outside the unit circle $|\zeta| = 1$, and, if $r(\zeta)$ has poles on the unit circle, then each such pole is simple, $p = 1$, and $q = \infty$.*

2.2. *Stability on finite past sequences.* The natural domain of the finite past filter $r(W)_+$ is S_{fp}, the linear space of all sequences with finite past. In this section we give necessary and sufficient conditions for the (p, q)-stability and boundedness of $r(W)_+$ on S_{fp},

LEMMA 6. *The finite past filter $r(W)_+$ is (p, q)-stable on S_{fp} if and only if it is (p, q)-stable on S_+.*

PROOF. Clearly stability on S_{fp} implies stability on the subspace S_+. For the converse, assume that $r(W)_+$ is (p, q)-stable on S_+, and that $x \in S_{[k, \infty)} \cap \ell^p$ for some $k \in \mathbf{Z}$. Then $W^{-k}x \in \ell_+^p$, and since $r(W)_+$ commutes with W,

$$r(W)_+x = r(W)_+ W^k W^{-k} x = W^k r(W)_+ W^{-k} x \in \ell^q.$$

THEOREM 2. *The finite past filter $r(W)_+$ is (p, q)-bounded on S_{fp} if and only if $1 \leq p \leq q \leq \infty$, the non-zero poles of $r(\zeta)$ lie on or outside the circle $|\zeta| = R$, and, if $r(\zeta)$ has poles on the circle $|\zeta| = R$, then each such pole is simple, $p = 1$, $q = \infty$, and W is power dominated.*

PROOF. The theorem can be proved by extending the range of k from $\mathbf{Z}^+$ to $\mathbf{Z}$ in the proofs of Lemma 4 and Theorem 1.

Necessary and sufficient conditions for the stability of unweighted finite past on S_{fp} follow immediately from Theorem 2.

COROLLARY 3. *The unweighted finite past filter $r(S)_+$ is (p, q)-bounded on S_{fp} if and only if $1 \leq p \leq q \leq \infty$, the non-zero poles of $r(\zeta)$ lie on or outside the unit circle $|\zeta| = 1$, and, if $r(\zeta)$ has poles on the unit circle, then each such pole is simple, $p = 1$, and $q = \infty$.*

Because the necessary and sufficient conditions in Theorems 1 and 2 are different, there is no automatic continuity result for arbitrary weighted finite past operators on S_{fp}.

EXAMPLE 1. Let W be the weighted shift with weights (w_n) given by

$$w_n = 1/2 \quad \text{for } n \geq 0 \text{ and } w_n = 2 \quad \text{for } n < 0.$$

Then $R = 2$ and $R_+ = 1/2$. So $(1 - W)_+^{-1}$ is (p, q)-stable on $\mathcal{S}_{fp}$ by Theorem 1 and Lemma 6. But $(1 - W)_+^{-1}$ is not (p, q)-bounded on $\mathcal{S}_{fp}$ by Theorem 2.

3. **Finite future filters.** Equation (13) defines the finite past operator $r(W)_+$ in terms of a Laurent series expansion of the rational function $r(\zeta)$. Other Laurent series expansions of $r(\zeta)$ determine other operators, which are also filters for (5) but on different domains. In particular, the expansion of $r(\zeta)$ in a neighbourhood of ∞ determines a filter which in a natural sense is a dual of $r(W)_+$.

We say that a sequence x is *anticausal* if it is supported on $(-\infty, 0]$, and that x has a *finite future* if $x \in \mathcal{S}_{(-\infty, k]}$ for some $k \in \mathbf{Z}$. We denote by $\mathcal{S}_-$ the set of all anticausal sequences, and by $\mathcal{S}_{ff}$ the set of all finite future sequences. Thus $\mathcal{S}_- = \mathcal{S}_{(-\infty, 0]}$, and $\mathcal{S}_{ff} = \bigcup_{k \in \mathbf{Z}} \mathcal{S}_{(-\infty, k]}$. The *finite future operator* $r(W)_-$ is defined on $\mathcal{S}_{ff}$ by

$$r(W)_- x = \sum_{j=r-s}^{\infty} g_j W^{-j} x \quad \text{for each } x \in \mathcal{S}_{ff},$$

where $\sum_{j=r-s}^{\infty} g_j \zeta^{-j}$ is the Laurent series expansion of the transfer function $r(\zeta)$, which is valid in a neighbourhood of $\zeta = \infty$. This expansion converges pointwise for each $x \in \mathcal{S}_{ff}$. In fact, if $x \in \mathcal{S}_{(-\infty, k]}$ then $r(W)_- x \in \mathcal{S}_{(-\infty, k-r+s]}$, and

$$\left(r(W)_- x\right)_n = \beta_n \sum_{j=r-s}^{k-n} g_j \beta_{n+j}^{-1} x_{n+j} \quad \text{for each } n \leq k - r + s.$$

It is not difficult to verify that

$$r(W)_- x = g * x \quad \text{for each } x \in \mathcal{S}_{ff},$$

where $g = a_{ff}^{-1} * b$, and where a_{ff}^{-1} is the unique finite future inverse of a in the field $\mathcal{S}_{ff}$. So

$$a * r(W)_- x = a * g * x = a * a_{ff}^{-1} * b * x = b * x \quad \text{for each } x \in \mathcal{S}_{ff},$$

and hence $r(W)_-$ is a finite future filter for (5).

The uniqueness of $r(W)_-$ as a finite future filter for (5) follows from the fact that $\ker a(W) \cap \mathcal{S}_{ff} = \{0\}$.

The analysis of $r(W)_-$ is simplified by a natural correspondence between finite past and finite future sequences. Let H be the *reversal mapping*, defined on $\mathcal{S}$ by

$$(Hx)_n = x_{-n} \quad \text{for each } n \in \mathbf{Z} \text{ and each } x = (x_n) \in \mathcal{S}.$$

The map H is isometric on each ℓ^p space, $H^2 = 1$, and $H\mathcal{S}_{fp} = \mathcal{S}_{ff}$.

Let $\tilde{W} = HW^{-1}H$. Then $\tilde{W}$ is a weighted shift whose weight sequence $(\tilde{w}_n)$ is given by

$$\tilde{w}_n = w_{-n-1}^{-1} \quad \text{for each } n \in \mathbf{Z}.$$

Let $\tilde{a}$ and $\tilde{b}$ be the polynomials defined by

$$\tilde{a}(\zeta) = \zeta^{r+s} a(\zeta^{-1}) \quad \text{and} \quad \tilde{b}(\zeta) = \zeta^{r+s} b(\zeta^{-1}),$$

and let

$$\tilde{r}(\zeta) = \tilde{b}(\zeta) \big/ \tilde{a}(\zeta) = r(\zeta^{-1}).$$

It is easy to check that

$$\tilde{a}(\tilde{W}) H r(W)_{-} H x = H\tilde{a}(W^{-1}) r(W)_{-} H x = HW^{-r-s} a(W) r(W)_{-} H x$$
$$= HW^{-r-s} b(W) H x = \tilde{b}(\tilde{W})x,$$

for each $x \in S_{fp}$, and so $H r(W)_{-} H$ is a filter for the equation

(20) $$\tilde{a}(\tilde{W})y = \tilde{b}(\tilde{W})x.$$

Furthermore S_{fp} is invariant under $H r(W)_{-} H$, and so by the uniqueness of finite past filters, $H r(W)_{-} H = \tilde{r}(\tilde{W})_{+}$, the finite past filter for (20).

Since H is an isometry on each ℓ^p, the stability and boundedness properties of $r(W)_{-}$ on S_{-} and on S_{ff} match those of $\tilde{r}(\tilde{W})_{+}$ on S_{+} and on S_{fp}, respectively, and by the results of the previous section these are determined by the location of the non-zero poles of $\tilde{r}(\zeta)$ relative to the spectra of $\tilde{W}_{+}$ and $\tilde{W}$. The spectrum of $\tilde{W}$, as an operator from ℓ^p into ℓ^p, is an annulus centred at 0 in the complex plane. The outer radius $\tilde{R}$ is given by

$$\tilde{R} = \lim_{n \to \infty} \left(\sup_{k \in \mathbf{Z}} \beta_{k-n} \beta_k^{-1} \right)^{1/n} = \lim_{n \to \infty} \left(\inf_{k \in \mathbf{Z}} \beta_k \beta_{k-n}^{-1} \right)^{-1/n} = L^{-1}.$$

Similarly, the spectrum of $\tilde{W}_{+}$ is a disc centred at the origin, whose radius $\tilde{R}_{+}$ is given by

$$\tilde{R}_{+} = \lim_{n \to \infty} \left(\sup_{k \leq 0} \beta_{k-n} \beta_k^{-1} \right)^{1/n} = \lim_{n \to \infty} \left(\inf_{k \leq 0} \beta_k \beta_{k-n}^{-1} \right)^{-1/n} = L_{-}^{-1}$$

Clearly ζ_0 is a pole of multiplicity ν of $\tilde{r}(\zeta)$ if and only if ζ_0^{-1} is a pole of multiplicity ν of $r(\zeta)$. So we have the following necessary and sufficient conditions for the (p,q)-boundedness of the finite future filter $r(W)_{-}$ on S_{-} and S_{ff}.

THEOREM 3. *The finite future filter $r(W)_{-}$ is (p,q)-bounded on S_{-} if and only if $1 \leq p \leq q \leq \infty$, the poles of $r(\zeta)$ lie on or inside the circle $|\zeta| = L_{-}$, and, if $r(\zeta)$ has poles on the circle $|\zeta| = L_{-}$, then each such pole is simple, $p = 1$, $q = \infty$, and W_{-}^{-1} is power dominated.*

THEOREM 4. *The finite future filter $r(W)_{-}$ is (p,q)-bounded on S_{ff} if and only if $1 \leq p \leq q \leq \infty$, the poles of $r(\zeta)$ lie on or inside the circle $|\zeta| = L$, and, if $r(\zeta)$ has poles on the circle $|\zeta| = L$, then each such pole is simple, $p = 1$, $q = \infty$, and W^{-1} is power dominated.*

4. **Two-sided filters.** In the previous two sections we have studied the behaviour of the 'one-sided' finite past and finite future filters for (5). We turn now to the problem of finding stable *two-sided* filters for (5), that is, filters whose domains include ℓ^p for some $p \geq 1$. In view of Lemma 2, we may restrict attention to the case in which $r(\zeta)$ has a single pole, and so we look for stable two-sided filters for equation (10).

One way of obtaining such a filter is by using continuity to extend the domain of either $(1 - \lambda W)_+^{-\mu}$ or $(1 - \lambda W)_-^{-\mu}$. Suppose, for example, that $(1 - \lambda W)_+^{-\mu}$ is (p, q)-bounded on S_{fp}. (By Theorem 1 this will occur if $|\lambda^{-1}| > R$, or possibly if $|\lambda^{-1}| = R$ in the exceptional $(1, \infty)$ case). If $p < \infty$, then $S_{fp} \cap \ell^p$ is dense in ℓ^p, and so there is a unique (p, q)-bounded extension of $(1 - \lambda W)_+^{-\mu}$ which is a filter for (10) and whose domain includes ℓ^p. On the other hand, if $(1 - \lambda W)_+^{-\mu}$ is (∞, ∞)-bounded on S_{fp}, then $|\lambda^{-1}| > R$ by Theorem 1, and so the series for $(1 - \lambda W)_+^{-\mu}$ converges in operator norm and defines a filter for (10) whose domain includes ℓ^∞.

Similarly, if $(1 - \lambda W)_-^{-\mu}$ is (p, q)-bounded on S_{ff} then its domain can be extended to include ℓ^p. (This will occur if $|\lambda^{-1}| < L$, or possibly if $|\lambda^{-1}| = L$ in the exceptional $(1, \infty)$ case).

If $L < |\lambda^{-1}| < R$, then $(1 - \lambda W)_+^{-\mu}$ and $(1 - \lambda W)_-^{-\mu}$ do not have bounded extensions. However it may still be possible to construct a (p, q)-bounded filter for (10) using both $(1 - \lambda W)_+^{-\mu}$ and $(1 - \lambda W)_-^{-\mu}$. Suppose that $L \leq R_+ < |\lambda^{-1}| < L_- \leq R$. Then $(1 - \lambda W)_+^{-\mu}$ and $(1 - \lambda W)_-^{-\mu}$ are (p, q)-stable on S_+ and S_- respectively, and we can define

$$T_0 = (1 - \lambda W)_+^{-\mu} P_+ + (1 - \lambda W)_-^{-\mu} P_-,$$

where P_+ is the projection of S onto S_+ defined, for any $x \in S$, by $(P_+ x)_n = x_n$ if $n \geq 0$ and $(P_+ x)_n = 0$ if $n < 0$, and where $P_- = 1 - P_+$. Then $P_+ x \in \ell_+^p$ for any $x \in \ell^p$, and because $(1 - \lambda W)_+^{-\mu}$ is (p, q)-stable on S_+, it follows that $(1 - \lambda W)_+^{-\mu} P_+ x \in \ell^q$. Similarly, $(1 - \lambda W)_-^{-\mu} P_- x \in \ell^q$. Furthermore, since $(1 - \lambda W)^{\mu+\nu}(1 - \lambda W)_+^{-\mu} P_+ x = (1 - \lambda W)^\nu P_+ x$ and $(1 - \lambda W)^{\mu+\nu}(1 - \lambda W)_-^{-\mu} P_- x = (1 - \lambda W)^\nu P_- x$, it follows that

$$(1 - \lambda W)^{\mu+\nu} T_0 x = (1 - \lambda W)^\nu (P_+ x + P_- x) = (1 - \lambda W)^\nu x.$$

So T_0 is a (p, q)-stable filter for (10). In fact, since $(1 - \lambda W)_+^{-\mu}$ and $(1 - \lambda W)_-^{-\mu}$ are automatically (p, q)-bounded on S_+ and S_- respectively, T_0 is (p, q)-bounded (on S).

For each $j \geq 1$ and each complex number λ, we denote by $\Phi(j, \lambda)$ the sequence whose n'th term is $\binom{n}{j-1} \beta_n \lambda^n$. The sequences $\Phi(j, \lambda), j = 1, 2, \ldots, \mu$ form a basis of $\ker(1 - \lambda W)^\mu$, and

$$(1 - \lambda W)\Phi(j, \lambda) = \Phi(j - 1, \lambda) \quad \text{for } j > 1 \text{ and } (1 - \lambda W)\Phi(1, \lambda) = 0.$$

There is also a simple connection between the sequence $\Phi(\mu, \lambda)$ and the impulse responses of $(1 - \lambda W)_+^{-\mu}$ and $(1 - \lambda W)_-^{-\mu}$. Let $\Phi_+(\mu, \lambda) = P_+ \Phi(\mu, \lambda)$, and let $\Phi_-(j, \lambda) = P_- \Phi(j, \lambda)$. Then

$$\Phi_+(j, \lambda) = (1 - \lambda W)_+^{-j} \delta^{(0)} \quad \text{and} \quad \Phi_-(j, \lambda) = -(1 - \lambda W)_-^{-j} \delta^{(0)}.$$

EXAMPLE 2. Suppose that $w_n = 1/2$ for all $n \geq 0$ and $w_n = 2$ for all $n < 0$, as in Example 1. Then $R = R_- = L_- = 2$ and $L = L_+ = R_+ = 1/2$, and the spectrum of W, as a bounded linear operator from ℓ^p into ℓ^p, is the closed annulus $\{\zeta : 1/2 \leq |\zeta| \leq 2\}$. For any $1 \leq p \leq q \leq \infty$, $(1 - W)_+^{-1}$ is not (p, q)-bounded on $\mathcal{S}_{fp}$ and $(1 - W)_-^{-1}$ is not (p, q)-bounded on $\mathcal{S}_{ff}$. However

$$(21) \qquad\qquad T_0 = (1 - W)_+^{-1} P_+ + (1 - W)_-^{-1} P_-$$

is a two-sided (p, q)-bounded filter for the equation $(1 - W)y = x$. Furthermore, if T is any other (p, q)-stable filter for this equation, then $Tx = T_0 x + \psi(x)\Phi(1, 1)$, for some linear functional ψ on ℓ^p.

We now show that the three constructions of a (p, q)-stable filter for (10) just given essentially exhaust the possibilities. For this we need some preliminary results. The first two are extensions of Lemma 3. Proofs based upon the closed graph theorem can be found in [1].

LEMMA 7. *Suppose that C is causal on $\mathcal{S}_+$, and that for each $x \in \ell_+^p$, $Cx \in \ell^q + E$, where E is a finite-dimensional subspace of $\mathcal{S}$. Then C is (p, q)-stable on $C^{-1}\ell^q = \{x \in \ell_+^p : Cx \in \ell^q\}$.*

LEMMA 8. *Suppose that C is causal and T is (p, q)-stable on $\mathcal{S}_+$, and suppose that $T - C$ has finite rank. Then if C is (p, q)-stable on $\mathcal{S}_+ \cap \mathcal{S}_{00}$, then C is (p, q)-bounded on $\mathcal{S}_+$.*

The third result concerns the boundedness of $(1 - \lambda W)_+^{-\mu}$ and $(1 - \lambda W)_-^{-\mu}$. It shows that, in the presence of a (p, q)-stable filter for (10), $(1 - \lambda W)_+^{-\mu}$ is (p, q)-bounded on $\mathcal{S}_+$ if and only if its impulse response is in ℓ^q, and similarly for $(1 - \lambda W)_-^{-\mu}$.

LEMMA 9. *Suppose that (10) has a (p, q)-stable filter. Then $(1 - \lambda W)_+^{-\mu}$ is (p, q)-bounded on $\mathcal{S}_+$ if and only $(1 - \lambda W)_+^{-\mu}\delta^{(0)} \in \ell^q$, and $(1 - \lambda W)_-^{-\mu}$ is (p, q)-bounded on $\mathcal{S}_-$ if and only if $(1 - \lambda W)_-^{-\mu}\delta^{(0)} = \Phi_-(\mu, \lambda) \in \ell^q$.*

PROOF. Clearly $(1 - \lambda W)_+^{-\mu}\delta^{(0)} = \Phi_+(\mu, \lambda) \in \ell^q$ if $(1 - \lambda W)_+^{-\mu}$ is (p, q)-bounded on $\mathcal{S}_+$. So suppose that T is a (p, q)-stable filter for (10) and that $\Phi_+(\mu, \lambda) \in \ell^q$. Then $(1 - \lambda W)_+^{-\mu} g(W)\delta^{(0)} = g(W)\Phi_+(\mu, \lambda) \in \ell^q$ for each polynomial $g(\zeta)$. So $(1 - \lambda W)_+^{-\mu}$ is (p, q)-stable on $\mathcal{S}_+ \cap \mathcal{S}_{00}$. Now T and $(1 - \lambda W)_+^{-\mu}$ are both filters for (10), and so $(T - (1 - \lambda W)_+^{-\mu})x \in \ker(1 - \lambda W)^{\mu+\nu}$, for each $x \in \ell_+^p$. Since $\ker(1 - \lambda W)^{\mu+\nu}$ is finite-dimensional, it follows from Lemma 8 that $(1 - \lambda W)_+^{-\mu}$ is (p, q)-bounded on $\mathcal{S}_+$.

Similar arguments work for $(1 - \lambda W)_-^{-\mu}$.

We are now able to prove the general result concerning the existence of (p, q)-stable filters for (5).

THEOREM 5. *There is a (p, q)-stable filter for (5) if and only if, for each non-zero pole λ^{-1} with multiplicity μ of the transfer function $r(\zeta)$, either*

1. *$(1 - \lambda W)_+^{-\mu}$ is (p, q)-bounded on $\mathcal{S}_{fp}$, or*
2. *$(1 - \lambda W)_-^{-\mu}$ is (p, q)-bounded on $\mathcal{S}_{ff}$, or*
3. *$(1 - \lambda W)_+^{-\mu}$ is (p, q)-bounded on $\mathcal{S}_+$ and $(1 - \lambda W)_-^{-\mu}$ is (p, q)-bounded on $\mathcal{S}_-$.*

PROOF. It follows from Lemma 2 that it is sufficient to prove the theorem for filters for equation (10). So suppose that T is a (p, q)-stable filter for (10) and that $(1 - \lambda W)_+^{-\mu}$ is not (p, q)-bounded on S_+. We must show that $(1 - \lambda W)_-^{-\mu}$ is (p, q)-bounded on S_{ff}.

Since $(1 - \lambda W)_+^{-\mu}$ is not (p, q)-bounded on S_+, $\Phi_+(\mu, \lambda) \notin \ell^q$ by Lemma 9, and $|\lambda R_+| \geq 1$ by Lemma 4. Suppose that $\Phi_+(1, \lambda) \in \ell^q$. Then by the same lemmas $(1 - \lambda W)_+^{-1}$ is (p, q)-bounded on S_+ and $|\lambda R_+| \leq 1$. So $|\lambda R_+| = 1$, and $\mu = 1$ by Lemma 4. But this is a contradicts $\Phi_+(\mu, \lambda) \notin \ell^q$, and so $\Phi_+(1, \lambda) \notin \ell^q$.

Since $T\delta^{(0)} - \Phi_+(\mu, \lambda) = (T - (1 - \lambda W)_+^{-\mu})\delta^{(0)} \in \ker(1 - \lambda W)^{\mu+\nu}$, there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_{\mu+\nu}$ such that $T\delta^{(0)} = \Phi_+(\mu, \lambda) + \sum_{j=1}^{\mu+\nu} c_j \Phi(j, \lambda)$. Applying the projection P_+ gives

$$P_+ T\delta^{(0)} = \Phi_+(\mu, \lambda) + \sum_{j=1}^{\mu+\nu} c_j \Phi_+(j, \lambda) \in \ell^q,$$

and since the sequences $\Phi_+(j, \lambda)$, $j = 1, 2, \ldots, \mu + \nu$ are linearly independent over ℓ^q, it follows that $c_j = 0$ for $j \neq \mu$ and $c_\mu = -1$. Therefore, $T\delta^{(0)} = \Phi_+(\mu, \lambda) - \Phi(\mu, \lambda) = -\Phi_-(\mu, \lambda)$. So by Lemma 9, $(1 - \lambda W)_-^{-\mu}$ is (p, q)-bounded on S_-.

It remains to be shown that $(1 - \lambda W)_-^{-\mu}$ is (p, q)-bounded on $S_+ \cap S_{ff}$. Let $S_{0+} = S_+ \cap S_{ff} = S_+ \cap S_{00}$. Then $S_{0+} = S_{[0,\mu)} \oplus (1 - \lambda W)^\mu S_{0+}$. We shall show that $(1 - \lambda W)_-^{-\mu}$ is (p, q)-bounded on $S_{[0,\mu)}$ and on $(1 - \lambda W)^\mu S_{0+}$, and that the projection of S_{0+} onto $S_{[0,\mu)}$ along $(1 - \lambda W)S_{0+}$ is (p, p)-bounded.

Since $(1 - \lambda W)_-^{-\mu}\delta^{(k)} \in \ell^q$ for $k = 0, 1, \ldots, \mu - 1$, $(1 - \lambda W)_-^{-\mu}$ is (p, q)-bounded on $S_{[0,\mu)}$. For each $u \in S_{0+}$,

$$(22) \qquad (1 - \lambda W)_-^{-\mu}(1 - \lambda W)^\mu u = (1 - \lambda W)_+^{-\mu}(1 - \lambda W)^\mu u = u \in \ell^q.$$

By Lemma 7 $(1 - \lambda W)_+^{-\mu}$ is (p, q)-bounded on $X_+ = \{x \in \ell_+^p : (1 - \lambda W)_+^{-\mu} x \in \ell^q\}$. Equation (22) shows that $(1 - \lambda W)^\mu S_{0+} \subseteq X_+$, and since $(1 - \lambda W)_-^{-\mu}$ and $(1 - \lambda W)_+^{-\mu}$ agree on $(1 - \lambda W)^\mu S_{0+}$, it follows that $(1 - \lambda W)_-^{-\mu}$ is (p, q)-bounded on $(1 - \lambda W)^\mu S_{0+}$.

Each $x \in S_{0+}$ uniquely determines polynomials $g(\zeta)$ and $h(\zeta)$ such that

$$x = \big((1 - \lambda W)^\mu g(W) + h(W)\big)\delta^{(0)},$$

where $\deg h(\zeta) < \mu$. Let E be the projection of S_{0+} onto $S_{[0,\mu)}$ along $(1 - \lambda W)^\mu S_{0+}$ defined by

$$Ex = E\big((1 - \lambda W)^\mu g(W) + h(W)\big)\delta^{(0)} = h(W)\delta^{(0)}.$$

Suppose that E is not (p, p)-bounded on S_{0+}. Then there are sequences $x^{(n)}$, $n = 1, 2, 3, \ldots$, in S_{0+} such that $\|Ex^{(n)}\|_p = 1$ for each n, and $\|x^{(n)}\|_p \to 0$ as $n \to \infty$. Since $S_{[0,\mu)}$ is finite-dimensional we may assume, by taking a subsequence if necessary, that $Ex^{(n)} \to y \in S_{[0,\mu)}$, where $\|y\|_p = 1$. Write $x^{(n)} = \big((1 - \lambda W)^\mu g^{(n)}(W) + h^{(n)}(W)\big)\delta^{(0)}$ and $y = \sum_{j=1}^{\mu} c_j(1 - \lambda W)^{\mu-j}\delta^{(0)}$. Then

$$(23) \qquad \|(1 - \lambda W)^\mu g^{(n)}(W)\delta^{(0)} + y\|_p \to 0 \quad \text{as } n \to \infty,$$

and so $g^{(n)}(W)\delta^{(0)} + (1 - \lambda W)_+^{-\mu}y \to 0$ pointwise as $n \to \infty$.

Now $(1-\lambda W)_+^{-\mu}y = \sum_{j=1}^{\mu} c_j\Phi_+(j,\lambda) \neq 0$, and since the $\Phi_+(j,\lambda)$, for $j = 1, 2, \ldots, \mu$, are linearly independent over ℓ^q, it follows that $(1-\lambda W)_+^{-\mu}y \notin \ell^q$. So $\|g^{(n)}(W)\delta^{(0)}\|_q \to \infty$ as $n \to \infty$. But $g^{(n)}(W)\delta^{(0)} = (1-\lambda W)_+^{-\mu}(1-\lambda W)^\mu g^{(n)}(W)\delta^{(0)}$, and since $(1-\lambda W)_+^{-\mu}$ is (p,q)-bounded on $(1-\lambda W)^\mu \mathcal{S}_{0+}$, $\|(1-\lambda W)^\mu g^{(n)}(W)\delta^{(0)}\|_p \to \infty$. Since this contradicts (23), we conclude that E is (p,p)-bounded and the proof of the theorem is complete.

EXAMPLE 3. Suppose that $w_n = 1/2$ for all $n \geq 0$ and $w_n = 2$ for all $n < 0$, as in Examples 1 and 2, let $a(\zeta) = (1-3\zeta)(1-\zeta)(1-\zeta/3)$ and let $b(\zeta) = 16$. Then the transfer function $r(\zeta)$ has poles at $1/3$, 1, and 3. Since $r(\zeta) = 27(1-3\zeta)^{-1} - 12(1-\zeta)^{-1} + (1-\zeta/3)^{-1}$, the operator T defined by

$$T = 27(1-3W)_-^{-1} - 12T_0 + (1-W/3)_+^{-1},$$

where T_0 is as given in (21), is a filter for

(24) $$(1-3W)(1-W)(1-W/3)y = 16x$$

which, for any $1 \leq p \leq q \leq \infty$, is (p,q)-bounded on $\mathcal{S}_{00}$. Since $(1-W/3)_+^{-1}$ and $(1-3W)_-^{-1}$ have bounded extensions to ℓ^p, and since T_0 is (p,q)-bounded on ℓ^p, T has an extension to ℓ^p which is a two-sided (p,q)-bounded filter for (24).

REMARK. The conditions given in Theorem 5 for the existence of a (p,q)-stable filter for (5) can be expressed in terms of the positions of the non-zero poles of the transfer function. For $1 \leq p \leq q \leq \infty$ and $(p,q) \neq (1,\infty)$, there is a (p,q)-stable filter for (5) if and only if each non-zero pole λ^{-1} satisfies one of the inequalities $|\lambda^{-1}| < L$, $|\lambda^{-1}| > R$, or $R_+ < |\zeta| < L_-$. The first and second of these inequalities place λ^{-1} outside the spectrum of W. However the third possibility, namely that $R_+ < |\zeta| < L_-$, allows for a (p,q)-stable filter even if the transfer function has poles inside the spectrum.

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Department of Mathematics and Statistics
Murdoch University
Murdoch, WA 6150
Australia
e-mail: harrison@prodigal.murdoch.edu.au
e-mail: ward@prodigal.murdoch.edu.au

Department of Mathematics and Statistics
University of Canterbury
Christchurch
New Zealand
e-mail: L.Eaton@math.canterbury.ac.nz