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Jump phenomena of the n-th eigenvalue of discrete Sturm–Liouville problems with application to the continuous case

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In this paper, we characterize jump phenomena of the n-th eigenvalue of self-adjoint discrete Sturm–Liouville problems in any dimension. For a fixed Sturm–Liouville equation, we completely characterize jump phenomena of the n-th eigenvalue. For a fixed boundary condition, unlike in the continuous case, the n-th eigenvalue exhibits jump phenomena and we describe the singularity under a non-degenerate assumption. Compared with the continuous case in Hu *et al.* (2019, *J. Differ. Equ.* **266**, 4106–4136) and Kong *et al.* (1999, *J. Differ. Equ.* **156**, 328–354), the jump set here is involved with coefficients of the Sturm–Liouville equations. This, along with arbitrariness of the dimension, causes difficulty when dividing the jump areas. We study the singularity by partitioning and analysing the local coordinate systems, and provide a Hermitian matrix which can determine the areas' division. To prove the asymptotic behaviour of the n-th eigenvalue, we generalize the method developed in Zhu and Shi (2016, *J. Differ. Equ.* **260**, 5987–6016) to any dimension. As an application, by transforming the continuous Sturm–Liouville problem of Atkinson type to a discrete one, we determine the number of eigenvalues and obtain complete characterization of jump phenomena of the n-th eigenvalue for the Atkinson type.

Keywords: Sturm–Liouville problem; the n-th eigenvalue; jump; asymptotic behaviour

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1. Introduction

Sturm–Liouville problems in the discrete version come from several physical models, including the vibrating string and random walk with discrete time process [**4**, **9**]. We briefly introduce these two models. Suppose that a weightless string bears l particles with masses $m_1, \dots, m_l$, and the horizontal distance between m_i and m_{i+1} is $1/c_i$, $1 \leqslant i \leqslant l-1$. Moreover, the string extends to length $1/c_l$ beyond m_l and

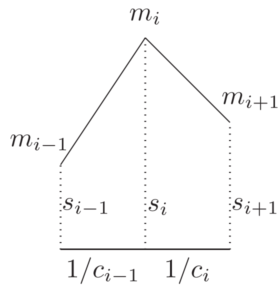

Figure 1. Vibrating string.

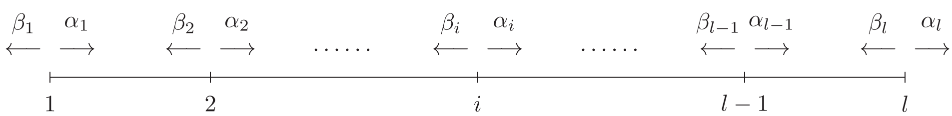

Figure 2. Random walk with discrete time process.

$1/c_0$ beyond m_1. Let s_i, $1 \leqslant i \leqslant l$, be the displacement of the particle m_i at a fixed time. Both ends are pinned down (i.e. $s_0 = s_{l+1} = 0$). Since the particle m_i does not move horizontally, we may assume that the horizontal component of the tension at m_i is both unit from the left and right, respectively. Then the restoring forces, induced by the vertical component of the tension from the left and right, are $c_{i-1}(s_i - s_{i-1})$ and $c_i(s_i - s_{i+1})$, respectively. See Fig. 1.

Therefore, by Newton's second law,

$$-m_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} s_i = c_{i-1}(s_i - s_{i-1}) + c_i(s_i - s_{i+1}) = -\nabla(c_i \Delta s_i), \quad 1 \leqslant i \leqslant l, \tag{1.1}$$

where $\Delta s_i = s_{i+1} - s_i$ and $\nabla s_i = s_i - s_{i-1}$. Taking $s_i = y_i \cos(\omega t)$, where y_i is the amplitude of m_i, we obtain from (1.1) that

$$-\nabla(c_i \Delta y_i) = \lambda m_i y_i, \quad 1 \leqslant i \leqslant l,$$

where $\lambda = \omega^2$. Since the boundary condition corresponds to the assumption that both ends are pinned down, this system becomes a self-adjoint discrete Sturm–Liouville problem.

Another model of the discrete Sturm–Liouville equation is random walking with discrete time process from probability theory.

Let a particle be in one of the l positions $1, \ldots, l$ at $t = 0$. Suppose that the particle is in position i at $t = t_0$. The rule of this random walking is that the particle will move to $i+1$ at $t = t_0 + 1$ with a probability α_i, move to $i-1$ at $t = t_0 + 1$ with a probability β_i, and stay in position i with a probability $1 - \alpha_i - \beta_i$. See Fig. 2. Moreover, if the particle moves to the left of position 1, or to the right of position l, it is considered permanently lost. So it is reasonable to set $\alpha_0 = 0$ and $\beta_{l+1} = 0$. Define $p_{rs}(j)$ as the probability of the particle being in position s at $t = j$

and starting in position r at $t = 0$. Then we have $p_{rs}(0) = \delta_{rs}$, and

$$p_{rs}(j+1) = \alpha_{s-1}p_{r,s-1}(j) + \beta_{s+1}p_{r,s+1}(j) + (1-\alpha_s-\beta_s)p_{rs}(j), \quad j \geqslant 0, \tag{1.2}$$

where $\delta_{rs} = 1$ if $r = s$, and $\delta_{rs} = 0$ if $r \neq s$. Let $P(j) = (p_{rs}(j))_{1\leqslant r,s\leqslant l}$, $j \geqslant 0$, and

$$T = \begin{pmatrix} -\alpha_1-\beta_1 & \alpha_1 & & & \\ \beta_2 & -\alpha_2-\beta_2 & \alpha_2 & & \\ & \beta_3 & -\alpha_3-\beta_3 & \ddots & \\ & & \ddots & \ddots & \alpha_{l-1} \\ & & & \beta_l & -\alpha_l-\beta_l \end{pmatrix}.$$

Then $P(0) = I_l$ and (1.2) is equivalent to

$$P(j+1) = P(j)(I_l+T),\ j \geqslant 0.$$

So $P(j+1) = (I_l+T)^j$, $j \geqslant 0$. However, the form $(I_l+T)^j$ provides little information on asymptotic form of $P(j)$ for large j. Instead, in the spectral theory, the eigenvalues and corresponding eigenfunctions of T play an important role in studying properties of $P(j)$ for any $j \geqslant 1$. To find an eigenvalue λ and the corresponding eigenfunction $(y_1, y_2, \ldots, y_l)$ of $-T$, we need to study the self-adjoint discrete Sturm–Liouville equation

$$-\nabla(g_j\Delta y_j) = \lambda a_j y_j, \quad 1 \leqslant j \leqslant l,$$

with the boundary condition $y_0 = y_{l+1} = 0$, where $g_j = \alpha_j a_j$ and $g_{j-1} = \beta_j a_j$.

Motivated by these two interesting models and recent interest on discrete equations [**5**, **6**, **9**], in this paper we consider a general self-adjoint discrete d-dimensional Sturm–Liouville problem for any $d \geqslant 1$. It consists of a symmetric discrete Sturm–Liouville equation

$$-\nabla(P_i\Delta y_i) + Q_i y_i = \lambda W_i y_i, \quad 1 \leqslant i \leqslant N, \tag{1.3}$$

and a self-adjoint boundary condition

$$A\begin{pmatrix} -y_0 \\ y_N \end{pmatrix} + B\begin{pmatrix} P_0\Delta y_0 \\ P_N\Delta y_N \end{pmatrix} = 0, \tag{1.4}$$

where $\Delta y_i = y_{i+1} - y_i$, $\nabla y_i = y_i - y_{i-1}$, $y = \{y_i\}_{i=0}^{N+1}$ is a sequence of d-dimensional complex-valued vectors; $P = \{P_j\}_{j=0}^{N}$, $Q = \{Q_i\}_{i=1}^{N}$ and $W = \{W_i\}_{i=1}^{N}$ are sequences of $d \times d$ complex-valued matrices and satisfy

$$P_j, Q_i, W_i \text{ are Hermitian } P_j \text{ is invertible, } W_i \text{ positive definite,} \tag{1.5}$$

$0 \leqslant j \leqslant N$; $\lambda \in \mathbb{C}$ is the spectral parameter, $N \geqslant 2$; A and B are $2d \times 2d$ complex-valued matrices such that

$$\operatorname{rank}(A, B) = 2d, \quad AB^* = BA^*. \tag{1.6}$$

The spectrum of a self-adjoint discrete Sturm–Liouville problem consists of real and finite eigenvalues, and thus can be arranged in the non-decreasing order. The

n-th eigenvalue can be considered as a function defined on the space of self-adjoint discrete Sturm–Liouville problems or on its subset. This function is not continuous in general, see the 1-dimensional case in [22]. The n-th eigenvalue exhibits jump phenomena near the discontinuity points. Unlike only jumping to $-\infty$ in the continuous case, the n-th eigenvalue also blows up to $+\infty$ in the discrete case. So we call the set of all discontinuity points in the considered space to be the jump set, and call any element in the jump set to be a jump point.

The aim of this paper is to determine the jump set and to completely provide the asymptotic behaviour of the n-th eigenvalue near any fixed jump point for the discrete Sturm–Liouville problems. As applications, we consider the Sturm–Liouville problem of Atkinson type, transform it into a discrete Sturm–Liouville problem, and then apply the discrete method to completely characterize jump phenomena of the n-th eigenvalue for the Atkinson type. Though the n-th eigenvalue jumps to $\pm\infty$ near the jump points in the Atkinson type as well as in the discrete case, the jump set in the Atkinson type is the same one as in the continuous case and is independent of coefficients of the Sturm–Liouville equations. This leads tremendous difference with the discrete case, where the jump set is involved heavily with coefficients of the equations.

Singularity of the n-th eigenvalue of Sturm–Liouville problems has attracted a lot of attention (see [7, 8, 10, 13, 16, 20, 22] and references therein) since Rellich [18]. Let us mention three contributions to finding the jump set of the n-th eigenvalue and providing all the asymptotic behaviour near each jump point. Kong, Wu and Zettl completely characterized it for the continuous 1-dimensional Sturm–Liouville problems, while Hu *et al.* gave the answer for the continuous d-dimensional case, where $d \geqslant 2$. Zhu and Shi obtained the desired result for the discrete 1-dimensional case. This paper is devoted to the discrete case in any dimension. We mention here that our result in theorem 4.4 for jump phenomena of the n-th eigenvalue on the boundary conditions is complete, while the conclusion in theorem 4.9 for jump phenomena on the equations is partial due to the non-degenerate assumption (4.22)–(4.23).

Compared with the continuous Sturm–Liouville problems, the n-th eigenvalue in the discrete case is not continuously dependent on the equations, and the criterion for continuity of the n-th eigenvalue is different due to the finiteness of the number of eigenvalues. This makes the method used in the continuous case [8, 13] unable to apply to the discrete case. On the other hand, compared with the 1-dimensional discrete case, the first difficulty for any dimensional case is how to divide areas in layers of the considered space such that the n-th eigenvalue has the same jump phenomena in any given area. Our method in this paper is to find some invertible elementary transformations converting the matrix, which determines the number of eigenvalues of the Sturm–Liouville problems, to a Hermitian matrix. The areas' division is then determined by the spectral information of this Hermitian matrix. The second difficulty is how to prove the asymptotic behaviour of the n-th eigenvalue. Our approach is first to prove the asymptotic behaviour in a certain direction using the monotonicity of continuous eigenvalue branches, and then combine the local topological property (geometric structure) of the considered space with the perturbation theory of eigenvalues to obtain the whole asymptotic behaviour. This can be regarded as a generalization of the method developed for 1-dimensional discrete case in [**22**] to any dimension. Finally, though our method for the Atkinson type is by transforming the Sturm–Liouville problem into a discrete one, it turns out to be no singularity of the n-th eigenvalue on the equations for the Atkinson type.

The rest of this paper is organized as follows. In § 2, topology on the space of Sturm–Liouville equations, and that on the space of boundary conditions are presented. Properties of eigenvalues are given in § 3. The number and multiplicity of eigenvalues are discussed in § 3.1, continuous eigenvalue branches are constructed and their properties are provided in § 3.2 and properties of the n-th eigenvalue are presented in § 3.3. In § 4, jump phenomena of the n-th eigenvalue on the boundary conditions are completely characterized for a fixed equation in § 4.1, while jump phenomena of the n-th eigenvalue on the equations are obtained for a fixed boundary condition under a non-degenerate assumption in § 4.2. Sturm–Liouville problem of the Atkinson type is transformed to a discrete one, and jump phenomena of the n-th eigenvalue are provided thoroughly in § 5. Conclusions are given in § 6.

Notation.

By $\mathbb{R}$ and $\mathbb{C}$ denote the set of all the real and complex numbers, respectively. The set of all $m \times n$ matrices over a field $\mathbb{F}$ is denoted by $\mathcal{M}_{m,n}(\mathbb{F})$, and $\mathcal{M}_{n,n}(\mathbb{F})$ is abbreviated to $\mathcal{M}_n(\mathbb{F})$. A^* is the complex conjugate transpose of $A \in \mathcal{M}_{m,n}(\mathbb{F})$, while A^T is the transpose of A. $\mathcal{H}_n(\mathbb{F})$ is the set of all $n \times n$ Hermitian matrices, while $\mathcal{P}_n(\mathbb{F})$ is the set of all $n \times n$ positive definite matrices over a field $\mathbb{F}$. For a matrix $S \in \mathcal{M}_n(\mathbb{F})$, its entries and columns are denoted by s_{ij} and $s_j = (s_{1j}, \ldots, s_{nj})^T$, respectively, $1 \leqslant i, j \leqslant n$. By I_n denote the $n \times n$ unit matrix. $\sharp(K)$ is the cardinality of the set K. By $r^-(A)$, $r^0(A)$and $r^+(A)$ denote the total multiplicity of negative, zero and positive eigenvalues of $A \in \mathcal{H}_n(\mathbb{C})$, respectively. Moreover, $L((a,b), \mathbb{C}^{n\times n})$ is the space of all $n \times n$ matrix-valued functions satisfying that every component of such a function is Lebesgue integrable on (a, b).

2. Space of self-adjoint discrete Sturm–Liouville problems

In this section, we introduce the topology on the space of self-adjoint discrete Sturm–Liouville problems.

The space of discrete Sturm–Liouville equations is

$$\Omega_N^{\mathbb{C}} := \{\boldsymbol{\omega} = \left(\{P_j^{-1}\}_{j=0}^N, \{Q_i\}_{i=1}^N, \{W_i\}_{i=1}^N\right) \in (\mathcal{M}_d(\mathbb{C}))^{3N+1} : (1.5) \text{ holds}\}$$

with the topology induced by $\mathbb{C}^{(3N+1)d^2}$.

Note that the space of self-adjoint boundary conditions is the same as the continuous case. Following [**8**], it is exactly the quotient space

$$\mathcal{B}^{\mathbb{C}} := \mathrm{GL}(2d, \mathbb{C}) \backslash \mathcal{L}_{2d,4d}(\mathbb{C}), \tag{2.1}$$

where

$$\mathcal{L}_{2d,4d}(\mathbb{C}) := \{(A, B) \in \mathcal{M}_{2d,4d}(\mathbb{C}) : \operatorname{rank}(A, B) = 2d, AB^* = BA^*\}$$

and

$$\mathrm{GL}(2d, \mathbb{C}) := \{T \in \mathcal{M}_{2d}(\mathbb{C}) : \det T \neq 0\}.$$

The boundary condition in $\mathcal{B}^{\mathbb{C}}$ is denoted by $[A\,|\,B] := \{(TA\,|\,TB) : T \in \mathrm{GL}(2d, \mathbb{C})\}$. Bold faced capital Latin letters, such as $\mathbf{A}$, are also used for boundary conditions.

Next we introduce the following form for the local coordinate systems on $\mathcal{B}^{\mathbb{C}}$. Let K be any subset of $\{1, 2, \ldots, 2d\}$. Denote

$$K_1 = K \cap \{1, 2, \ldots, d\}, \quad K_2 = K \cap \{d+1, d+2, \ldots, 2d\}. \tag{2.2}$$

By E_K denote the $4d \times 4d$ matrix generated from I_{4d} by multiplying -1 to the $(k+2d)$-th column and then exchanging the k-th and the $(k+2d)$-th columns for each $k \in K$. Then it has the following form:

$$E_K = \begin{pmatrix} E_{K,1} \\ E_{K,2} \end{pmatrix} = \begin{pmatrix} E_1 & 0 & I_d - E_1 & 0 \\ 0 & E_2 & 0 & I_d - E_2 \\ E_1 - I_d & 0 & E_1 & 0 \\ 0 & E_2 - I_d & 0 & E_2 \end{pmatrix}, \tag{2.3}$$

where $E_{K,1}, E_{K,2} \in \mathcal{M}_{2d,4d}(\mathbb{C})$, $E_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ and $E_2 = \{\beta_1, \beta_2, \ldots, \beta_d\}$ are $d \times d$ diagonal matrices with

$$\alpha_i = \begin{cases} 0 & \text{if } i \in K_1, \\ e_i & \text{if } i \notin K_1, \end{cases} \quad \beta_i = \begin{cases} 0 & \text{if } d+i \in K_2, \\ e_i & \text{if } d+i \notin K_2, \end{cases} \tag{2.4}$$

and e_i is the i-th column of I_d. Then

$$E_K^* J_{2d} E_K = J_{2d}, \quad E_K E_K^* = I_{4d}, \tag{2.5}$$

where

$$J_{2d} = \begin{pmatrix} 0 & -I_{2d} \\ I_{2d} & 0 \end{pmatrix}.$$

We define

$$\mathcal{O}_K^{\mathbb{C}} := \{[(S\,|\,I_{2d})E_K] : \; S \in \mathcal{H}_{2d}(\mathbb{C})\}. \tag{2.6}$$

For $\mathbf{A} = [(S\,|\,I_{2d})E_K] \in \mathcal{O}_K^{\mathbb{C}}$, we denote S by $S(\mathbf{A})$ to indicate its dependence on $\mathbf{A}$ if necessary. It is clear that $\mathcal{O}_K^{\mathbb{C}}$ defined here coincides with that defined in (2.1) of [8]. It follows from theorem 2.1 in [8] that

$$\mathcal{B}^{\mathbb{C}} = \bigcup_{K \subset \{1,2,\ldots,2d\}} \mathcal{O}_K^{\mathbb{C}}.$$

Moreover, $\mathcal{B}^{\mathbb{C}}$ is a connected and compact real-analytic manifold of dimension $4d^2$. The readers are also referred to [2, 3, 14, 17] for more details.

The product space $\Omega_N^{\mathbb{C}} \times \mathcal{B}^{\mathbb{C}}$ is the space of self-adjoint discrete Sturm–Liouville problems, and $(\boldsymbol{\omega}, \mathbf{A})$ is used to stand for an element in $\Omega_N^{\mathbb{C}} \times \mathcal{B}^{\mathbb{C}}$ in the sequel.

3. Properties of eigenvalues

In this section, we study properties of eigenvalues of the self-adjoint discrete Sturm–Liouville problems.

3.1. The number and multiplicity of eigenvalues

Let

$$l[0, N+1] := \left\{ y = \{y_i\}_{i=0}^{N+1} :\ y_i \in \mathbb{C}^d,\ 0 \leqslant i \leqslant N+1 \right\}.$$

The initial value problem of (1.3) has a unique solutions. More precisely,

LEMMA 3.1. *Let $z_{i_0}, \tilde{z}_{i_0} \in \mathbb{C}^d$ for some $1 \leqslant i_0 \leqslant N$. Then, for each $\lambda \in \mathbb{C}$, (1.3) has a unique solution $y(\lambda) \in l[0, N+1]$ satisfying $y_{i_0}(\lambda) = z_{i_0}$, $P_{i_0}\Delta y_{i_0}(\lambda) = \tilde{z}_{i_0}$.*

Proof. This can be deduced by the invertibility of P_j for $0 \leqslant j \leqslant N$ and the iteration of

$$P_i \Delta y_i = P_{i-1}\Delta y_{i-1} - (\lambda W_i - Q_i) y_i, \quad 1 \leqslant i \leqslant N. \qquad \square$$

Recall that λ is called an eigenvalue of the discrete Sturm–Liouville problem $(\boldsymbol{\omega}, \mathbf{A})$ if there exists $y \in l[0, N+1]$ which is non-trivial and solves (1.3)–(1.4). Here y is called an eigenfunction corresponding to λ, and it is said to be normalized if $\sum_{i=1}^{N} y_i^* W_i y_i = 1$. By $\sigma(\boldsymbol{\omega}, \mathbf{A})$ denote the spectral set of $(\boldsymbol{\omega}, \mathbf{A})$. For any $\lambda \in \mathbb{C}$, let $\phi^j(\lambda) = \{\phi_i^j(\lambda)\}_{i=0}^{N+1}$, $j = 1, \ldots, 2d$, be the fundamental solutions to (1.3) determined by the initial data

$$\begin{pmatrix} \phi_0^1(\lambda) & \cdots & \phi_0^{2d}(\lambda) \\ P_0\Delta\phi_0^1(\lambda) & \cdots & P_0\Delta\phi_0^{2d}(\lambda) \end{pmatrix} = I_{2d}.$$

Denote

$$\Phi(\lambda) := \begin{pmatrix} -\phi_0^1(\lambda) & \cdots & -\phi_0^{2d}(\lambda) \\ \phi_N^1(\lambda) & \cdots & \phi_N^{2d}(\lambda) \end{pmatrix}, \quad \Psi(\lambda) := \begin{pmatrix} P_0\Delta\phi_0^1(\lambda) & \cdots & P_0\Delta\phi_0^{2d}(\lambda) \\ P_N\Delta\phi_N^1(\lambda) & \cdots & P_N\Delta\phi_N^{2d}(\lambda) \end{pmatrix}.$$

We write $\Phi(\lambda)$ and $\Psi(\lambda)$ as $\Phi_{\boldsymbol{\omega}}(\lambda)$ and $\Psi_{\boldsymbol{\omega}}(\lambda)$ if necessary. Then the eigenvalues of $(\boldsymbol{\omega}, \mathbf{A})$ can be regarded as zeros of the polynomial $\Gamma_{(\boldsymbol{\omega},\mathbf{A})}$ as follows.

LEMMA 3.2. *$\lambda \in \sigma(\boldsymbol{\omega}, \mathbf{A})$ if and only of λ is a zero of*

$$\Gamma_{(\boldsymbol{\omega},\mathbf{A})}(\lambda) := \det(A\Phi(\lambda) + B\Psi(\lambda)).$$

Proof. The proof is similar to that of lemma 3.2 in [23]. $\square$

Let $\lambda \in \sigma(\boldsymbol{\omega}, \mathbf{A})$. The order of λ as a zero of $\Gamma_{(\boldsymbol{\omega},\mathbf{A})}$ is called its analytic multiplicity. The number of linearly independent eigenfunctions for λ is called its geometric multiplicity. Let $x_i = P_i\Delta y_i$ for $0 \leqslant i \leqslant N$. Then the Sturm–Liouville equation (1.3)

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can be transformed to a discrete linear Hamiltonian system:

$$J_d\Delta\begin{pmatrix}y_i\\x_i\end{pmatrix}=\left(\begin{pmatrix}-Q_{i+1} & 0\\0 & P_i^{-1}\end{pmatrix}+\lambda\begin{pmatrix}W_{i+1} & 0\\0 & 0\end{pmatrix}\right)R\begin{pmatrix}y_i\\x_i\end{pmatrix},\quad 0\leqslant i\leqslant N-1,$$

where $R(y_i^T,x_i^T)^T=(y_{i+1}^T,x_i^T)^T$ is the partial right shift operator and

$$J_d=\begin{pmatrix}0 & -I_d\\I_d & 0\end{pmatrix}.$$

Then by theorem 4.1 in [21], we get the relationship of analytic and geometric multiplicities of λ:

LEMMA 3.3. *The analytic and geometric multiplicities of $\lambda\in\sigma(\boldsymbol{\omega},\mathbf{A})$ are the same.*

Therefore, we do not distinguish these two multiplicities of λ. Let $\sharp_1(\sigma(\boldsymbol{\omega},\mathbf{A})\cap I)$ be the number of eigenvalues in $I\subset\mathbb{R}$, counting multiplicities, of $(\boldsymbol{\omega},\mathbf{A})$. Since $\sigma(\boldsymbol{\omega},\mathbf{A})\subset\mathbb{R}$ by [19], we have $\sharp_1(\sigma(\boldsymbol{\omega},\mathbf{A})\cap\mathbb{R})=\sharp_1(\sigma(\boldsymbol{\omega},\mathbf{A}))$. The next lemma determines $\sharp_1(\sigma(\boldsymbol{\omega},\mathbf{A}))$.

LEMMA 3.4.

$$\sharp_1(\sigma(\boldsymbol{\omega},\mathbf{A}))=(N-2)d+\operatorname{rank}(A_1P_0^{-1}+B_1,B_2),\tag{3.1}$$

where $A_j,B_j\in\mathcal{M}_{2d\times d}$ $(j=1,2)$ are given by

$$\mathbf{A}=[A\mid B]=[(A_1,A_2)|(B_1,B_2)].\tag{3.2}$$

Proof. By theorem 4.1 in [19],

$$\sharp_1(\sigma(\boldsymbol{\omega},\mathbf{A}))=(N-2)d+\operatorname{rank}(A_1+B_1P_0,B_2).$$

Then (3.1) is obtained by $(A_1+B_1P_0,B_2)\left[\begin{smallmatrix}P_0^{-1} & 0\\0 & I_d\end{smallmatrix}\right]=(A_1P_0^{-1}+B_1,B_2)$. □

Note that $(N-2)d\leqslant\sharp_1(\sigma(\boldsymbol{\omega},\mathbf{A}))\leqslant Nd$.

3.2. Continuous eigenvalue branch

In this subsection, we construct continuous eigenvalue branches. Then we study their derivative formulae and monotonicity in some directions.

The first lemma is the small perturbation theory of eigenvalues.

LEMMA 3.5. *Let $(\boldsymbol{\omega}_0,\mathbf{A}_0)\in\mathcal{O}\subset\Omega_N^{\mathbb{C}}\times\mathcal{B}^{\mathbb{C}}$, and $c_1,c_2\in\mathbb{R}\setminus\sigma(\boldsymbol{\omega}_0,\mathbf{A}_0)$ with $c_1<c_2$. Then there exists a neighbourhood $\mathcal{U}\subset\mathcal{O}$ of $(\boldsymbol{\omega}_0,\mathbf{A}_0)$ such that for each $(\boldsymbol{\omega},\mathbf{A})\in\mathcal{U}$, $\sharp_1(\sigma(\boldsymbol{\omega},\mathbf{A})\cap(c_1,c_2))=\sharp_1(\sigma(\boldsymbol{\omega}_0,\mathbf{A}_0)\cap(c_1,c_2))$ and $c_1,c_2\notin\sigma(\boldsymbol{\omega},\mathbf{A})$.*

Proof. Using lemma 3.2, the proof is by a standard perturbation procedure for zeros of the analytic function $\Gamma_{(\boldsymbol{\omega}_0,\mathbf{A}_0)}$. □

By lemma 3.5 and a similar approach to theorem 3.5 in [23], we then construct the continuous eigenvalue branches.

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LEMMA 3.6. *Let* $(\boldsymbol{\omega}_0, \mathbf{A}_0) \in \Omega_N^{\mathbb{C}} \times \mathcal{B}^{\mathbb{C}}$ *and* $\lambda_* \in \sigma(\boldsymbol{\omega}_0, \mathbf{A}_0)$ *with multiplicity* m*. Fix a small* $\varepsilon > 0$ *such that* $\sigma(\boldsymbol{\omega}_0, \mathbf{A}_0) \cap [\lambda_* - \varepsilon, \lambda_* + \varepsilon] = \{\lambda_*\}$*. Then there is a connected neighbourhood* $\mathcal{U}$ *of* $(\boldsymbol{\omega}_0, \mathbf{A}_0)$ *and continuous functions* $\Lambda_i : \mathcal{U} \to \mathbb{R}$, $1 \leqslant i \leqslant m$, *such that* $\lambda_* - \varepsilon < \Lambda_1(\boldsymbol{\omega}, \mathbf{A}) \leqslant \cdots \leqslant \Lambda_m(\boldsymbol{\omega}, \mathbf{A}) < \lambda_* + \varepsilon$ *and* $\lambda_* \pm \varepsilon \notin \sigma(\boldsymbol{\omega}, \mathbf{A})$ *for all* $(\boldsymbol{\omega}, \mathbf{A}) \in \mathcal{U}$, *where* $\{\Lambda_i(\boldsymbol{\omega}, \mathbf{A})\}_{i=1}^m \subset \sigma(\boldsymbol{\omega}, \mathbf{A})$.

Here $\Lambda_i : \mathcal{U} \to \mathbb{R}$, $1 \leqslant i \leqslant m$, are called the continuous eigenvalue branches through λ_*. We write $\Lambda_i(\boldsymbol{\omega})$ when $\mathbf{A}$ is fixed, and write $\Lambda_i(\mathbf{A})$ when $\boldsymbol{\omega}$ is fixed. Then we shall make a continuous choice of eigenfunctions for the eigenvalues along a continuous simple eigenvalue branch ($m = 1$).

LEMMA 3.7. *Let* u_0 *be an eigenfunction for a simple eigenvalue* $\lambda_* \in \sigma(\boldsymbol{\omega}_0, \mathbf{A}_0)$, *and* Λ *be the continuous eigenvalue branch defined on* $\mathcal{U}$ *through* λ_**. Then there exists a neighbourhood* $\mathcal{U}_1 \subset \mathcal{U}$ *of* $(\boldsymbol{\omega}_0, \mathbf{A}_0)$ *such that for any* $(\boldsymbol{\omega}, \mathbf{A}) \in \mathcal{U}_1$, *there is an eigenfunction* $u_{\Lambda(\boldsymbol{\omega}, \mathbf{A})}$ *for* $\Lambda(\boldsymbol{\omega}, \mathbf{A})$ *satisfying that* $u_{\Lambda(\boldsymbol{\omega}, \mathbf{A})} = u_0$, *and* $u_{\Lambda(\boldsymbol{\omega}, \mathbf{A})} \to u_{\Lambda(\boldsymbol{\omega}_0, \mathbf{A}_0)}$ *in* $\mathbb{C}^{(N+2)d}$ *as* $\mathcal{U}_1 \ni (\boldsymbol{\omega}, \mathbf{A}) \to (\boldsymbol{\omega}_0, \mathbf{A}_0)$.

Proof. The proof is similar to that of lemma 4.3 in [**23**], and thus we omit the details. □

Besides lemma 3.7, we also need the following lemma to deduce the derivative formulae for continuous simple eigenvalue branches.

LEMMA 3.8. *Let* y *be an eigenfunction for* $\lambda \in \sigma(\boldsymbol{\omega}, \mathbf{A})$ *and* z *be an eigenfunction for* $\tilde{\lambda} \in \sigma(\tilde{\boldsymbol{\omega}}, \mathbf{A})$, *where* $\boldsymbol{\omega} = (P^{-1}, Q, W), \tilde{\boldsymbol{\omega}} = (\tilde{P}^{-1}, \tilde{Q}, \tilde{W})$ *and* $\mathbf{A} = [A \,|\, B]$*. Then*

$$(\Delta z_0)^* \tilde{P}_0 y_0 - z_0^* P_0 \Delta y_0 = (\Delta z_N)^* \tilde{P}_N y_N - z_N^* P_N \Delta y_N. \tag{3.3}$$

Proof. For convenience, denote

$$(A,\, B) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{2d} \end{pmatrix}, \quad Y = \begin{pmatrix} -y_0 \\ y_N \\ P_0 \Delta y_0 \\ P_N \Delta y_N \end{pmatrix}, \quad Z = \begin{pmatrix} -z_0 \\ z_N \\ \tilde{P}_0 \Delta z_0 \\ \tilde{P}_N \Delta z_N \end{pmatrix}, \tag{3.4}$$

where $\alpha_i \in \mathcal{M}_{1,4d}(\mathbb{C})$, $i = 1, \ldots, 2d$. Then

$$(A,\, B) J_{2d} (A,\, B)^* = 0, \quad (A,\, B) Y = 0, \quad (A,\, B) Z = 0. \tag{3.5}$$

Since $\operatorname{rank}(A,\, B) = 2d$, the first equation in (3.5) yields that each solution of the equation $(A,\, B)X = 0$ is a linear combination of $J_{2d}\alpha_i^*$, $1 \leqslant i \leqslant 2d$. From the last two equations in (3.5), we know that there exists $c_i, d_i \in \mathbb{C}$, $1 \leqslant i \leqslant 2d$, such that $Y = \sum_{i=1}^{2d} c_i J_{2d} \alpha_i^*$ and $Z = \sum_{i=1}^{2d} d_i J_{2d} \alpha_i^*$. The first equation in (3.5) also implies that

$$\alpha_i J_{2d} \alpha_j^* = 0, \quad 1 \leqslant i, j \leqslant 2d.$$

So

$$Z^* J_{2d} Y = \left(\sum_{i=1}^{2d} d_i J_{2d} \alpha_i^* \right)^* J_{2d} \left(\sum_{i=1}^{2d} c_i J_{2d} \alpha_i^* \right) = 0,$$

which is equivalent to (3.3). The proof is complete. □

Note that the method used in lemma 4.4 of [**22**] depends on separated and coupled boundary conditions, and thus cannot be applied to lemma 3.8 here for mixing boundary conditions when $d \geqslant 2$. With the help of lemma 3.8, we give the derivative formulae of the continuous simple eigenvalue branch with respect to coefficients of the Sturm–Liouville equations.

LEMMA 3.9. *Fix $\mathbf{A} \in \mathcal{B}^{\mathbb{C}}$. Let $\boldsymbol{\omega} = (P^{-1}, Q, W) \in \Omega_N^{\mathbb{C}}$, λ_* be a simple eigenvalue of $(\boldsymbol{\omega}, \mathbf{A})$, $y \in l[0, N+1]$ be a normalized eigenfunction for λ_*, and Λ be the continuous simple eigenvalue branch over $\Omega_N^{\mathbb{C}}$ through λ_*. Then*

$$\mathrm{d}\Lambda|_{\boldsymbol{\omega}}(H, K, L) = -\sum_{i=0}^{N-1} (P_i \Delta y_i)^* H_i (P_i \Delta y_i) + \sum_{i=1}^{N} y_i^* K_i y_i - \lambda_* \sum_{i=1}^{N} y_i^* L_i y_i \tag{3.6}$$

for all $(H, K, L) = ((H_0, \ldots, H_N), (K_1, \ldots, K_N), (L_1, \ldots, L_N)) \in (\mathcal{H}_d(\mathbb{C}))^{3N+1}$ and $(P^{-1} + H, Q + K, W + L) \in \Omega_N^{\mathbb{C}}$.

Proof. Let $\boldsymbol{\sigma} \in \Omega_N^{\mathbb{C}}$ with $\boldsymbol{\sigma} = (P^{-1} + H, Q + K, W + L) =: (\tilde{P}^{-1}, \tilde{Q}, \tilde{W})$. By lemma 3.7, we can choose an eigenfunction $z = z(\cdot, \boldsymbol{\sigma})$ for $\Lambda = \Lambda(\boldsymbol{\sigma})$ with $\boldsymbol{\sigma}$ sufficiently close to $\boldsymbol{\omega}$ in $\Omega_N^{\mathbb{C}}$ such that $z \to y$ as $\boldsymbol{\sigma} \to \boldsymbol{\omega}$. Then it follows from (1.3) that

$$\begin{aligned}
&[\Lambda(\boldsymbol{\sigma}) - \Lambda(\boldsymbol{\omega})] \sum_{i=1}^{N} z_i^* W_i y_i \\
&\quad = \sum_{i=1}^{N} [z_i^* \nabla (P_i \Delta y_i) - (\nabla (\tilde{P}_i \Delta z_i))^* y_i] - \Lambda(\boldsymbol{\sigma}) \sum_{i=1}^{N} z_i^* L_i y_i + \sum_{i=1}^{N} z_i^* K_i y_i \\
&\quad = -\sum_{i=0}^{N-1} \Delta z_i^* P_i \Delta y_i + \sum_{i=1}^{N} \nabla (z_i^* P_i \Delta y_i) + \sum_{i=0}^{N-1} \Delta z_i^* \tilde{P}_i \Delta y_i - \sum_{i=1}^{N} \nabla (\Delta z_i^* \tilde{P}_i y_i) \\
&\qquad - \Lambda(\boldsymbol{\sigma}) \sum_{i=1}^{N} z_i^* L_i y_i + \sum_{i=1}^{N} z_i^* K_i y_i \\
&\quad = \sum_{i=0}^{N-1} (\tilde{P}_i \Delta z_i)^* (P_i^{-1} - \tilde{P}_i^{-1}) (P_i \Delta y_i) + z_N^* P_N \Delta y_N - z_0^* P_0 \Delta y_0 \\
&\qquad - (\Delta z_N)^* \tilde{P}_N y_N + (\Delta z_0)^* \tilde{P}_0 y_0 - \Lambda(\boldsymbol{\sigma}) \sum_{i=1}^{N} z_i^* L_i y_i + \sum_{i=1}^{N} z_i^* K_i y_i.
\end{aligned}$$

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By lemma 3.8, we get

$$[\Lambda(\boldsymbol{\sigma}) - \Lambda(\boldsymbol{\omega})] \sum_{i=1}^{N} z_i^* W_i y_i$$
$$= - \sum_{i=0}^{N-1} (\tilde{P}_i \Delta z_i)^* H_i (P_i \Delta y_i) - \Lambda(\boldsymbol{\sigma}) \sum_{i=1}^{N} z_i^* L_i y_i + \sum_{i=1}^{N} z_i^* K_i y_i,$$

which yields that (3.6) holds. This completes the proof. □

Let us fix all the components of $\boldsymbol{\omega}$ except P_j^{-1}, and write the perturbed term P_j^{-1} by $P_j^{-1}(\boldsymbol{\omega})$ to indicate its dependence on $\boldsymbol{\omega}$ for a given $0 \leqslant j \leqslant N-1$. $Q_i(\boldsymbol{\omega})$ has the similar meaning for $1 \leqslant i \leqslant N$. Then we get the following monotonicity result.

Corollary 3.10. *Fix $\mathbf{A} \in \mathcal{B}^{\mathbb{C}}$. Let Λ be a continuous eigenvalue branch defined on $\mathcal{V} \subset \Omega_N^{\mathbb{C}}$. If $P_j^{-1}(\boldsymbol{\sigma}) - P_j^{-1}(\boldsymbol{\omega})$ is positive semi-definite for a given $0 \leqslant j \leqslant N-1$, then $\Lambda(\boldsymbol{\sigma}) \leqslant \Lambda(\boldsymbol{\omega})$. If $Q_i(\boldsymbol{\sigma}) - Q_i(\boldsymbol{\omega})$ is positive semi-definite for a given $1 \leqslant i \leqslant N$, then $\Lambda(\boldsymbol{\omega}) \leqslant \Lambda(\boldsymbol{\sigma})$.*

Proof. We only prove it for the case that $P_j^{-1}(\boldsymbol{\sigma}) - P_j^{-1}(\boldsymbol{\omega})$ is positive semi-definite, since the other is similar. Let $\boldsymbol{\omega}(s) = s\boldsymbol{\sigma} + (1-s)\boldsymbol{\omega}$ for $0 \leqslant s \leqslant 1$. Since $\Gamma_{(\boldsymbol{\omega}(s),\mathbf{A})}(\lambda) = \det(A\Phi_{\boldsymbol{\omega}(s)}(\lambda) + B\Psi_{\boldsymbol{\omega}(s)}(\lambda))$ is a polynomial of the two variables s and λ, we have that either there exist finite points $s_1, \ldots, s_{n_0} \in [0,1]$ such that $\Lambda(\boldsymbol{\omega}(s))$ is a simple eigenvalue for $(\boldsymbol{\omega}(s), \mathbf{A})$ with $s \in [0,1] \setminus \{s_1, \ldots, s_{n_0}\}$, or $\Lambda(\boldsymbol{\omega}(s))$ is a multiple eigenvalue for all $(\boldsymbol{\omega}(s), \mathbf{A})$ with $s \in [0,1]$, see § 13 in chapter 5 of [**11**]. For the former case, choose any $s_0 \in [0,1]\setminus\{s_1, \ldots, s_{n_0}\}$. Since $\Lambda(\boldsymbol{\omega}(s_0))$ is simple, by lemma 3.9 we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\Lambda(\boldsymbol{\omega}(s_0)) = -(P_j \Delta y_j)^* (P_j^{-1}(\boldsymbol{\sigma}) - P_j^{-1}(\boldsymbol{\omega}))(P_j \Delta y_j) \leqslant 0,$$

where $y \in l[0, N+1]$ is a normalized eigenfunction for $\Lambda(\boldsymbol{\omega}(s_0))$. This implies that $\Lambda(\boldsymbol{\omega}(\cdot))$ is non-increasing on $[0,1]$. Thus, $\Lambda(\boldsymbol{\sigma}) = \Lambda(\boldsymbol{\omega}(1)) \leqslant \Lambda(\boldsymbol{\omega}(0)) = \Lambda(\boldsymbol{\omega})$. For the latter case, there exists $\boldsymbol{\tau} \in \Omega_N^{\mathbb{C}}$ such that $\Lambda(\boldsymbol{\omega} + t\boldsymbol{\tau})$ is a simple eigenvalue for $(\boldsymbol{\omega} + t\boldsymbol{\tau}, \mathbf{A})$, where $t \in (0, t_0)$ and $t_0 > 0$ is small enough. Then it follows from the former case that $\Lambda(\boldsymbol{\omega}(\cdot) + t\boldsymbol{\tau})$ is non-increasing on $[0,1]$ for any fixed $t \in (0, t_0)$. Thus,

$$\Lambda(\boldsymbol{\sigma}) = \Lambda(\boldsymbol{\omega}(1)) = \lim_{t \to 0^+} \Lambda(\boldsymbol{\omega}(1) + t\boldsymbol{\tau}) \leqslant \lim_{t \to 0^+} \Lambda(\boldsymbol{\omega}(0) + t\boldsymbol{\tau}) = \Lambda(\boldsymbol{\omega}(0)) = \Lambda(\boldsymbol{\omega}).$$

□

Then we give the derivative formula of a continuous simple eigenvalue branch with respect to boundary conditions.

Lemma 3.11. *Fix $\boldsymbol{\omega} \in \Omega_N^{\mathbb{C}}$. Let λ_* be a simple eigenvalue of $(\boldsymbol{\omega}, \mathbf{A})$ for some $\mathbf{A} \in \mathcal{O}_K^{\mathbb{C}}$, $y \in l[0, N+1]$ be a normalized eigenfunction for λ_*, and Λ be the continuous*

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simple eigenvalue branch through λ_*. *Then*

$$\mathrm{d}\Lambda|_{\mathbf{A}}(H) = Y^* E_{K,1}^* H E_{K,1} Y$$

for $H \in \mathcal{H}_{2d}(\mathbb{C})$, *where* $E_{K,1}$ *and* Y *are given in* (2.3) *and* (3.4), *respectively.*

Proof. By (2.6), there exists $S \in \mathcal{H}_{2d}(\mathbb{C})$ such that $\mathbf{A} = [S \mid I_{2d}]E_K$. Let $\mathbf{B} = [S + H \mid I_{2d}]E_K$ with $H \in \mathcal{H}_{2d}$. Then there exists an eigenfunction $\tilde{y} = y_{\Lambda(\mathbf{B})}$ for $\Lambda(\mathbf{B})$ such that $\tilde{y} \to y$ in $\mathbb{C}^{(N+2)d}$ as $\mathbf{B} \to \mathbf{A}$. $\tilde{Y}$ has the similar meaning as Y. Note that $\tilde{y}$ and y satisfy

$$-\nabla(P_i \Delta \tilde{y}_i) + Q_i \tilde{y}_i = \Lambda(\mathbf{B}) W_i \tilde{y}_i, \quad -\nabla(P_i \Delta y_i) + Q_i y_i = \Lambda(\mathbf{A}) W_i y_i, \ 1 \leqslant i \leqslant N,$$

and thus

$$(\Lambda(\mathbf{B}) - \Lambda(\mathbf{A}))\tilde{y}_i^* W_i y_i = -\Delta[y_{i-1}, \tilde{y}_{i-1}],$$

where $[y_i, \tilde{y}_i] = (\Delta \tilde{y}_i)^* P_i y_i - \tilde{y}_i^* P_i \Delta y_i$. From the boundary conditions $\mathbf{A}$ and $\mathbf{B}$, we have

$$S E_{K,1} Y + E_{K,2} Y = 0 \text{ and } (S + H) E_{K,1} \tilde{Y} + E_{K,2} \tilde{Y} = 0. \tag{3.7}$$

It then follows from (2.5) and (3.7) that

$$\begin{aligned}
(\Lambda(\mathbf{B}) - \Lambda(\mathbf{A})) \sum_{i=1}^{N} \tilde{y}_i^* W_i y_i &= [y_0, \tilde{y}_0] - [y_N, \tilde{y}_N] = \tilde{Y}^* J_{2d}^* Y \\
&= \tilde{Y}^* E_K^* J_{2d}^* E_K Y = -\tilde{Y}^* E_{K,2}^* E_{K,1} Y + \tilde{Y}^* E_{K,1}^* E_{K,2} Y \\
&= \tilde{Y}^* E_{K,1}^* (S + H) E_{K,1} Y - \tilde{Y}^* E_{K,1}^* S E_{K,1} Y \\
&= \tilde{Y}^* E_{K,1}^* H E_{K,1} Y.
\end{aligned}$$

This completes the proof. □

By the derivative formula of a continuous simple eigenvalue branch in lemma 3.11, the following result can be obtained with a similar argument to corollary 3.10.

Corollary 3.12. *Fix* $\boldsymbol{\omega} \in \Omega_N^{\mathbb{C}}$. *Let* Λ *be a continuous eigenvalue branch defined on* $\mathcal{U} \subset \mathcal{O}_K^{\mathbb{C}}$. *Then* $\Lambda(\mathbf{A}) \leqslant \Lambda(\mathbf{B})$ *if* $\mathbf{A}, \mathbf{B} \in \mathcal{U}$ *and* $S(\mathbf{B}) - S(\mathbf{A})$ *is positive semi-definite.*

The monotonicity of continuous eigenvalue branches in corollaries 3.10 and 3.12 plays an important role in studying the asymptotic behaviour of the n-th eigenvalue in a certain direction, see (4.19) and (4.39).

3.3. Properties of the n-th eigenvalue

Based on lemma 3.4, the eigenvalues of $(\boldsymbol{\omega}, \mathbf{A}) \in \Omega_N^{\mathbb{C}} \times \mathcal{B}^{\mathbb{C}}$ can be arranged in the following non-decreasing order:

$$\lambda_1(\boldsymbol{\omega}, \mathbf{A}) \leqslant \lambda_2(\boldsymbol{\omega}, \mathbf{A}) \leqslant \cdots \leqslant \lambda_{\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A}))}(\boldsymbol{\omega}, \mathbf{A}).$$

Therefore, for any $1 \leqslant n \leqslant Nd$, the n-th eigenvalue can be regarded as a function defined on $\Omega_N^{\mathbb{C}} \times \mathcal{B}^{\mathbb{C}}$ or on its subset, called the n-th eigenvalue function. Firstly, we provide a criterion for all these functions to be continuous on a subset of $\Omega_N^{\mathbb{C}} \times \mathcal{B}^{\mathbb{C}}$.

LEMMA 3.13. *Let $\mathcal{O}$ be a connected subset of $\Omega_N^{\mathbb{C}} \times \mathcal{B}^{\mathbb{C}}$. If $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A})) \equiv k_0$, $(\boldsymbol{\omega}, \mathbf{A}) \in \mathcal{O}$, for some $k_0 > 0$, then the restrictions of λ_n, $1 \leqslant n \leqslant k_0$, to $\mathcal{O}$ are continuous. Moreover, they are locally continuous eigenvalue branches on $\mathcal{O}$.*

Then we list several other properties of the n-th eigenvalue function in order to study its asymptotic behaviour. The following lemma strengthens the result in theorem 2.2 of [**23**].

LEMMA 3.14. *Let $\mathcal{O} \subset \Omega_N^{\mathbb{C}} \times \mathcal{B}^{\mathbb{C}}$, $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A})) = m_1 + m_2 + m_3$ for all $(\boldsymbol{\omega}, \mathbf{A}) \in \mathcal{O}$, and $\sharp_1(\sigma(\boldsymbol{\omega}_0, \mathbf{A}_0)) = m_2$ for some $(\boldsymbol{\omega}_0, \mathbf{A}_0) \in \bar{\mathcal{O}} \setminus \mathcal{O}$, where $m_i \geqslant 0$, $1 \leqslant i \leqslant 3$. If*

$$\lim_{\mathcal{O} \ni (\boldsymbol{\omega}, \mathbf{A}) \to (\boldsymbol{\omega}_0, \mathbf{A}_0)} \lambda_n(\boldsymbol{\omega}, \mathbf{A}) = -\infty, \quad 1 \leqslant n \leqslant m_1, \tag{3.8}$$

and

$$\lim_{\mathcal{O} \ni (\boldsymbol{\omega}, \mathbf{A}) \to (\boldsymbol{\omega}_0, \mathbf{A}_0)} \lambda_n(\boldsymbol{\omega}, \mathbf{A}) = +\infty, \quad m_1 + m_2 + 1 \leqslant n \leqslant m_1 + m_2 + m_3, \tag{3.9}$$

then

$$\lim_{\mathcal{O} \ni (\boldsymbol{\omega}, \mathbf{A}) \to (\boldsymbol{\omega}_0, \mathbf{A}_0)} \lambda_n(\boldsymbol{\omega}, \mathbf{A}) = \lambda_{n-m_1}(\boldsymbol{\omega}_0, \mathbf{A}_0), \quad m_1 + 1 \leqslant n \leqslant m_1 + m_2.$$

Proof. Let $c_1, c_2 \in \mathbb{R}$ such that $\sharp_1(\sigma(\boldsymbol{\omega}_0, \mathbf{A}_0) \cap (c_1, c_2)) = m_2$. Then we get by lemma 3.5 that there exists a neighbourhood $\mathcal{U} \subset \mathcal{O}$ of $(\boldsymbol{\omega}_0, \mathbf{A}_0)$ such that $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A}) \cap (c_1, c_2)) = m_2$ and $c_1, c_2 \notin \sigma(\boldsymbol{\omega}, \mathbf{A})$ for all $(\boldsymbol{\omega}, \mathbf{A}) \in \mathcal{U}$. It follows from (3.8)–(3.9) that $\mathcal{U}$ can be shrunk such that $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A}) \cap (-\infty, c_1)) = m_1$ and $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A}) \cap (c_2, \infty)) = m_3$ for all $(\boldsymbol{\omega}, \mathbf{A}) \in \mathcal{U}$. This implies that $\sigma(\boldsymbol{\omega}, \mathbf{A}) \cap (c_1, c_2) = \{\lambda_n(\boldsymbol{\omega}, \mathbf{A}) : m_1 + 1 \leqslant n \leqslant m_1 + m_2\}$ for all $(\boldsymbol{\omega}, \mathbf{A}) \in \mathcal{U}$. Then the conclusion holds again by lemma 3.5. □

LEMMA 3.15. *Let $\mathcal{O}$ be a connected subset of $\Omega_N^{\mathbb{C}} \times \mathcal{B}^{\mathbb{C}}$ and $c_1, c_2 \in \mathbb{R}$ with $c_1 < c_2$. Assume that for all $(\boldsymbol{\omega}, \mathbf{A}) \in \mathcal{O}$, $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A})) = k$, $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A}) \cap (c_1, c_2)) = m$ with $m < k$, and $c_1, c_2 \notin \sigma(\boldsymbol{\omega}, \mathbf{A})$. Then the other $k - m$ eigenvalues out of $[c_1, c_2]$, denoted by $\hat{\lambda}_1(\boldsymbol{\omega}, \mathbf{A}) \leqslant \cdots \leqslant \hat{\lambda}_{k-m}(\boldsymbol{\omega}, \mathbf{A})$, have the following properties.*

(1) *Let $E_i = \{\hat{\lambda}_i(\boldsymbol{\omega}, \mathbf{A}) : (\boldsymbol{\omega}, \mathbf{A}) \in \mathcal{O}\}$. Then for all $1 \leqslant i \leqslant k - m$,*

$$\text{either } E_i \subset (-\infty, c_1) \text{ or } E_i \subset (c_2, +\infty),$$

and there exists $1 \leqslant i_0 \leqslant k$ such that $\hat{\lambda}_i = \lambda_{i_0}$ is continuous on $\mathcal{O}$.

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(2) *Let* $(\boldsymbol{\omega}_0, \mathbf{A}_0) \in \bar{\mathcal{O}} \setminus \mathcal{O}$, $\sharp_1(\sigma(\boldsymbol{\omega}_0, \mathbf{A}_0)) = m$, *and* $\sigma(\boldsymbol{\omega}_0, \mathbf{A}_0) \subset (c_1, c_2)$. *If* $E_{i_0} \subset (-\infty, c_1)$ *for some* $1 \leqslant i_0 \leqslant k - m$, *then*

$$\lim_{\mathcal{O} \ni (\boldsymbol{\omega}, \mathbf{A}) \to (\boldsymbol{\omega}_0, \mathbf{A}_0)} \hat{\lambda}_i(\boldsymbol{\omega}, \mathbf{A}) = -\infty, \quad 1 \leqslant i \leqslant i_0.$$

If $E_{j_0} \subset (c_2, +\infty)$ *for some* $1 \leqslant j_0 \leqslant k - m$, *then*

$$\lim_{\mathcal{O} \ni (\boldsymbol{\omega}, \mathbf{A}) \to (\boldsymbol{\omega}_0, \mathbf{A}_0)} \hat{\lambda}_j(\boldsymbol{\omega}, \mathbf{A}) = +\infty, \quad j_0 \leqslant j \leqslant k - m.$$

The following result indicates that the monotonicity of λ_n in a certain direction determines its asymptotic behaviour in this direction.

LEMMA 3.16. *Let* $\mathcal{O} = \{(\boldsymbol{\omega}, \mathbf{A})_\nu \in \Omega_N^{\mathbb{C}} \times \mathcal{B}^{\mathbb{C}} : \nu \in (\nu_0 - \epsilon, \nu_0 + \epsilon)\}$, *where* $(\boldsymbol{\omega}, \mathbf{A})_\nu$ *is continuously dependent on* $\nu \in (\nu_0 - \epsilon, \nu_0 + \epsilon)$ *for some* $\epsilon > 0$. *Assume that* $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A})_{\nu_0}) = m \geqslant 0$, *and for all* $\nu \in (\nu_0 - \epsilon, \nu_0 + \epsilon) \setminus \{\nu_0\}$, $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A})_\nu) = k > m$.

(1) *If* $\lambda_n(\nu) := \lambda_n((\boldsymbol{\omega}, \mathbf{A})_\nu)$ *is non-increasing on* $(\nu_0 - \epsilon, \nu_0)$ *for all* $1 \leqslant n \leqslant k$, *then*

$$\lim_{\nu \to \nu_0^-} \lambda_n(\nu) = -\infty, \quad 1 \leqslant n \leqslant k - m,$$

$$\lim_{\nu \to \nu_0^-} \lambda_n(\nu) = \lambda_{n-(k-m)}(\nu_0), \quad k - m + 1 \leqslant n \leqslant k.$$

(2) *If* $\lambda_n(\nu)$ *is non-decreasing on* $(\nu_0 - \epsilon, \nu_0)$ *for all* $1 \leqslant n \leqslant k$, *then*

$$\lim_{\nu \to \nu_0^-} \lambda_n(\nu) = \lambda_n(\nu_0), \quad 1 \leqslant n \leqslant m, \quad \lim_{\nu \to \nu_0^-} \lambda_n(\nu) = +\infty, \ m + 1 \leqslant n \leqslant k.$$

(3) *If* $\lambda_n(\nu)$ *is non-increasing on* $(\nu_0, \nu_0 + \epsilon)$ *for all* $1 \leqslant n \leqslant k$, *then*

$$\lim_{\nu \to \nu_0^+} \lambda_n(\nu) = \lambda_n(\nu_0), \quad 1 \leqslant n \leqslant m, \quad \lim_{\nu \to \nu_0^+} \lambda_n(\nu) = +\infty, \ m + 1 \leqslant n \leqslant k.$$

(4) *If* $\lambda_n(\nu)$ *is non-decreasing on* $(\nu_0, \nu_0 + \epsilon)$ *for all* $1 \leqslant n \leqslant k$, *then*

$$\lim_{\nu \to \nu_0^+} \lambda_n(\nu) = -\infty, \quad 1 \leqslant n \leqslant k - m,$$

$$\lim_{\nu \to \nu_0^+} \lambda_n(\nu) = \lambda_{n-(k-m)}(\nu_0), \ k - m + 1 \leqslant n \leqslant k.$$

Note that the analyses in the proofs of lemmas 3.13, 3.15 and 3.16 are independent of the dimension of the Sturm–Liouville problem (1.3)–(1.4). Thus they are indeed a straightforward generalization of theorems 2.1, 2.3 and lemma 2.7 in [**22**].

4. Jump phenomena of the n-th eigenvalue of discrete Sturm–Liouville problems

In this section, we completely characterize jump phenomena of the n-th eigenvalue on the boundary conditions for a fixed equation. Then we characterize jump phenomena of the n-th eigenvalue on the equations for a fixed boundary condition under a non-degenerate assumption.

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4.1. Jump phenomena of the n-th eigenvalue on the boundary conditions

Fix a Sturm–Liouville equation $\boldsymbol{\omega} = (P^{-1}, Q, W)$ such that $P_0^{-1} \in \mathcal{P}_d(\mathbb{C})$ in this subsection. Let $K \subseteq \{1, 2, \dots, 2d\}$. For any boundary condition $\mathbf{A} = [A \mid B] \in \mathcal{O}_K^{\mathbb{C}}$, it follows from (2.6) that there exists $S \in \mathcal{H}_{2d}(\mathbb{C})$ such that $\mathbf{A} = [(S \mid I_{2d})E_K]$. Let us write $S = S(\mathbf{A})$ in the partitioned form:

$$S(\mathbf{A}) = \begin{pmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{pmatrix},$$

where $S_1, S_3 \in \mathcal{H}_d(\mathbb{C})$ and $S_2 \in \mathcal{M}_d(\mathbb{C})$. Then it follows that

$$\begin{aligned}
(A,\, B) &= (S,\, I_{2d})E_K \\
&= \begin{pmatrix} S_1 & S_2 & I_d & 0 \\ S_2^* & S_3 & 0 & I_d \end{pmatrix} \begin{pmatrix} E_1 & 0 & I_d - E_1 & 0 \\ 0 & E_2 & 0 & I_d - E_2 \\ E_1 - I_d & 0 & E_1 & 0 \\ 0 & E_2 - I_d & 0 & E_2 \end{pmatrix} \\
&= \begin{pmatrix} S_1E_1 + E_1 - I_d & S_2E_2 & S_1(I_d - E_1) + E_1 & S_2(I_d - E_2) \\ S_2^*E_1 & S_3E_2 + E_2 - I_d & S_2^*(I_d - E_1) & S_3(I_d - E_2) + E_2 \end{pmatrix}.
\end{aligned}$$

Recall that $A_j, B_j \in \mathcal{M}_{2d\times d}$, $j = 1, 2$, are defined in (3.2). Then we have

$$\begin{aligned}
&(A_1P_0^{-1} + B_1,\, B_2) \\
&\quad = \begin{pmatrix} S_1(E_1P_0^{-1} + I_d - E_1) + (E_1 - I_d)P_0^{-1} + E_1 & S_2(I_d - E_2) \\ S_2^*(E_1P_0^{-1} + I_d - E_1) & S_3(I_d - E_2) + E_2 \end{pmatrix}. \qquad (4.1)
\end{aligned}$$

From the structure of E_1 and the fact that $P_0^{-1} \in \mathcal{P}_d(\mathbb{C})$, we infer that $E_1P_0^{-1} + I_d - E_1$ is invertible. Then it follows that

$$\begin{aligned}
&(A_1P_0^{-1} + B_1, B_2)\begin{pmatrix} (E_1P_0^{-1} + I_d - E_1)^{-1} & 0 \\ 0 & I_d \end{pmatrix} \qquad (4.2) \\
&\quad = \begin{pmatrix} S_1 + [(E_1 - I_d)P_0^{-1} + E_1](E_1P_0^{-1} + I_d - E_1)^{-1} & S_2(I_d - E_2) \\ S_2^* & S_3(I_d - E_2) + E_2 \end{pmatrix}.
\end{aligned}$$

Recall that K_2 is defined in (2.2), and e_i is the i-th column of I_d. For convenience, we set $r = \sharp(K_2)$. Let us write $K_2 = \{k_1 + d, k_2 + d, \dots, k_r + d\}$ with $1 \leqslant k_1 < k_2 < \cdots < k_r \leqslant d$, and

$$E_0 = (e_{k_1}, \dots, e_{k_r}), \qquad (4.3)$$

if $K_2 \neq \emptyset$. For any $\mathbf{A} \in \mathcal{B}^{\mathbb{C}}$, we define

$$D(\mathbf{A}) = (A_1P_0^{-1} + B_1, B_2). \qquad (4.4)$$

For any $\mathbf{A} \in \mathcal{O}_K^{\mathbb{C}}$, we define

$$S_K^D(\mathbf{A}) = \begin{pmatrix} S_1 + [(E_1 - I_d)P_0^{-1} + E_1](E_1P_0^{-1} + I_d - E_1)^{-1} & S_2E_0 \\ E_0^*S_2^* & E_0^*S_3E_0 \end{pmatrix} \text{ if } K_2 \neq \emptyset, \qquad (4.5)$$

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$$S_K^D(\mathbf{A}) = S_1 + [(E_1 - I_d)P_0^{-1} + E_1](E_1 P_0^{-1} + I_d - E_1)^{-1} \text{ if } K_2 = \emptyset. \tag{4.6}$$

Then the following result holds.

Lemma 4.1. *Let* $\mathbf{A} \in \mathcal{O}_K^{\mathbb{C}}$. *Then*

$$\operatorname{rank} D(\mathbf{A}) = \operatorname{rank} S_K^D(\mathbf{A}) + d - r, \tag{4.7}$$

and $S_K^D(\mathbf{A}) \in \mathcal{H}_{d+r}(\mathbb{C})$.

Proof. By (2.4), $\operatorname{rank}(E_2) = d - r$ and thus (4.7) holds. To prove $S_K^D(\mathbf{A}) \in \mathcal{H}_{d+r}(\mathbb{C})$, it suffices to show that

$$[(E_1 - I_d)P_0^{-1} + E_1](E_1 P_0^{-1} + I_d - E_1)^{-1} \in \mathcal{H}_d(\mathbb{C}). \tag{4.8}$$

Direct computation gives

$$\begin{aligned}(P_0^{-1}E_1 + I_d - E_1)&[(E_1 - I_d)P_0^{-1} + E_1] \\ &= [P_0^{-1}(E_1 - I_d) + E_1](E_1 P_0^{-1} + I_d - E_1).\end{aligned}$$

Since $(P_0^{-1}E_1 + I_d - E_1)$ and $(E_1 P_0^{-1} + I_d - E_1)$ are invertible, we have

$$\begin{aligned}[(E_1 - I_d)P_0^{-1} &+ E_1](E_1 P_0^{-1} + I_d - E_1)^{-1} \\ &= (P_0^{-1}E_1 + I_d - E_1)^{-1}[P_0^{-1}(E_1 - I_d) + E_1],\end{aligned}$$

which yields (4.8). □

Define

$$\mathcal{B}_k := \{\mathbf{A} \in \mathcal{B}^{\mathbb{C}} \mid r^0(D(\mathbf{A})) = k\}, \quad 0 \leqslant k \leqslant 2d, \tag{4.9}$$

$$\mathcal{B}_K^{(r^0,r^+,r^-)} := \{\mathbf{A} \in \mathcal{O}_K^{\mathbb{C}} \mid r^0 = r^0(S_K^D(\mathbf{A})), \quad r^{\pm} = r^{\pm}(S_K^D(\mathbf{A}))\} \tag{4.10}$$

for nonnegative integers $r^0, r^{\pm}$ with $r^0 + r^- + r^+ = d + r$. Equation (4.9) gives the $2d + 1$ layers in $\mathcal{B}^{\mathbb{C}}$, while (4.10) divides $\mathcal{O}_K^{\mathbb{C}}$ into different areas. Theorem 4.4 below indicates that the n-th eigenvalue exhibits the same jump phenomena in any given area. By lemma 3.4, we have the following result.

Lemma 4.2. $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A})) = Nd - k$ *for* $\mathbf{A} \in \mathcal{B}_k$, *and* $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A})) = Nd - r^0$ *for* $\mathbf{A} \in \mathcal{B}_K^{(r^0,r^+,r^-)}$.

Lemma 4.3. *Let* $\mathbf{A} \in \mathcal{B}_K^{(r_1^0,r_1^+,r_1^-)}$. *Then*

$$\mathcal{U}_\varepsilon^{(r^0,r^+,r^-)} := \mathcal{U}_\varepsilon \cap \mathcal{B}_K^{(r^0,r^+,r^-)}$$

with $\mathcal{U}_\varepsilon = \{\mathbf{B} \in \mathcal{O}_K^{\mathbb{C}} : \|S(\mathbf{B}) - S(\mathbf{A})\|_{\mathcal{M}_{2d}} < \varepsilon\}$ *is path connected for any* $r^0 \leqslant r_1^0$, $r^{\pm} \geqslant r_1^{\pm}$ *satisfying* $r^0 + r^+ + r^- = d + r$, *and* $\varepsilon > 0$ *sufficiently small.*

Proof. The proof is similar to lemma 7.2 in [8]. □

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We are now in a position to give the complete characterization of jump phenomena of the n-th eigenvalue on the boundary conditions.

THEOREM 4.4. *Fix* $\boldsymbol{\omega} \in \Omega_N^{\mathbb{C}}$.

(1) *Let* $0 \leqslant k \leqslant 2d$. *Then the restriction of* λ_n *to* $\mathcal{B}_k$ *is continuous for any* $1 \leqslant n \leqslant Nd - k$.

(2) *Consider the restriction of* λ_n *to* $\mathcal{O}_K^{\mathbb{C}}$, *where* $K \subset \{1, \ldots, 2d\}$. *Let* $0 \leqslant r^0 < r_1^0 \leqslant d + r$ *and* $r^{\pm} \geqslant r_1^{\pm}$. *Then for any* $\mathbf{A} \in \mathcal{B}_K^{(r_1^0, r_1^+, r_1^-)}$, *we have*

$$\lim_{\mathcal{B}_K^{(r^0, r^+, r^-)} \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = -\infty, \quad 1 \leqslant n \leqslant r^+ - r_1^+, \tag{4.11}$$

$$\lim_{\mathcal{B}_K^{(r^0, r^+, r^-)} \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = \lambda_{n-(r^+ - r_1^+)}(\mathbf{A}),$$
$$r^+ - r_1^+ < n \leqslant Nd - r^0 - (r^- - r_1^-), \tag{4.12}$$

$$\lim_{\mathcal{B}_K^{(r^0, r^+, r^-)} \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = +\infty, \quad Nd - r^0 - (r^- - r_1^-) < n \leqslant Nd - r^0. \tag{4.13}$$

Consequently, the jump set is $\cup_{1 \leqslant k \leqslant 2d} \mathcal{B}_k$.

Proof. (1) is a direct consequence of lemmas 3.13 and 4.2. Now, we prove (2). Note that $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A})) = Nd - r_1^0$ by lemma 4.2. Choose $c_1, c_2 \in \mathbb{R}$ such that $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A}) \cap (c_1, c_2)) = Nd - r_1^0$. By lemma 3.5, there exists $\varepsilon > 0$ such that for all $\mathbf{B} \in \mathcal{U}_\varepsilon$ defined in lemma 4.3, we have $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{B}) \cap (c_1, c_2)) = Nd - r_1^0$ and $c_1, c_2 \notin \sigma(\boldsymbol{\omega}, \mathbf{B})$. It follows from lemma 4.3 that $\mathcal{U}_\varepsilon^{(r^0, r^+, r^-)}$ is path connected. By lemma 4.2, $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{B})) = Nd - r^0$ for $\mathbf{B} \in \mathcal{U}_\varepsilon^{(r^0, r^+, r^-)}$, and thus $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{B}) \cap ((-\infty, c_1) \cup (c_2, +\infty))) = r_1^0 - r^0$. Let $\sigma(\boldsymbol{\omega}, \mathbf{B}) \cap ((-\infty, c_1) \cup (c_2, +\infty)) := \{\hat{\lambda}_1(\mathbf{B}) \leqslant \cdots \leqslant \hat{\lambda}_{r_1^0 - r^0}(\mathbf{B})\}$ for $\mathbf{B} \in \mathcal{U}_\varepsilon^{(r^0, r^+, r^-)}$. By lemma 3.15 (1), either $\hat{\lambda}_n(\mathcal{U}_\varepsilon^{(r^0, r^+, r^-)}) \subset (-\infty, c_1)$ or $\hat{\lambda}_n(\mathcal{U}_\varepsilon^{(r^0, r^+, r^-)}) \subset (c_2, +\infty)$ for all $1 \leqslant n \leqslant r_1^0 - r^0$. Then we divide our proof in two steps.

Step 1. We show that

$$\lim_{\mathcal{B}_K^{(r^0, r^+, r^-)} \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = -\infty, \quad 1 \leqslant n \leqslant r^+ - r_1^+, \tag{4.14}$$

$$\lim_{\mathcal{B}_K^{(r^0, r^+, r^-)} \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = \lambda_{n-(r^+ - r_1^+)}(\mathbf{A}), \quad r^+ - r_1^+ < n \leqslant Nd - r^0, \tag{4.15}$$

for $r^+ > r_1^+, r^- = r_1^-$; and

$$\lim_{\mathcal{B}_K^{(r^0, r^+, r^-)} \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = \lambda_n(\mathbf{A}), \quad 1 \leqslant n \leqslant (Nd - r^0) - (r^- - r_1^-), \tag{4.16}$$

$$\lim_{\mathcal{B}_K^{(r^0, r^+, r^-)} \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = +\infty, \quad (Nd - r^0) - (r^- - r_1^-) < n \leqslant Nd - r^0 \tag{4.17}$$

for $r^+ = r_1^+, r^- > r_1^-$.

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Consider $r^+ > r_1^+$ and $r^- = r_1^-$. In this case, $r_1^0 - r^0 = r^+ - r_1^+$. Note that there exists a unitary matrix $M \in \mathcal{M}_{d+r}$ such that

$$S_K^D(\mathbf{A}) = M \begin{pmatrix} M_+ & & \\ & M_- & \\ & & 0_{r_1^0} \end{pmatrix} M^*,$$

where $M_+ = \operatorname{diag}\{\mu_1, \ldots, \mu_{r_1^+}\}$ with $\mu_i > 0,\ 1 \leqslant i \leqslant r_1^+$, and $M_- = \operatorname{diag}\{\nu_1, \ldots, \nu_{r_1^-}\}$ with $\nu_j < 0,\ 1 \leqslant j \leqslant r_1^-$. Recall that $S = S(\mathbf{A})$. If $K_2 \neq \emptyset$, we define $\mathbf{B}_t = [S(\mathbf{B}_t) \,|\, I_{2d}]E_K$ with

$$\begin{aligned} &(s_{ij}(\mathbf{B}_t))_{i,j\in\{1,\ldots,d,k_1+d,\ldots,k_r+d\}} \\ &= \begin{pmatrix} S_1 & S_2E_0 \\ E_0^*S_2^* & E_0^*S_3E_0 \end{pmatrix} + M \begin{pmatrix} 0_{r_1^+ + r_1^-} & & \\ & tI_{r^+ - r_1^+} & \\ & & 0_{r^0} \end{pmatrix} M^*, \end{aligned} \tag{4.18}$$

$t \geqslant 0$ is sufficiently small, and $s_{ij}(\mathbf{B}_t) = s_{ij}$ if $i \in \{d+1, \ldots, 2d\} \setminus K_2$ or $j \in \{d+1, \ldots, 2d\} \setminus K_2$. If $K_2 = \emptyset$, we only modify (4.18) as

$$(s_{ij}(\mathbf{B}_t))_{i,j\in\{1,\ldots,d\}} = S_1 + M \begin{pmatrix} 0_{r_1^+ + r_1^-} & & \\ & tI_{r^+ - r_1^+} & \\ & & 0_{r^0} \end{pmatrix} M^*$$

in the definition of $\mathbf{B}_t$. Then $\mathbf{B}_0 = \mathbf{A}$,

$$S_K^D(\mathbf{B}_t) = M \begin{pmatrix} M_+ & & & \\ & M_- & & \\ & & tI_{r^+ - r_1^+} & \\ & & & 0_{r^0} \end{pmatrix} M^*, t > 0,$$

and thus $r^0(S_K^D(\mathbf{B}_t)) = r^0$, $r^\pm(S_K^D(\mathbf{B}_t)) = r^\pm$, which gives $\mathbf{B}_t \in \mathcal{U}_\varepsilon^{(r^0,r^+,r^-)}$. Moreover, $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{B}_t)) = Nd - r^0$ for $t > 0$, and $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{B}_0)) = Nd - r_1^0$. It follows from lemma 3.13 that for any fixed $1 \leqslant n \leqslant Nd - r^0$, $\lambda_n(\mathbf{B}_\cdot)$ is locally a continuous eigenvalue branch for $t > 0$. Since

$$S_K^D(\mathbf{B}_{t_2}) - S_K^D(\mathbf{B}_{t_1}) = M \begin{pmatrix} 0_{r_1^+ + r_1^-} & & \\ & (t_2 - t_1)I_{r^+ - r_1^+} & \\ & & 0_{r^0} \end{pmatrix} M^* \tag{4.19}$$

is a positive semi-definite matrix, we get by corollary 3.12 that $\lambda_n(\mathbf{B}_{t_1}) \leqslant \lambda_n(\mathbf{B}_{t_2})$ with $0 < t_1 < t_2$ for all $1 \leqslant n \leqslant Nd - r^0$. Hence, by lemma 3.16 (4), $\lim_{t\to 0^+} \lambda_n(\mathbf{B}_t) = -\infty, 1 \leqslant n \leqslant r^+ - r_1^+$. Therefore, there exists $t_0 > 0$ such that $\mathbf{B}_{t_0} \in \mathcal{U}_\varepsilon^{(r^0,r^+,r^-)}$ and $\lambda_n(\mathbf{B}_{t_0}) < c_1$, $1 \leqslant n \leqslant r^+ - r_1^+$, which yields that $\hat{\lambda}_n(\mathbf{B}_{t_0}) = \lambda_n(\mathbf{B}_{t_0})$. According to lemma 3.15 (1), $\hat{\lambda}_n(\mathcal{U}_\varepsilon^{(r^0,r^+,r^-)}) = \lambda_n(\mathcal{U}_\varepsilon^{(r^0,r^+,r^-)}) \subset (-\infty, c_1)$, $1 \leqslant n \leqslant r^+ - r_1^+$, and (4.14) holds. Thanks to lemma 3.14, we get (4.15).

Consider $r^+ = r_1^+$ and $r^- > r_1^-$. Since (4.16)–(4.17) can be shown in a similar way, we omit the details.

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Step 2. Show that (4.11)–(4.13) hold for $r^{\pm} > r_1^{\pm}$.

In this case, $r_1^0 - r^0 = (r^+ - r_1^+) + (r^- - r_1^-)$. It follows from (4.14) and (4.17) that

$$\lim_{\mathcal{B}_K^{(r_1^0-(r^+-r_1^+),r^+,r_1^-)} \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = -\infty, \quad 1 \leqslant n \leqslant r^+ - r_1^+,$$
$$\lim_{\mathcal{B}_K^{(r_1^0-(r^--r_1^-),r_1^+,r^-)} \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = +\infty, \quad Nd - r_1^0 < n \leqslant Nd - r_1^0 + (r^- - r_1^-).$$

This implies that $\lambda_n(\tilde{\mathbf{B}}_1) \in (-\infty, c_1)$ with $1 \leqslant n \leqslant r^+ - r_1^+$, and $\lambda_n(\tilde{\mathbf{B}}_2) \in (c_2, +\infty)$ with $Nd - r_1^0 < n \leqslant Nd - r_1^0 + (r^- - r_1^-)$ for any fixed $\tilde{\mathbf{B}}_1 \in \mathcal{U}_\varepsilon^{(r_1^0-(r^+-r_1^+),r^+,r_1^-)}$ and $\tilde{\mathbf{B}}_2 \in \mathcal{U}_\varepsilon^{(r_1^0-(r^--r_1^-),r_1^+,r^-)}$.

Note that $(r^0, r^+, r^-) = (r_1^0 - (r^+ - r_1^+) - (r^- - r_1^-), r^+, r_1^- + (r^- - r_1^-))$. Then we infer from (4.16) that

$$\lim_{\mathcal{B}_K^{(r^0,r^+,r^-)} \ni \mathbf{B} \to \tilde{\mathbf{B}}_1} \lambda_n(\mathbf{B}) = \lambda_n(\tilde{\mathbf{B}}_1), \quad 1 \leqslant n \leqslant Nd - r^0 - (r^- - r_1^-).$$

Since $N \geqslant 2$, we get that

$$Nd - r^0 - (r^- - r_1^-) \geqslant r^+ - r_1^+. \tag{4.20}$$

Therefore, there exists $\tilde{\mathbf{B}}_3 \in \mathcal{U}_\varepsilon^{(r^0,r^+,r^-)}$ such that $\lambda_n(\tilde{\mathbf{B}}_3) \in (-\infty, c_1)$ with $1 \leqslant n \leqslant r^+ - r_1^+$.

On the other hand, $(r^0, r^+, r^-) = (r_1^0 - (r^+ - r_1^+) - (r^- - r_1^-), r_1^+ + (r^+ - r_1^+), r^-)$. Thus we get by (4.15) that

$$\lim_{\mathcal{B}_K^{(r^0,r^+,r^-)} \ni \mathbf{B} \to \tilde{\mathbf{B}}_2} \lambda_n(\mathbf{B}) = \lambda_{n-(r^+-r_1^+)}(\tilde{\mathbf{B}}_2), \quad r^+ - r_1^+ < n \leqslant Nd - r^0,$$

which, along with (4.20), yields that there exists $\tilde{\mathbf{B}}_4 \in \mathcal{U}_\varepsilon^{(r^0,r^+,r^-)}$ such that $\lambda_n(\tilde{\mathbf{B}}_4) \in (c_2, +\infty)$ with $Nd - r^0 - (r^- - r_1^-) < n \leqslant Nd - r^0$. Therefore, we have shown

$$\sharp_1(\sigma(\boldsymbol{\omega}, \tilde{\mathbf{B}}_3) \cap (-\infty, c_1)) = r^+ - r_1^+, \quad \sharp_1(\sigma(\boldsymbol{\omega}, \tilde{\mathbf{B}}_4) \cap (c_2, +\infty)) = r^- - r_1^-. \tag{4.21}$$

Note that $\mathcal{U}_\varepsilon^{(r^0,r^+,r^-)}$ is path connected and $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{B}) \cap (c_1, c_2)) = Nd - r_1^0 = Nd - r^0 - (r^+ - r_1^+) - (r^- - r_1^-)$ for all $\mathbf{B} \in \mathcal{U}_\varepsilon^{(r^0,r^+,r^-)}$. Thus we infer from (4.21) and lemma 3.15 (1) that $\lambda_n(\mathcal{U}_\varepsilon^{(r^0,r^+,r^-)}) \subset (-\infty, c_1)$ for all $1 \leqslant n \leqslant r^+ - r_1^+$, and $\lambda_n(\mathcal{U}_\varepsilon^{(r^0,r^+,r^-)}) \subset (c_2, +\infty)$ for all $Nd - r^0 - (r^- - r_1^-) < n \leqslant Nd - r^0$. Then it follows from lemma 3.15 (2) that (4.11) and (4.13) hold. This, along with lemma 3.14, implies that (4.12) holds. This completes the proof. □

Jump phenomena of the n-th eigenvalue on the Sturm–Liouville equations

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4.2. Jump phenomena of the n-th eigenvalue on the Sturm–Liouville equations

Fix a boundary condition $\mathbf{A} = [(A_1,\ A_2)|(B_1,\ B_2)] = [S \mid I_{2d}]E_K \in \mathcal{O}_K^{\mathbb{C}}$. In this subsection, we always assume that one of the following non-degenerate conditions holds:

$$R \text{ and } E_0^*(S_2^* E_1 R^{-1} S_2 - S_3)E_0 \text{ are invertible if } K_2 \neq \emptyset, \tag{4.22}$$

$$R \text{ is invertible if } K_2 = \emptyset, \tag{4.23}$$

where E_0 is defined in (4.3) and $R := S_1 E_1 + E_1 - I_d$. In particular, the assumption holds for any $\mathbf{A} \in \mathcal{O}_K^{\mathbb{C}}$ when $K_1 = \{1, \ldots, d\}$ and $K_2 = \emptyset$. For any $\boldsymbol{\omega} = (P^{-1}, Q, W) \in \Omega_N^{\mathbb{C}}$, we have by (4.1) that

$$(A_1 P_0^{-1} + B_1,\ B_2) = \begin{pmatrix} RP_0^{-1} + S_1(I_d - E_1) + E_1 & S_2(I_d - E_2) \\ S_2^*(E_1 P_0^{-1} + I_d - E_1) & S_3(I_d - E_2) + E_2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} R^{-1} & 0 \\ S_2^* E_1 R^{-1} & -I_d \end{pmatrix} (A_1 P_0^{-1} + B_1,\ B_2) \tag{4.24}$$
$$= \begin{pmatrix} P_0^{-1} + R^{-1}(S_1(I_d - E_1) + E_1) & R^{-1} S_2 (I_d - E_2) \\ S_2^*[E_1 - I_d + E_1 R^{-1} & (S_2^* E_1 R^{-1} S_2 - S_3) \\ (S_1(I_d - E_1) + E_1)] & (I_d - E_2) - E_2 \end{pmatrix}.$$

Next, we analyse the partitioned structure of the matrix above.

LEMMA 4.5. *$[E_1 - I_d + E_1 R^{-1}(S_1(I_d - E_1) + E_1)]R^* = I_d$ and $S_2^* E_1 R^{-1} S_2 - S_3 \in \mathcal{H}_d(\mathbb{C})$.*

Proof. Direct computation gives

$$[E_1 - I_d + E_1 R^{-1}(S_1(I_d - E_1) + E_1)]R^* = I_d - E_1 + E_1 R^{-1}(E_1 S_1 - S_1 + S_1 E_1).$$

Since

$$E_1 R^{-1} = (E_1 S_1 + E_1 - I_d)^{-1} E_1, \tag{4.25}$$

we have

$$\begin{aligned} &I_d - E_1 + E_1 R^{-1}(E_1 S_1 - S_1 + S_1 E_1) \\ &\quad = I_d - E_1 + (E_1 S_1 + E_1 - I_d)^{-1} E_1 (E_1 S_1 - S_1 + S_1 E_1) \\ &\quad = I_d - E_1 + (E_1 S_1 + E_1 - I_d)^{-1} E_1 S_1 E_1 \\ &\quad = I_d - E_1 + (E_1 S_1 + E_1 - I_d)^{-1}(E_1 S_1 + E_1 - I_d)E_1 = I_d. \end{aligned}$$

$S_2^* E_1 R^{-1} S_2 - S_3 \in \mathcal{H}_d(\mathbb{C})$ follows directly from (4.25). □

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By lemma 4.5, we have $S_2^*[E_1 - I_d + E_1 R^{-1}(S_1(I_d - E_1) + E_1)] = (R^{-1}S_2)^*$. In the case that $K_2 \neq \emptyset$, we define

$$\hat{T}_1 := R^{-1}S_2E_0, \quad \hat{T}_2 := E_0^*(S_2^*E_1R^{-1}S_2 - S_3)E_0.$$

Then by assumption (4.22) and lemma 4.5, $\hat{T}_2 \in \mathcal{H}_r(\mathbb{C})$ is invertible. Direct computation implies that

$$\begin{pmatrix} I_d & -\hat{T}_1\hat{T}_2^{-1} \\ 0 & \hat{T}_2^{-1} \end{pmatrix} \begin{pmatrix} P_0^{-1} + R^{-1}(S_1(I_d - E_1) + E_1) & \hat{T}_1 \\ \hat{T}_1^* & \hat{T}_2 \end{pmatrix} \begin{pmatrix} I_d & 0 \\ -\hat{T}_2^{-1}\hat{T}_1^* & I_r \end{pmatrix} \tag{4.26}$$
$$= \begin{pmatrix} P_0^{-1} + R^{-1}(S_1(I_d - E_1) + E_1) - \hat{T}_1\hat{T}_2^{-1}\hat{T}_1^* & 0 \\ 0 & I_r \end{pmatrix}.$$

In the case that $K_2 = \emptyset$, we have $E_2 = I_d$. Then the next transformation after (4.24) is

$$\begin{pmatrix} P_0^{-1} + R^{-1}(S_1(I_d - E_1) + E_1) & 0 \\ \tilde{R} & -I_d \end{pmatrix} \begin{pmatrix} I_d & 0 \\ \tilde{R} & -I_d \end{pmatrix} \tag{4.27}$$
$$= \begin{pmatrix} P_0^{-1} + R^{-1}(S_1(I_d - E_1) + E_1) & 0 \\ 0 & I_d \end{pmatrix},$$

where $\tilde{R} = S_2^*[E_1 - I_d + E_1R^{-1}(S_1(I_d - E_1) + E_1)]$. For any $\boldsymbol{\omega} \in \Omega_N^{\mathbb{C}}$, we define

$$T(\boldsymbol{\omega}) := P_0^{-1} + R^{-1}(S_1(I_d - E_1) + E_1) - \hat{T}_1\hat{T}_2^{-1}\hat{T}_1^* \text{ if } K_2 \neq \emptyset, \tag{4.28}$$
$$T(\boldsymbol{\omega}) := P_0^{-1} + R^{-1}(S_1(I_d - E_1) + E_1) \text{ if } K_2 = \emptyset, \tag{4.29}$$

and $F(\boldsymbol{\omega}) := (A_1P_0^{-1} + B_1,\, B_2)$ in both cases. Then $\operatorname{rank} F(\boldsymbol{\omega}) = \operatorname{rank} T(\boldsymbol{\omega}) + d$. Moreover, we have the following result.

LEMMA 4.6. *Let $\boldsymbol{\omega} \in \Omega_N^{\mathbb{C}}$. Then $T(\boldsymbol{\omega}) \in \mathcal{H}_d(\mathbb{C})$.*

Proof. Since

$$(S_1 - S_1E_1 + E_1)R^* = S_1E_1 + E_1S_1 - S_1 = R(S_1 - E_1S_1 + E_1),$$

and R is invertible, we have

$$R^{-1}(S_1 - S_1E_1 + E_1) = (S_1 - E_1S_1 + E_1)(R^*)^{-1}.$$

This implies that $P_0^{-1} + R^{-1}(S_1(I_d - E_1) + E_1) \in \mathcal{H}_d(\mathbb{C})$. Since $\hat{T}_2 \in \mathcal{H}_r(\mathbb{C})$ when $K_2 \neq \emptyset$, we get $T(\boldsymbol{\omega}) \in \mathcal{H}_d(\mathbb{C})$. □

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Let

$$l_1 = \max_{\boldsymbol{\omega} \in \Omega_N^{\mathbb{C}}} r^0(F(\boldsymbol{\omega})).$$

Then $l_1 \leqslant d$. Define

$$\mathcal{E}_k := \{\boldsymbol{\omega} \in \Omega_N^{\mathbb{C}} \mid r^0(F(\boldsymbol{\omega})) = k\}, \quad 0 \leqslant k \leqslant l_1, \tag{4.30}$$

$$\mathcal{E}^{(r^0,r^+,r^-)} := \{\boldsymbol{\omega} \in \Omega_N^{\mathbb{C}} \mid r^0 = r^0(T(\boldsymbol{\omega})), \quad r^\pm = r^\pm(T(\boldsymbol{\omega}))\}, \tag{4.31}$$

for attainable nonnegative integers r^0, r^+, r^- with $r^0 + r^+ + r^- = d$. The $l_1 + 1$ layers of $\Omega_N^{\mathbb{C}}$ are given in (4.30), while the areas' division is provided in (4.31). Note here that not all the nonnegative integers $0 \leqslant r^0, r^\pm \leqslant d$ satisfying $r^0 + r^+ + r^- = d$ can be achievable in general, since $P_0^{-1} \in \mathcal{P}_d(\mathbb{C})$ while it is not necessary that $P_0^{-1} - T(\boldsymbol{\omega}) \in \mathcal{P}_d(\mathbb{C})$. Similarly, it is possible that $l_1 < d$. The following result is a direct consequence of lemma 3.4.

Lemma 4.7. *$\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A})) = Nd - k$ for any $\boldsymbol{\omega} \in \mathcal{E}_k$, and $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A})) = Nd - r^0$ for any $\boldsymbol{\omega} \in \mathcal{E}^{(r^0,r^+,r^-)}$.*

Note that the transformations (4.24), (4.26) and (4.27) are independent of $\boldsymbol{\omega} \in \Omega_N^{\mathbb{C}}$. Moreover, the following result holds by the construction of $T(\boldsymbol{\omega})$ and a similar argument to that in the proof of lemma 7.2 in [8].

Lemma 4.8. *Let $\boldsymbol{\omega} \in \mathcal{E}^{(r_1^0,r_1^+,r_1^-)}$. Then $\mathcal{V}_\varepsilon^{(r^0,r^+,r^-)} := \{\boldsymbol{\sigma} \in \Omega_N^{\mathbb{C}} : \|\boldsymbol{\sigma} - \boldsymbol{\omega}\|_{\mathbb{C}^{(3N+1)d^2}} < \varepsilon\} \cap \mathcal{E}^{(r^0,r^+,r^-)}$ is path connected for any $0 \leqslant r^0 \leqslant r_1^0$, $r^\pm \geqslant r_1^\pm$ satisfying $r^0 + r^+ + r^- = d$, and $\varepsilon > 0$ sufficiently small.*

Theorem 4.9. *Fix $\mathbf{A} \in \mathcal{O}_K^{\mathbb{C}}$.*

(1) *Let $0 \leqslant k \leqslant l_1$. Then the restriction of λ_n to $\mathcal{E}_k$ is continuous for any $1 \leqslant n \leqslant Nd - k$.*

(2) *Consider the restriction of λ_n to $\Omega_N^{\mathbb{C}}$. Let $0 \leqslant r^0 < r_1^0 \leqslant l_1$ and $r^\pm \geqslant r_1^\pm$. Then for any $\boldsymbol{\omega} \in \mathcal{E}^{(r_1^0,r_1^+,r_1^-)}$, we have*

$$\lim_{\mathcal{E}^{(r^0,r^+,r^-)} \ni \boldsymbol{\sigma} \to \boldsymbol{\omega}} \lambda_n(\boldsymbol{\sigma}) = -\infty, \quad 1 \leqslant n \leqslant r^- - r_1^-, \tag{4.32}$$

$$\lim_{\mathcal{E}^{(r^0,r^+,r^-)} \ni \boldsymbol{\sigma} \to \boldsymbol{\omega}} \lambda_n(\boldsymbol{\sigma}) = \lambda_{n-(r^- - r_1^-)}(\boldsymbol{\omega}),$$
$$r^- - r_1^- < n \leqslant Nd - r^0 - (r^+ - r_1^+), \tag{4.33}$$

$$\lim_{\mathcal{E}^{(r^0,r^+,r^-)} \ni \boldsymbol{\sigma} \to \boldsymbol{\omega}} \lambda_n(\boldsymbol{\sigma}) = +\infty, \quad Nd - r^0 - (r^+ - r_1^+) < n \leqslant Nd - r^0. \tag{4.34}$$

Consequently, the jump set is $\cup_{1 \leqslant k \leqslant l_1} \mathcal{E}_k$.

Proof. By lemma 4.7, $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A})) = Nd - k$ for any $\boldsymbol{\omega} \in \mathcal{E}_k$. It follows from lemma 3.13 that (1) holds. Choose $c_1, c_2 \in \mathbb{R}$ such that $\sharp_1(\sigma(\boldsymbol{\omega}, \mathbf{A}) \cap (c_1, c_2)) =$

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$Nd - r_1^0$. Then lemmas 3.5, 3.15 (1) and 4.8 ensure that for $\boldsymbol{\sigma} \in \mathcal{V}_\varepsilon^{(r^0,r^+,r^-)}$ with $\varepsilon > 0$ small enough, all the eigenvalues of $(\boldsymbol{\sigma}, \mathbf{A})$ outside $[c_1, c_2]$, denoted by $\tilde{\lambda}_1(\boldsymbol{\sigma}) \leqslant \cdots \leqslant \tilde{\lambda}_{r_1^0 - r^0}(\boldsymbol{\sigma})$, satisfy that either $\tilde{\lambda}_n(\mathcal{V}_\varepsilon^{(r^0,r^+,r^-)}) \subset (-\infty, c_1)$ or $\tilde{\lambda}_n(\mathcal{V}_\varepsilon^{(r^0,r^+,r^-)}) \subset (c_2, +\infty)$, $1 \leqslant n \leqslant r_1^0 - r^0$. Then we divide our proof in two steps.

Step 1. We show that

$$\lim_{\mathcal{E}^{(r^0,r^+,r^-)} \ni \boldsymbol{\sigma} \to \boldsymbol{\omega}} \lambda_n(\boldsymbol{\sigma}) = -\infty, \; 1 \leqslant n \leqslant r^- - r_1^-, \tag{4.35}$$

$$\lim_{\mathcal{E}^{(r^0,r^+,r^-)} \ni \boldsymbol{\sigma} \to \boldsymbol{\omega}} \lambda_n(\boldsymbol{\sigma}) = \lambda_{n-(r^- - r_1^-)}(\boldsymbol{\omega}), \quad r^- - r_1^- < n \leqslant Nd - r^0 \tag{4.36}$$

for $r^- > r_1^-, r^+ = r_1^+$; and

$$\lim_{\mathcal{E}^{(r^0,r^+,r^-)} \ni \boldsymbol{\sigma} \to \boldsymbol{\omega}} \lambda_n(\boldsymbol{\sigma}) = \lambda_n(\boldsymbol{\omega}), \quad 1 \leqslant n \leqslant (Nd - r^0) - (r^+ - r_1^+), \tag{4.37}$$

$$\lim_{\mathcal{E}^{(r^0,r^+,r^-)} \ni \boldsymbol{\sigma} \to \boldsymbol{\omega}} \lambda_n(\boldsymbol{\sigma}) = +\infty, \quad (Nd - r^0) - (r^+ - r_1^+) < n \leqslant Nd - r^0 \tag{4.38}$$

for $r^- = r_1^-, r^+ > r_1^+$.

We only prove (4.35)–(4.36), since (4.37)–(4.38) can be proved similarly. Let $L \in \mathcal{M}_d(\mathbb{C})$ be a unitary matrix such that $T(\boldsymbol{\omega}) = L\,\mathrm{diag}\{\tilde{\mu}_1, \ldots, \tilde{\mu}_d\} L^*$, where $\tilde{\mu}_i$, $1 \leqslant i \leqslant d$, are the eigenvalues of $T(\boldsymbol{\omega})$ and $\tilde{\mu}_1 = \cdots = \tilde{\mu}_{r_1^0} = 0$. Recall that $P_0^{-1}(\boldsymbol{\omega})$ is used to indicate its dependence on $\boldsymbol{\omega}$, while all the components of $\boldsymbol{\omega}$ except P_0^{-1} are fixed. Define

$$P_0^{-1}(\boldsymbol{\sigma}_t) = P_0^{-1}(\boldsymbol{\omega}) + L \begin{pmatrix} tI_{r_1^0 - r^0} & \\ & 0_{d-(r_1^0 - r^0)} \end{pmatrix} L^* \tag{4.39}$$

with $t \leqslant 0$ small enough. Then $\boldsymbol{\sigma}_0 = \boldsymbol{\omega}$ and $\boldsymbol{\sigma}_t \in \mathcal{V}_\varepsilon^{(r^0,r^+,r^-)}$, $t < 0$. Since $P_0^{-1}(\boldsymbol{\sigma}_{t_2}) - P_0^{-1}(\boldsymbol{\sigma}_{t_1})$ is a positive semi-definite matrix for $t_1 < t_2 < 0$, we infer from corollary 3.10 and lemma 3.13 that $\lambda_n(\boldsymbol{\sigma}_{t_1}) \geqslant \lambda_n(\boldsymbol{\sigma}_{t_2})$ for each $1 \leqslant n \leqslant Nd - r^0$. Hence, by lemma 3.16 (1), $\lim_{t \to 0^-} \lambda_n(\boldsymbol{\sigma}_t) = -\infty, 1 \leqslant n \leqslant r^- - r_1^-$. Then we get by lemma 3.15 that $\tilde{\lambda}_n(\mathcal{V}_\varepsilon^{(r^0,r^+,r^-)}) = \lambda_n(\mathcal{V}_\varepsilon^{(r^0,r^+,r^-)}) \subset (-\infty, c_1)$, $1 \leqslant n \leqslant r^- - r_1^-$, satisfy (4.35). This, along with lemma 3.14, yields (4.36).

Step 2. Show that (4.32)–(4.34) hold for $r^\pm > r_1^\pm$.

By (4.35) and (4.38), we have $\lambda_n(\tilde{\boldsymbol{\sigma}}_1) \in (-\infty, c_1)$ with $1 \leqslant n \leqslant r^- - r_1^-$, and $\lambda_n(\tilde{\boldsymbol{\sigma}}_2) \in (c_2, +\infty)$ with $Nd - r_1^0 < n \leqslant Nd - r_1^0 + (r^+ - r_1^+)$ for any fixed $\tilde{\boldsymbol{\sigma}}_1 \in \mathcal{V}_\varepsilon^{(r_1^0 - (r^- - r_1^-), r_1^+, r^-)}$ and $\tilde{\boldsymbol{\sigma}}_2 \in \mathcal{V}_\varepsilon^{(r_1^0 - (r^+ - r_1^+), r^+, r_1^-)}$. Then we infer from (4.36)–(4.37) that

$$\lim_{\mathcal{E}^{(r^0,r^+,r^-)} \ni \boldsymbol{\sigma} \to \tilde{\boldsymbol{\sigma}}_1} \lambda_n(\boldsymbol{\sigma}) = \lambda_n(\tilde{\boldsymbol{\sigma}}_1), \quad 1 \leqslant n \leqslant Nd - r^0 - (r^+ - r_1^+),$$
$$\lim_{\mathcal{E}^{(r^0,r^+,r^-)} \ni \boldsymbol{\sigma} \to \tilde{\boldsymbol{\sigma}}_2} \lambda_n(\boldsymbol{\sigma}) = \lambda_{n-(r^- - r_1^-)}(\tilde{\boldsymbol{\sigma}}_2), \quad r^- - r_1^- < n \leqslant Nd - r^0.$$

Since $Nd - r^0 - (r^+ - r_1^+) \geqslant r^- - r_1^-$, we obtain that there exists $\tilde{\boldsymbol{\sigma}}_3 \in \mathcal{V}_\varepsilon^{(r^0,r^+,r^-)}$ such that $\lambda_n(\tilde{\boldsymbol{\sigma}}_3) \in (-\infty, c_1)$ with $1 \leqslant n \leqslant r^- - r_1^-$, and there exists $\tilde{\boldsymbol{\sigma}}_4 \in$

$\mathcal{V}_\varepsilon^{(r^0,r^+,r^-)}$ such that $\lambda_n(\tilde{\boldsymbol{\sigma}}_4) \in (c_2, +\infty)$ with $Nd - r^0 - (r^+ - r_1^+) < n \leqslant Nd - r^0$. This implies that

$$\sharp_1(\sigma(\boldsymbol{\sigma}, \mathbf{A}) \cap (-\infty, c_1)) = r^- - r_1^-, \quad \sharp_1(\sigma(\boldsymbol{\sigma}, \mathbf{A}) \cap (c_2, +\infty)) = r^+ - r_1^+$$

for all $\boldsymbol{\sigma} \in \mathcal{V}_\varepsilon^{(r^0,r^+,r^-)}$. Then lemma 3.15 (2) ensures that (4.32) and (4.34) hold. Finally, (4.33) is obtained by lemma 3.14. □

5. Applications to d-dimensional Sturm–Liouville problems of Atkinson type

Consider the d-dimensional Sturm–Liouville problem of Atkinson type with $d \geqslant 1$. The continuous Sturm–Liouville equation is

$$-(\hat{P}y')' + \hat{Q}y = \lambda \hat{W} y \text{ on } (a, b), \tag{5.1}$$

where $\hat{P}, \hat{Q}$ and $\hat{W}$ are $d \times d$ Hermitian matrix-valued functions on $[a, b]$, and

$$\hat{P}^{-1}, \hat{Q}, \hat{W} \in L((a, b), \mathbb{C}^{d\times d}).$$

The self-adjoint boundary condition is given by

$$A \begin{pmatrix} -y(a) \\ y(b) \end{pmatrix} + B \begin{pmatrix} (\hat{P}y')(a) \\ (\hat{P}y')(b) \end{pmatrix} = 0, \tag{5.2}$$

where A and B are $2d \times 2d$ complex matrices, where A and B satisfy (1.6). Let $u = y$ and $v = \hat{P}y'$. Then (5.1) is transformed to

$$\begin{cases} u' = \hat{P}^{-1} v, \\ v' = (\hat{Q} - \lambda \hat{W}) u, \end{cases} \tag{5.3}$$

on (a, b). Equation (5.1) is said to be of Atkinson type if there exists a partition of the interval (a, b),

$$a = a_0 < b_0 < a_1 < b_1 < \cdots < a_N < b_N = b$$

for some $N > 1$ such that

$$\hat{P}^{-1} \equiv 0 \text{ on } [a_i, b_i], \quad \hat{W}_i := \int_{a_i}^{b_i} \hat{W}(s)\mathrm{d}s \in \mathcal{P}_d(\mathbb{C}),\ 0 \leqslant i \leqslant N, \tag{5.4}$$

and

$$\hat{Q} \equiv \hat{W} \equiv 0 \text{ on } [b_{j-1}, a_j], \quad \hat{P}_j^{-1} := \int_{b_{j-1}}^{a_j} \hat{P}^{-1}(s)\mathrm{d}s \text{ is invertible }, \ 1 \leqslant j \leqslant N. \tag{5.5}$$

Note that (5.1) is a formal equation due to the definition of $\hat{P}^{-1}$ in (5.4). The conditions (5.4)–(5.5) for Atkinson type should be understood in the sense of (5.3), where $\hat{P}^{-1}$ is regarded as a notation, see also (1.3) and (2.2) in [**12**], or (1.2) and (H4) in [**15**]. A d-dimensional Sturm–Liouville problem is said to be of Atkinson

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type if it consists of (5.1) of Atkinson type and a self-adjoint boundary condition. A 1-dimensional case has been studied in [**1**, **4**, **12**, **15**]. In this section, we always assume that (5.1)–(5.2) is of Atkinson type. The space of Sturm–Liouville equations of Atkinson type is

$$\hat{\Omega} := \left\{ \left(\hat{P}^{-1}, \hat{Q}, \hat{W} \right) \in (L((a,b), \mathbb{C}^{d\times d}))^3 : \text{ (5.4)–(5.5) hold} \right\}$$

with topology induced by $(L((a,b), \mathbb{C}^{d\times d}))^3$. $\hat{\boldsymbol{\omega}} = (\hat{P}^{-1}, \hat{Q}, \hat{W})$ is used for an element in $\hat{\Omega}$. Note that the space of self-adjoint boundary conditions is also $\mathcal{B}^{\mathbb{C}}$ defined by (2.1). Set

$$\hat{Q}_i := \int_{a_i}^{b_i} \hat{Q}(s)\mathrm{d}s, \quad 0 \leqslant i \leqslant N.$$

It follows from (5.4)–(5.5) that if (u, v) is a solution of (5.3), then $u(t) \equiv u_i \in \mathcal{M}_{d\times 1}$ is a constant vector on $[a_i, b_i]$, $0 \leqslant i \leqslant N$, and $v(t) \equiv v_j \in \mathcal{M}_{d\times 1}$ is a constant vector on $[b_{j-1}, a_j]$, $1 \leqslant j \leqslant N$. Furthermore, we define

$$v_0 = v(a), \quad v_{N+1} = v(b), \;\; u_{-1} = u_0 - v_0, \;\; u_{N+1} = u_N + v_{N+1}. \tag{5.6}$$

We construct a d-dimensional discrete Sturm–Liouville problem as follows:

$$-\nabla(\hat{P}_{i+1}\Delta u_i) + \hat{Q}_i u_i = \lambda \hat{W}_i u_i, \quad 0 \leqslant i \leqslant N, \tag{5.7}$$

where $\hat{P}_{N+1} = \hat{P}_0 = I_d$, and a boundary condition

$$A \begin{pmatrix} -u_0 \\ u_N \end{pmatrix} + B \begin{pmatrix} \Delta u_{-1} \\ \Delta u_N \end{pmatrix} = 0, \tag{5.8}$$

where A and B are given in (5.2). By writing A and B into the form (3.2), direct computation implies that (5.8) is equivalent to the standard discrete boundary condition:

$$(A_1,\ A_2) \begin{pmatrix} -u_{-1} \\ u_N \end{pmatrix} + (B_1 - A_1,\ B_2) \begin{pmatrix} \Delta u_{-1} \\ \Delta u_N \end{pmatrix} = 0. \tag{5.9}$$

Now we show that (5.1)–(5.2) is equivalent to the constructed discrete Sturm–Liouville problem above.

LEMMA 5.1.

(1) (5.7)–(5.8) *is a self-adjoint discrete Sturm–Liouville problem.*

(2) (5.1)–(5.2)*) is equivalent to* (5.7)–(5.8).

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Proof. Firstly, we show that (1) holds. Since A and B satisfy (1.6), we have

$$A_1B_1^* + A_2B_2^* = B_1A_1^* + B_2A_2^*, \text{ and rank } (A, B) = 2d.$$

Thus

$$\begin{aligned}(A_1, A_2)\begin{pmatrix} B_1^* - A_1^* \\ mathbf B_2^* \end{pmatrix} &= A_1B_1^* - A_1A_1^* + A_2B_2^* \\ &= B_1A_1^* - A_1A_1^* + B_2A_2^* = (B_1 - A_1, B_2)\begin{pmatrix} A_1^* \\ mathbf A_2^* \end{pmatrix},\end{aligned}$$

and

$$\text{rank}(A_1, A_2, B_1 - A_1, B_2) = \text{rank}\left((A, B)\begin{pmatrix} I_d & & -I_d & \\ & I_d & & \\ & & I_d & \\ & & & I_d \end{pmatrix}\right) = 2d.$$

It follows that (5.9) is a self-adjoint boundary condition. Since $\hat{P}_{j+1}, \hat{Q}_i, \hat{W}_i$ are Hermitian, $\hat{P}_{j+1}$ is invertible, and $\hat{W}_i \in \mathcal{P}_d(\mathbb{C})$ for $0 \leqslant i \leqslant N$ and $-1 \leqslant j \leqslant N$, we have

$$\boldsymbol{\tau} := \left(\{\hat{P}_j^{-1}\}_{j=0}^{N+1}, \{\hat{Q}_i\}_{i=0}^{N}, \{\hat{W}_i\}_{i=0}^{N}\right) \in \Omega_{N+1}^{\mathbb{C}}.$$

Hence, (1) holds.

Next, we prove (2). It suffices to show that (5.3) with (5.2) is equivalent to (5.7)–(5.8). Let (u, v) be a solution of (5.3). Since $v \equiv v_i$ is a constant vector on $[b_{i-1}, a_i]$, we have

$$u_i - u_{i-1} = u(a_i) - u(b_{i-1}) = \int_{b_{i-1}}^{a_i} u'(s)\mathrm{d}s = \int_{b_{i-1}}^{a_i} \hat{P}^{-1}(s)v(s)\mathrm{d}s = \hat{P}_i^{-1}v_i$$

for any $1 \leqslant i \leqslant N$, which, together with (5.6) and the fact that $\hat{P}_{N+1} = \hat{P}_0 = I_d$, yields that

$$\hat{P}_i(u_i - u_{i-1}) = v_i, \quad 0 \leqslant i \leqslant N + 1. \tag{5.10}$$

Since $u \equiv u_j$ is a constant vector on $[a_j, b_j]$, we obtain

$$\begin{aligned} v_{j+1} - v_j &= v(b_j) - v(a_j) \\ &= \int_{a_j}^{b_j} v'(s)\mathrm{d}s = \int_{a_j}^{b_j} (\hat{Q}(s) - \lambda\hat{W}(s))u(s)\mathrm{d}s = (\hat{Q}_j - \lambda\hat{W}_j)u_j \end{aligned} \tag{5.11}$$

for any $0 \leqslant j \leqslant N$. Then (5.7) is obtained by combining (5.10)–(5.11).

Conversely, let $\{u_i\}_{i=-1}^{N+1}$ be a solution of (5.7) and define $v_i = \hat{P}_i(u_i - u_{i-1})$ for $0 \leqslant i \leqslant N+1$. Let $u(t) = u_i$ for all $t \in [a_i, b_i]$ and $0 \leqslant i \leqslant N$, $v(t) = v_j$ for all $t \in$

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$[b_{j-1}, a_j]$ and $1 \leqslant j \leqslant N$, $v(a) = v_0$, and

$$u(t) = u(b_{j-1}) + \int_{b_{j-1}}^{t} \hat{P}^{-1}(s) v_j \mathrm{d}s, \quad t \in [b_{j-1}, a_j],$$

$$v(t) = v(a_i) + \int_{a_i}^{t} (\hat{Q}(s) - \lambda \hat{W}(s)) u_i \mathrm{d}s, \quad t \in [a_i, b_i].$$

Then (u, v) is a solution of (5.3).

Moreover, $y(a) = u_0$, $y(b) = u_N$, $(\hat{P}y')(a) = v(a) = u_0 - u_{-1} = \Delta u_{-1}$ and $(\hat{P}y')(b) = v(b) = u_{N+1} - u_N = \Delta u_N$. Thus (5.2) is equivalent to (5.8). □

By transforming the Sturm–Liouville problem of Atkinson type to the discrete case, we can now determine the number of eigenvalues in the following lemma, which generalizes theorems 2.1 and 3.1 in [**15**] for 1-dimension to any dimension.

LEMMA 5.2. *Let* $(\hat{\boldsymbol{\omega}}, \mathbf{A}) \in \hat{\Omega} \times \mathcal{B}^{\mathbb{C}}$ *with* $\mathbf{A}$ *given in* (3.2)*. Then the eigenvalues of* $(\hat{\boldsymbol{\omega}}, \mathbf{A})$*, including multiplicities, are the same as those of* $(\boldsymbol{\tau}, \mathbf{A}) = (\boldsymbol{\tau}, \mathbf{C})$*, and*

$$\sharp_1(\sigma(\hat{\boldsymbol{\omega}}, \mathbf{A})) = \sharp_1(\sigma(\boldsymbol{\tau}, \mathbf{A})) = \sharp_1(\sigma(\boldsymbol{\tau}, \mathbf{C})) = (N-1)d + \mathrm{rank}(B),$$

where $\boldsymbol{\tau} \in \Omega_{N+1}^{\mathbb{C}}$ *is the transformed discrete Sturm–Liouville equation by* $\hat{\boldsymbol{\omega}}$*, and*

$$\mathbf{C} = [(A_1, A_2)|(B_1 - A_1, B_2)].$$

REMARK 5.3. Note that here $\mathbf{A}$ is under the basis $(-u_0, u_N, \Delta u_{-1}, \Delta u_N)$ when we write $(\boldsymbol{\tau}, \mathbf{A})$, while $\mathbf{C}$ is under the standard basis $(-u_{-1}, u_N, \Delta u_{-1}, \Delta u_N)$ when we write $(\boldsymbol{\tau}, \mathbf{C})$. In this sense, $(\boldsymbol{\tau}, \mathbf{A}) = (\boldsymbol{\tau}, \mathbf{C})$.

Proof. By lemma 5.1 (2), $(\hat{\boldsymbol{\omega}}, \mathbf{A})$ is equivalent to $(\boldsymbol{\tau}, \mathbf{A}) = (\boldsymbol{\tau}, \mathbf{C})$, and $\hat{P}_0 = I_d$. Then applying lemma 3.4 to $(\boldsymbol{\tau}, \mathbf{C})$, we have

$$\begin{aligned}\sharp_1(\sigma(\hat{\boldsymbol{\omega}}, \mathbf{A})) &= \sharp_1(\sigma(\boldsymbol{\tau}, \mathbf{C})) \\ &= ((N+1) - 2)d + \mathrm{rank}(A_1 \hat{P}_0^{-1} + (B_1 - A_1), B_2) = (N-1)d + \mathrm{rank}(B_1, B_2).\end{aligned}$$

This completes the proof. □

Then we study jump phenomena of the n-th eigenvalue of d-dimensional Sturm–Liouville problems of Atkinson type. We first claim in proposition 5.4 below that there is no singularity of the n-th eigenvalue on the equations. In fact, for a fixed $\mathbf{A} \in \mathcal{B}^{\mathbb{C}}$, we infer from lemma 5.2 that $\sharp_1(\sigma(\hat{\boldsymbol{\omega}}, \mathbf{A}))$ is independent of $\hat{\boldsymbol{\omega}} \in \hat{\Omega}$. This, together with lemma 3.13, implies the following result.

PROPOSITION 5.4. *Fix* $\mathbf{A} = [A \mid B] \in \mathcal{B}^{\mathbb{C}}$*. Then the* n*-th eigenvalue is continuous on the whole space of Sturm–Liouville equations of Atkinson type* $\hat{\Omega}$ *for all* $1 \leqslant n \leqslant \sharp_1(\sigma(\hat{\boldsymbol{\omega}}, \mathbf{A})) = (N-1)d + \mathrm{rank}(B)$*.*

Next, we consider jump phenomena of the n-th eigenvalue on the boundary conditions. Lemma 5.2 indicates that it suffices to study jump phenomena of the n-th eigenvalue of $(\boldsymbol{\tau}, \mathbf{C}) = (\boldsymbol{\tau}, \mathbf{A})$ for the fixed $\boldsymbol{\tau}$. The coupled term $B_1 - A_1$ in the standard boundary condition $\mathbf{C}$ makes it hard to apply theorem 4.4 to $(\boldsymbol{\tau}, \mathbf{C})$ directly. We shall apply the method developed in § 3 and 4 to the discrete Sturm–Liouville problem $(\boldsymbol{\tau}, \mathbf{A})$ and provide a direct proof here. In order to study jump phenomena in a certain direction, we need the derivative formula of a continuous simple eigenvalue branch.

LEMMA 5.5. *Fix $\boldsymbol{\tau} \in \Omega_{N+1}^{\mathbb{C}}$. Let λ_* be a simple eigenvalue of $(\boldsymbol{\tau}, \mathbf{A})$ for $\mathbf{A} \in \mathcal{O}_K^{\mathbb{C}}$, $u \in l[-1, N+1]$ be a normalized eigenfunction for λ_*, and Λ be the continuous simple eigenvalue branch through λ_*. Then we have the following derivative formula*

$$\mathrm{d}\Lambda|_{\mathbf{A}}(H) = Z^* E_{K,1}^* H E_{K,1} Z$$

for $H \in \mathcal{H}_{2d}(\mathbb{C})$, where

$$Z^T = (-u_0^T, u_N^T, (\Delta u_{-1})^T, (\Delta u_N)^T).$$

REMARK 5.6. Lemma 3.11 is unable to be directly applied here due to the different basis. Note carefully that Z in lemma 5.5 is different from Y in lemma 3.11.

Proof. Recall that there exists $S \in \mathcal{H}_{2d}(\mathbb{C})$ such that $\mathbf{A} = [S \mid I_{2d}]E_K$. Let $\mathbf{B} = [S + H \mid I_{2d}]E_K$ with $H \in \mathcal{H}_{2d}(\mathbb{C})$. Then by lemma 3.7, there exists an eigenfunction $\tilde{u} = \{\tilde{u}_i\}_{i=-1}^{N+1}$ for $\Lambda(\mathbf{B})$ such that $\tilde{u} \to u$ in $\mathbb{C}^{(N+2)d}$ as $\mathbf{B} \to \mathbf{A}$. Note that $\tilde{u}$ and u satisfy

$$\begin{aligned} &-\nabla(\hat{P}_{i+1}\Delta\tilde{u}_i) + \hat{Q}_i\tilde{u}_i = \Lambda(\mathbf{B})\hat{W}_i\tilde{u}_i, \\ &-\nabla(\hat{P}_{i+1}\Delta u_i) + \hat{Q}_i u_i = \Lambda(\mathbf{A})\hat{W}_i u_i, \quad 0 \leqslant i \leqslant N, \end{aligned}$$

and thus

$$\begin{aligned} (\Lambda(\mathbf{B}) - \Lambda(\mathbf{A}))\sum_{i=0}^{N} \tilde{u}_i^* \hat{W}_i u_i &= \lfloor u_{-1}, \tilde{u}_{-1} \rfloor - \lfloor u_N, \tilde{u}_N \rfloor \\ &= \tilde{u}_0^* u_{-1} - \tilde{u}_{-1}^* u_0 - \lfloor u_N, \tilde{u}_N \rfloor \\ &= (\Delta\tilde{u}_{-1})^* u_0 - \tilde{u}_0^*(\Delta u_{-1}) - \lfloor u_N, \tilde{u}_N \rfloor, \end{aligned}$$

where $\lfloor u_i, \tilde{u}_i \rfloor = (\Delta\tilde{u}_i)^* \hat{P}_{i+1} u_i - \tilde{u}_i^* \hat{P}_{i+1} \Delta u_i$. $\mathbf{A}$ and $\mathbf{B}$ tell us that $S E_{K,1} Z + E_{K,2} Z = 0$ and $(S + H)E_{K,1}\tilde{Z} + E_{K,2}\tilde{Z} = 0$. Then we infer from (2.5) that

$$(\Lambda(\mathbf{B}) - \Lambda(\mathbf{A}))\sum_{i=0}^{N} \tilde{u}_i^* \hat{W}_i u_i = \tilde{Z}^* E_K^* J_{2d}^* E_K Z = \tilde{Z}^* E_{K,1}^* H E_{K,1} Z.$$

This completes the proof. □

As a consequence, we get the following conclusion.

COROLLARY 5.7. *Let Λ be a continuous eigenvalue branch defined on $\mathcal{U} \subset \mathcal{O}_K^{\mathbb{C}}$. Then $\Lambda(\mathbf{A}) \leqslant \Lambda(\mathbf{B})$ if $\mathbf{A}, \mathbf{B} \in \mathcal{U}$ and $S(\mathbf{B}) - S(\mathbf{A})$ is positive semi-definite.*

Let the k-th layer in $\mathcal{B}^{\mathbb{C}}$ be defined as

$$\hat{\Sigma}_k := \{\mathbf{A} \in \mathcal{B}^{\mathbb{C}} \mid r^0(B) = k\}, \quad 0 \leqslant k \leqslant 2d.$$

Then the following result is a direct consequence of lemma 5.2.

COROLLARY 5.8. *Fix* $\hat{\boldsymbol{\omega}} \in \hat{\Omega}$. *Then*

(1) $\sharp_1(\sigma(\hat{\boldsymbol{\omega}}, \mathbf{A})) = (N+1)d - k$ *for all* $\mathbf{A} \in \hat{\Sigma}_k$.

(2) $\sharp_1(\sigma(\hat{\boldsymbol{\omega}}, \mathbf{A})) = (N+1)d$ *for all* $\mathbf{A} \in \mathcal{O}^{\mathbb{C}}_{\emptyset}$.

For a nonempty subset $K = \{n_1, \ldots, n_{m_0}\} \subset \{1, \ldots, 2d\}$, we define $\hat{E}_0 = (\hat{e}_{n_1}, \ldots, \hat{e}_{n_{m_0}})$, and

$$S^A_K(\mathbf{A}) = \hat{E}^*_0 S(\mathbf{A}) \hat{E}_0, \tag{5.12}$$

where $\mathbf{A} \in \mathcal{O}^{\mathbb{C}}_K$, and $\hat{e}_i$ is the i-th column of I_{2d}. The divided area is defined by

$$J^{(r^0,r^+,r^-)}_K := \{\mathbf{A} \in \mathcal{O}^{\mathbb{C}}_K | r^0(S^A_K(\mathbf{A})) = r^0, r^{\pm}(S^A_K(\mathbf{A})) = r^{\pm}\}$$

for three nonnegative integers r^0, r^+ and r^- satisfying $r^0 + r^+ + r^- = m_0$. Then we are ready to provide the complete characterization of jump phenomena of the n-th eigenvalue for the Atkinson type.

THEOREM 5.9. *Fix* $\hat{\boldsymbol{\omega}} \in \hat{\Omega}$.

(1) *Let* $0 \leqslant k \leqslant 2d$. *Then the restriction of* λ_n *to* $\hat{\Sigma}_k$ *is continuous for any* $1 \leqslant n \leqslant (N+1)d - k$. *Moreover, the restriction of* λ_n *to* $\mathcal{O}^{\mathbb{C}}_{\emptyset}$ *is continuous for any* $1 \leqslant n \leqslant (N+1)d$.

(2) *Consider the restriction of* λ_n *to* $\mathcal{O}^{\mathbb{C}}_K$, *where* $\emptyset \neq K \subset \{1, \ldots, 2d\}$. *Let* $0 \leqslant r^0 < r^0_1 \leqslant \sharp(K)$ *and* $r^{\pm} \geqslant r^{\pm}_1$. *Then for any* $\mathbf{A} \in J^{(r^0_1, r^+_1, r^-_1)}_K$, *we have*

$$\lim_{J^{(r^0,r^+,r^-)}_K \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = -\infty, \quad 1 \leqslant n \leqslant r^+ - r^+_1, \tag{5.13}$$

$$\lim_{J^{(r^0,r^+,r^-)}_K \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = \lambda_{n-(r^+ - r^+_1)}(\mathbf{A}),$$
$$r^+ - r^+_1 < n \leqslant (N+1)d - r^0 - (r^- - r^-_1), \tag{5.14}$$

$$\lim_{J^{(r^0,r^+,r^-)}_K \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = +\infty,$$
$$(N+1)d - r^0 - (r^- - r^-_1) < n \leqslant (N+1)d - r^0. \tag{5.15}$$

Consequently, the jump set is $\cup_{1 \leqslant k \leqslant 2d} \hat{\Sigma}_k$.

REMARK 5.10. Note that $S^A_K(\mathbf{A})$ is independent of the Sturm–Liouville equations of Atkinson type, while $S^D_K(\mathbf{A})$ defined in (4.5)–(4.6) is indeed involved heavily with the coefficient P^{-1}_0 of the discrete equations.

Proof. We study the equivalent discrete Sturm–Liouville problem $(\boldsymbol{\tau}, \mathbf{A})$. (1) is straightforward by lemma 3.13 and corollary 5.8. Next, we show that (2) holds. Choose $c_1, c_2 \in \mathbb{R}$ such that $\sharp_1(\sigma(\boldsymbol{\tau}, \mathbf{A}) \cap (c_1, c_2)) = (N+1)d - r_1^0$. It follows from lemma 3.5 that $\sharp_1(\sigma(\boldsymbol{\tau}, \mathbf{B}) \cap (c_1, c_2)) = (N+1)d - r_1^0$ with $c_1, c_2 \notin \sigma(\boldsymbol{\tau}, \mathbf{B})$ for all $\mathbf{B} \in \mathcal{U}_\varepsilon$, defined in lemma 4.3, and $\varepsilon > 0$ small enough. Lemma 7.2 in [8] implies that $\hat{\mathcal{U}}_\varepsilon^{(r^0,r^+,r^-)} = \mathcal{U}_\varepsilon \cap J_K^{(r^0,r^+,r^-)}$ is path connected. Note that

$$\sharp_1(\sigma(\boldsymbol{\tau}, \mathbf{B}) \cap ((-\infty, c_1) \cup (c_2, +\infty))) = r_1^0 - r^0 \tag{5.16}$$

for $\mathbf{B} \in \hat{\mathcal{U}}_\varepsilon^{(r^0,r^+,r^-)}$.

We show that if $r^+ > r_1^+$ and $r^- = r_1^-$, then (4.14′)–(4.15′) hold. Similarly, if $r^+ = r_1^+$ and $r^- > r_1^-$, then (4.16′)–(4.17′) hold. Here (4.14′)–(4.17′) are defined as (4.14)–(4.17) with $\mathcal{B}_K^{(r^0,r^+,r^-)}$ and N replaced by $J_K^{(r^0,r^+,r^-)}$ and $N+1$. Let $\hat{M} \in \mathcal{M}_{m_0}$ be the unitary matrix such that $S_K^A(\mathbf{A}) = \hat{M}\operatorname{diag}\{\tilde{\nu}_1, \ldots, \tilde{\nu}_{m_0}\}\hat{M}^*$, where $m_0 = \sharp(K)$ and $\tilde{\nu}_1 = \cdots = \tilde{\nu}_{r_1^0} = 0$. Define $\hat{\mathbf{B}}_t = [S(\hat{\mathbf{B}}_t) \,|\, I_{2d}]E_K$ with

$$S_K^A(\hat{\mathbf{B}}_t) = S_K^A(\mathbf{A}) + \hat{M}\begin{pmatrix} tI_{r^+-r_1^+} & \\ & 0_{m_0-(r^+-r_1^+)} \end{pmatrix}\hat{M}^*,$$

$t \geqslant 0$ is sufficiently small, and $s_{ij}(\hat{\mathbf{B}}_t) = s_{ij}(\mathbf{A})$ if $i \in \{1, \ldots, 2d\} \setminus K$ or $j \in \{1, \ldots, 2d\} \setminus K$. Since $S(\hat{\mathbf{B}}_{t_2}) - S(\hat{\mathbf{B}}_{t_1})$ is a positive semi-definite matrix for $t_2 > t_1 > 0$, it follows from lemma 3.13 and corollary 5.7 that $\lambda_n(\hat{\mathbf{B}}_\cdot)$ is non-decreasing on $t \in (0, \varepsilon)$ for each $1 \leqslant n \leqslant (N+1)d - r^0$, where $\varepsilon > 0$ is small enough. Hence, by lemma 3.16 (4), $\lim_{t\to 0^+} \lambda_n(\hat{\mathbf{B}}_t) = -\infty, 1 \leqslant n \leqslant r^+ - r_1^+ = r_1^0 - r^0$. This, along with lemma 3.15 and (5.16), yields that $\lambda_n(\hat{\mathcal{U}}_\varepsilon^{(r^0,r^+,r^-)}) \subset (-\infty, c_1)$, $1 \leqslant n \leqslant r^+ - r_1^+$, and (4.14′) holds. Then we get by lemma 3.14 that (4.15′) holds.

Finally, we prove (5.13)–(5.15) for $r^\pm > r_1^\pm$. It follows from (4.14′) and (4.17′) that $\lambda_n(\tilde{\mathbf{A}}_1) \in (-\infty, c_1)$ with $1 \leqslant n \leqslant r^+ - r_1^+$, and $\lambda_n(\tilde{\mathbf{A}}_2) \in (c_2, +\infty)$ with $(N+1)d - r_1^0 < n \leqslant (N+1)d - r_1^0 + (r^- - r_1^-)$ for any fixed $\tilde{\mathbf{A}}_1 \in \hat{\mathcal{U}}_\varepsilon^{(r_1^0-(r^+-r_1^+),r^+,r_1^-)}$ and $\tilde{\mathbf{A}}_2 \in \hat{\mathcal{U}}_\varepsilon^{(r_1^0-(r^--r_1^-),r_1^+,r^-)}$. Furthermore, we have by (4.16′) that there exists $\tilde{\mathbf{A}}_3 \in \hat{\mathcal{U}}_\varepsilon^{(r^0,r^+,r^-)}$ such that $\lambda_n(\tilde{\mathbf{A}}_3) \in (-\infty, c_1)$ with $1 \leqslant n \leqslant r^+ - r_1^+$. It follows from (4.15′) that there exists $\tilde{\mathbf{A}}_4 \in \hat{\mathcal{U}}_\varepsilon^{(r^0,r^+,r^-)}$ such that $\lambda_n(\tilde{\mathbf{A}}_4) \in (c_2, +\infty)$ with $r^+ - r_1^+ \leqslant (N+1)d - r^0 - (r^- - r_1^-) < n \leqslant (N+1)d - r^0$. Then we get by lemma 3.15 (1) that $\lambda_n(\hat{\mathcal{U}}_\varepsilon^{(r^0,r^+,r^-)}) \subset (-\infty, c_1)$ for $1 \leqslant n \leqslant r^+ - r_1^+$, and $\lambda_n(\hat{\mathcal{U}}_\varepsilon^{(r^0,r^+,r^-)}) \subset (c_2, +\infty)$ for $(N+1)d - r^0 - (r^- - r_1^-) < n \leqslant (N+1)d - r^0$. Thanks to lemma 3.15 (2), we get (5.13) and (5.15). Then (5.14) is a direct consequence of lemma 3.14. The proof is complete. □

6. Conclusions and comparisons of jump phenomena of the *n*-th eigenvalue among continuous case, discrete case and Atkinson type

In this section, we compare jump phenomena of the n-th eigenvalue among the Sturm–Liouville problems for the continuous case (1.1)–(1.2) in [8], the discrete case (1.3)–(1.4), and the Atkinson type (5.1)–(5.2).

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(i) Comparison of jump phenomena on boundary conditions.

According to theorem 7.1 in [8], theorems 4.4 and 5.9, the jump phenomena on the boundary conditions are determined by the constructed Hermitian matrices, which are $S_K^C(\mathbf{A})$ given in (4.2) of [8] for the continuous case, $S_K^D(\mathbf{A})$ defined in (4.5)–(4.6) for the discrete case, and $S_K^A(\mathbf{A})$ defined in (5.12) for the Atkinson type, where $\mathbf{A} \in \mathcal{O}_K^{\mathbb{C}}$.

For the continuous case, theorem 7.1 in [8] tells us that the first m_c eigenvalues jump to $-\infty$ as a path of boundary conditions from the lower layer of $\mathcal{O}_K^{\mathbb{C}}$ tends to a given boundary condition in the upper layer. Here the jump number m_c is exactly the number of transitional eigenvalues (from positive to zero) of the determined Hermitian matrices. It is further shown that this number is the Maslov index of the path of boundary conditions in a forthcoming paper.

For the discrete case, theorem 4.4 indicates that not only the first m_d^- eigenvalues jump to $-\infty$, but the last m_d^+ eigenvalues also blow up to $+\infty$ as a path of boundary conditions from the lower layer tends to a given boundary condition in the upper layer. Here the jump number m_d^- has the similar meaning as m_c in the continuous case, while m_d^+ is the number of transitional eigenvalues (from negative to zero) of the determined Hermitian matrices.

For the Atkinson type, theorem 5.9 renders both similar jump phenomena to $\pm\infty$ with numbers $m_a^\pm$ as in the discrete case. However, $m_d^\pm \neq m_a^\pm$ in general, which is due to the fact that the determined Hermitian matrices are different, i.e. $S_K^D(\mathbf{A}) \neq S_K^A(\mathbf{A})$. It is also interesting to see that the determined Hermitian matrices for the continuous case and the Atkinson type are the same, i.e.$S_K^C(\mathbf{A}) = S_K^A(\mathbf{A})$. The jump set in the Atkinson type coincides with that in the continuous case. This, in particular, provides a direct consequence: $m_c = m_a^-$.

The determined Hermitian matrix is independent of coefficients of the Sturm–Liouville equations for the continuous case and the Atkinson type, while the coefficient P_0^{-1} involves heavily in the Hermitian matrix for the discrete case. In addition, the order of the determined Hermitian matrix is $d + \sharp(K_2)$ for the discrete case, while it is $\sharp(K)$ for the continuous case and the Atkinson type. This implies that the maximal jump number in the discrete case is always no less than that in the continuous case and the Atkinson type.

(ii) Comparison of jump phenomena on the equations.

Based on theorem 6.1 in [8], theorem 4.9and proposition 5.4, the n-th eigenvalue has no singularity on coefficients of the Sturm–Liouville equations for the continuous case and the Atkinson type, while indeed exhibits jump phenomena when coefficients of the Sturm–Liouville equations vary for the discrete case.

For the discrete case, theorem 4.9 also provides jump phenomena to both $\pm\infty$ with jump numbers $\tilde{m}_d^\pm$ as a path of equations from the lower layer of $\Omega_N^{\mathbb{C}}$ tends to a given equation in the upper layer. The determined Hermitian matrix is given by $T(\boldsymbol{\omega})$ defined in (4.28)–(4.29). $\tilde{m}_d^-$ is the number of transitional eigenvalues (from negative to zero) of the determined Hermitian matrices, while $\tilde{m}_d^+$ is the number of transitional eigenvalues (from positive to zero) of the determined Hermitian matrices. Here the reverse direction for the transitional eigenvalues in the definitions of $m_d^\pm$ and $\tilde{m}_d^\pm$ is essentially due to the opposite monotonicity of the continuous eigenvalue branches, see corollaries 3.10 and 3.12.

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(iii) Comparison of the method in the proof of jump phenomena.

Compared with the continuous cases in [8] and [13], the jump set in the discrete case is involved heavily with coefficients of the Sturm–Liouville equations. Moreover, the finiteness of spectrum for the discrete case or the Atkinson type makes the method for the continuous case (e.g. continuity principle in [8, 13]) invalid here. Compared with the 1-dimensional discrete case in [22], the first difficulty is how to divide areas in each layer of the considered space such that the n-th eigenvalue has the same jump phenomena in a given area. We study jump phenomena by partitioning and analysing the local coordinate systems, and provide a Hermitian matrix which can determine the areas' division. As mentioned in the Introduction, our approach to proving the asymptotic behaviour of the n-th eigenvalue here should be taken as a generalization of the method developed for 1-dimensional discrete case in [22] to any dimension.

Finally, we list several determined Hermitian matrices as follows in 2-dimensions to exhibit how the difference is between the continuous case (Atkinson type) and the discrete case. The orders of the determined Hermitian matrices for the discrete case are larger than those for the continuous case (Atkinson type) in (2)–(3), (5)–(6) and (8). On the other hand, these orders are the same in (4), (7) and (9). Even though, for example, the maximal jump number is 1 in the continuous case and the Atkinson type, while it is 3 in the discrete case when $K = \{3\}$. However, it is both 2 in any case when $K = \{1, 2\}$.

Let

$$P_0^{-1} = \begin{pmatrix} p_1 & p_2 \\ \bar{p}_2 & p_3 \end{pmatrix}$$

for the discrete case.

(1) $K = \emptyset$.

$$S_K^D(\mathbf{A}) = \begin{pmatrix} s_{11} + \dfrac{p_3}{p_1 p_3 - |p_2|^2} & s_{12} - \dfrac{p_2}{p_1 p_3 - |p_2|^2} \\ \bar{s}_{12} - \dfrac{\bar{p}_2}{p_1 p_3 - |p_2|^2} & s_{22} + \dfrac{p_1}{p_1 p_3 - |p_2|^2} \end{pmatrix}$$

and there are no $S_K^C(\mathbf{A})$ and $S_K^A(\mathbf{A})$, since there is no singularity for the continuous case and the Atkinson type when $K = \emptyset$.

(2) $K = \{1\}$.

$$S_K^C(\mathbf{A}) = S_K^A(\mathbf{A}) = (s_{11}), \quad S_K^D(\mathbf{A}) = \begin{pmatrix} s_{11} - \dfrac{p_1 p_3 - |p_2|^2}{p_3} & s_{12} - \dfrac{p_2}{p_3} \\ \bar{s}_{12} - \dfrac{\bar{p}_2}{p_3} & s_{22} + \dfrac{1}{p_3} \end{pmatrix}.$$

(3) $K = \{3\}$.

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$$S_K^C(\mathbf{A}) = S_K^A(\mathbf{A}) = (s_{33}),$$

$$S_K^D(\mathbf{A}) = \begin{pmatrix} s_{11} + \dfrac{p_3}{p_1p_3 - |p_2|^2} & s_{12} - \dfrac{p_2}{p_1p_3 - |p_2|^2} & s_{13} \\ \bar{s}_{12} - \dfrac{\bar{p}_2}{p_1p_3 - |p_2|^2} & s_{22} + \dfrac{p_1}{p_1p_3 - |p_2|^2} & s_{23} \\ \bar{s}_{13} & \bar{s}_{23} & s_{33} \end{pmatrix}.$$

(4) $K = \{1, 2\}$.

$$S_K^C(\mathbf{A}) = S_K^A(\mathbf{A}) = (s_{ij})_{1\leqslant i,j\leqslant 2}, \quad S_K^D(\mathbf{A}) = \begin{pmatrix} s_{11} - p_1 & s_{12} - p_2 \\ \bar{s}_{12} - \bar{p}_2 & s_{22} - p_3 \end{pmatrix}.$$

(5) $K = \{1, 3\}$.

$$S_K^C(\mathbf{A}) = S_K^A(\mathbf{A}) = (s_{ij})_{i,j\in\{1,3\}},$$

$$S_K^D(\mathbf{A}) = \begin{pmatrix} s_{11} - \dfrac{p_1p_3 - |p_2|^2}{p_3} & s_{12} - \dfrac{p_2}{p_3} & s_{13} \\ \bar{s}_{12} - \dfrac{\bar{p}_2}{p_3} & s_{22} + \dfrac{1}{p_3} & s_{23} \\ \bar{s}_{13} & \bar{s}_{23} & s_{33} \end{pmatrix}.$$

(6) $K = \{3, 4\}$.

$$S_K^C(\mathbf{A}) = S_K^A(\mathbf{A}) = (s_{ij})_{3\leqslant i,j\leqslant 4},$$

$$S_K^D(\mathbf{A}) = \begin{pmatrix} s_{11} + \dfrac{p_3}{p_1p_3 - |p_2|^2} & s_{12} - \dfrac{p_2}{p_1p_3 - |p_2|^2} & s_{13} & s_{14} \\ \bar{s}_{12} - \dfrac{\bar{p}_2}{p_1p_3 - |p_2|^2} & s_{22} + \dfrac{p_1}{p_1p_3 - |p_2|^2} & s_{23} & s_{24} \\ \bar{s}_{13} & \bar{s}_{23} & s_{33} & s_{34} \\ \bar{s}_{14} & \bar{s}_{24} & \bar{s}_{34} & s_{44} \end{pmatrix}.$$

(7) $K = \{1, 2, 3\}$.

$$S_K^C(\mathbf{A}) = S_K^A(\mathbf{A}) = (s_{ij})_{1\leqslant i,j\leqslant 3}, \quad S_K^D(\mathbf{A}) = \begin{pmatrix} s_{11} - p_1 & s_{12} - p_2 & s_{13} \\ \bar{s}_{12} - \bar{p}_2 & s_{22} - p_3 & s_{23} \\ \bar{s}_{13} & \bar{s}_{23} & s_{33} \end{pmatrix}.$$

(8) $K = \{1, 3, 4\}$.

$$S_K^C(\mathbf{A}) = S_K^A(\mathbf{A}) = (s_{ij})_{i,j\in\{1,3,4\}},$$
$$S_K^D(\mathbf{A}) = \begin{pmatrix} s_{11} - \dfrac{p_1p_3 - |p_2|^2}{p_3} & s_{12} - \dfrac{p_2}{p_3} & s_{13} & s_{14} \\ \bar{s}_{12} - \dfrac{\bar{p}_2}{p_3} & s_{22} + \dfrac{1}{p_3} & s_{23} & s_{24} \\ \bar{s}_{13} & \bar{s}_{23} & s_{33} & s_{34} \\ \bar{s}_{14} & \bar{s}_{24} & \bar{s}_{34} & s_{44} \end{pmatrix}.$$

(9) $K = \{1, 2, 3, 4\}$.

$$S_K^C(\mathbf{A}) = S_K^A(\mathbf{A}) = (s_{ij})_{1\leqslant i,j\leqslant 4}, \quad S_K^D(\mathbf{A}) = \begin{pmatrix} s_{11} - p_1 & s_{12} - p_2 & s_{13} & s_{14} \\ \bar{s}_{12} - \bar{p}_2 & s_{22} - p_3 & s_{23} & s_{24} \\ \bar{s}_{13} & \bar{s}_{23} & s_{33} & s_{34} \\ \bar{s}_{14} & \bar{s}_{24} & \bar{s}_{34} & s_{44} \end{pmatrix}.$$

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