Canad. Math. Bull. Vol. **25** (1), 1982

THE LIMIT OF BIASED VARISOLVENT CHEBYSHEV APPROXIMATION

BY
CHARLES B. DUNHAM

ABSTRACT. Best biased and one-sided Chebyshev approximation with respect to a varisolvent approximating function on an interval are considered. The uniform limit of best biased approximations is the (unique) best one-sided approximation if the best one-sided approximation is of maximum degree. Examples are given where the best one-sided approximation is not of maximum degree and failure of uniform convergence and of existence occurs.

Let $[\alpha, \beta]$ be a closed interval and let $C[\alpha, \beta]$ be the space of continuous functions on $[\alpha, \beta]$. For given r in $[0, \infty]$ define

$$\begin{aligned} d_r(y) &= y \qquad y \le 0 \\ &= ry \qquad y > 0 \end{aligned}$$

and for $g \in C[\alpha, \beta]$ define the r-biased Chebyshev norm to be

$$\|g\|_r = \sup\{|d_r(g(x))| : \alpha \le x \le \beta\}.$$

The $\| \ \|_\infty$ norm is also called the one-sided (from above) norm. Let F be an approximating function unisolvent of variable degree on $[\alpha, \beta]$ with parameter space P and bounded degree. The r-biased Chebyshev problem is given $f \in C[\alpha, \beta]$ to find $A^* \in P$ for which $e_r(A) = \|f - F(A, .)\|_r$ attains its infimum $\rho_r(f)$ over $A \in P$. Such a parameter A^* is called best with respect to the r-biased norm and $F(A^*, .)$ is called a best approximation with respect to the r-biased Chebyshev norm.

Varisolvent approximating functions (approximating functions unisolvent of variable degree) are studied in [**9**, Chapter 7] with respect to ordinary Chebyshev approximation. We will assume that the difficulty pointed out in [**1**; **3**] does not occur: we assume

HYPOTHESIS A. For given $A \in P$ and $\varepsilon > 0$ there exists $B, C \in P$ such that

$$F(A, .) - \varepsilon < F(B, .) < F(A, .) < F(C, .) < F(A, .) + \varepsilon.$$

This is a necessary condition for an alternating theory [**9**, 21].

Received by the editors February 22, 1980 and, in revised form, August 5, 1980.
AMS Subject Classification (1980) 41A50.

https://doi.org/10.4153/CMB-1982-007-1 Published online by Cambridge University Press

r-biased Chebyshev approximation, $0<r<\infty$, is introduced in [**5**, 224] under different notation and a general characterization of best approximations is given.

THEOREM. *Let F be of degree n at A. $F(A,.)$ is a best r-biased approximation to f if and only if $d_r(F-F(A,.))$ alternates n times on $[\alpha,\beta]$. A best r-biased approximation is unique.*

If there exists no $F(A,.)\geq f$, the one-sided problem is vacuous. We henceforth assume existence of such an $F(A,.)$.

THEOREM. *Let F be of degree n at A. $F(A,.)\geq f$ is a best one-sided approximation to f if and only if there is a set $x_0,\ldots,x_n$, $\alpha\leq x_0<\cdots<x_n\leq\beta$ such that $f-F(A,.)$ takes alternately the value $-e_\infty(A)$ and 0 on the set. Best one-sided approximations are unique.*

LEMMA 1. *Let $F(A,.)$ be the best one-sided approximation to f on $[\alpha,\beta]$ and F be of degree n at A. Let $\{x_0,\ldots,x_n\}$ be an ordered set of points such that $f-F(A,.)$ is alternately $-e_\infty(A)$ and 0. Let $\delta>1/r$ and $\|f-F(B,.)\|_r\leq e_\infty(A)$. Then*

$$
(1)\qquad \begin{aligned} F(B,x_i)-F(A,x_i)&\geq-\delta\,\|f-F(A,.)\|_\infty \text{ if } f(x_i)-F(A,x_i)=0\\ &\leq\delta\,\|f-F(A,.)\|_\infty \text{ if } f(x_i)-F(A,x_i)=-e_\infty(A)\end{aligned}
$$

Proof. Suppose $F(B,x_i)-F(A,x_i)<-\delta\,\|f-F(A,.)\|_\infty$ and $f(x_i)-F(A,x_i)=0$. Then $|f(x_i)-F(B,x_i)|_r\geq r\delta\,\|f-F(A,.)\|_\infty>\|f-F(A,.)\|_\infty$. Suppose $F(B,x_i)-F(A,x_i)>0$ and $f(x_i)-F(A,x_i)=-e_\infty(A)$, then $f(x_i)-F(B,x_i)<-e_\infty(A)$, hence $\|f-F(B,.)\|_r>e_\infty(A)$.

Let $\|\ \|$ denote the ordinary Chebyshev norm on $[\alpha,\beta]$, which is equal to $\|\ \|_1$.

LEMMA 2. *Let F be of degree n (maximal) at A then for given $\delta>0$ there exists $\eta(\delta)>0$ such that $\|F(A,.)-F(B,.)\|<\eta(\delta)$ if (1) holds and $\eta(\delta)\to 0$ as $\delta\to 0$.*

This lemma was first stated in [**4**] and proven in [**7**].

LEMMA 3. *Let F be unisolvent of degree m at A_k, $k=0,1,\ldots$ and let $\{F(A_k,.)\}$ converge pointwise to $F(A_0,.)$ on m distinct points then $\{F(A_k,.)\}$ converges uniformly to $F(A_0,.)$.*

This lemma is a generalization of a result of Tornheim. It was first stated in [**4**] and proven in [**7**].

LEMMA 4. *Let $F(A,.)$ be the one-sided best approximation to f and $f\neq F(A,.)$, then for $r<\infty$, $\rho_r(f)<e_\infty(A)$.*

https://doi.org/10.4153/CMB-1982-007-1 Published online by Cambridge University Press

Proof. Since $\|g\|_r \le \|g\|_\infty$ for $g \in C[\alpha, \beta]$, we have $\rho_r(f) \le e_\infty(A)$. If $\rho_r(f) = e_\infty(A)$ then $F(A, .)$ is a best r-biased approximation to f. But $f - F(A, .) \le 0$ and so A cannot be best by the alternating characterization of [**5**].

THEOREM. *Let F be unisolvent of variable degree. Let f have a best one-sided approximation $F(A, .)$ and F be of degree n (maximal) at A. There exists M such that $r > M$ implies that there is a best approximation to f with respect to $\|\ \|_r$. Let $r(k)$ be an increasing sequence with limit ∞ and $F(A_k, .)$ be best with respect to $\|\ \|_{r(k)}$ then $\{F(A_k, .)\}$ converges uniformly to $F(A, .)$.*

Proof. The theorem is obvious if $f \equiv F(A, .)$, so we assume that f is not an approximant.

Let $x_0, \ldots, x_n$ be as in Lemma 1. By definition of solvency of degree n at A there exists $\gamma > 0$ such that if $|y_k - F(A, x_k)| < \gamma$, $k = 1, \ldots, n$, then there exists a parameter B satisfying

$$F(B, x_k) = y_k \qquad k = 1, \ldots, n. \tag{2}$$

Using property Z and maximality of n, it is easily seen that F is unisolvent of degree n at such B, and hence B is completely determined by (2). Choose δ such that $\eta(\delta) < \gamma/2$ then by Lemmas 1 and 2, if $r > 1/\delta$ and $\|f - F(B, .)\|_r \le e_\infty(A)$, we have $\|F(A, .) - F(B, .)\| < \gamma/2$. Now let $\|f - F(B_k, .)\|_r$, be a decreasing sequence with limit $\rho_r(f)$, which is less than $e_\infty(A)$ by Lemma 4, then for all k sufficiently large $\|F(A, .) - F(B_k, .)\| < \gamma/2$. Then n-tuples of values at the points $x_1, \ldots, x_n$ of the approximants $F(B_k, .)$ form, therefore, a bounded sequence with subsequence converging to an accumulation point $(y_1, \ldots, y_n)$ which determines a parameter B at which F is unisolvent of degree n. By Lemma 3, $\{F(B_k, .)\}$ converges uniformly on $[\alpha, \beta]$ to $F(B, .)$, hence for all $x \in [\alpha, \beta]$, $|f(x) - F(B, x)| \le \rho_r(f)$ and so $F(B, .)$ is a best approximation to f with respect to $\|\ \|_r$. The first part of the theorem is shown. Now let $\{r(k)\} \to \infty$, then for all k sufficiently large a best approximation $F(A_k, .)$ with respect to the $r(k)$ norm exists. From Lemmas 1 and 2 it follows that $\{F(A_k, .)\}$ converges uniformly to $F(A, .)$.

If (F, P) is unisolvent, all approximations are of maximum degree and we always have uniform convergence of biased approximations to the one-sided approximation.

We now give an example where F is unisolvent of less than maximum degree at the best one-sided approximation and uniform convergence does not occur. Consider the case when $F(A, x) = a_1 \exp(a_2 x)$. It follows from results of Barrar and Loeb [**2**, 594] and of Meinardus and Schwedt [**8**, 312–313] that F is unisolvent of degree 1 at parameters corresponding to the zero function and degree 2 at parameters corresponding to nonzero functions.

https://doi.org/10.4153/CMB-1982-007-1 Published online by Cambridge University Press

EXAMPLE 1. Let $[\alpha, \beta]=[0, 1]$ and $f(x)=x-1$. As $f(1)=0$, $f\leq 0$, and 0 is of degree 1, 0 is the best one-sided approximation to f. As $f\leq 0$, 0 is not a best approximation with respect to the $\| \|_r$ norm, $0<r<\infty$. Let $F(A_k, .)$ be best to f with respect to $\| \|_k$, then $f-F(A_k, .)$ oscillates twice [**5**, 227], hence $F(A_k, .)$ is non-constant. Now

$$\frac{d^2}{dx^2}(f(x)-F(A_k, x))=-F''(A_k, x)=-a_1a_2^2\exp(a_2x).$$

As $F(A_k, .)<0$, $a_1<0$, hence

$$\frac{d^2}{dx^2}(f(x)-F(A_k, x))>0, \qquad 0\leq x\leq 1,$$

and $F(A_k, 0)<f(0)=-1$. Hence $F(A_k, .)\not\rightarrow 0$ and convergence does not occur.

Best biased approximations need not exist if the best one-sided approximation is not of maximum degree.

THEOREM. *Given varisolvent* (F, P) *and u continuous on* $[\alpha, \beta]$, *define* $P_u=\{A : F(A, .)>u\}$. (F, P_u) *is a varisolvent family with the same degrees.*

This follows directly from the definition of varisolvence.

EXAMPLE 2. Take the same problem as in the previous example except we let $u=-1$ and approximate by (F, P_u). Suppose $F(A_k, .)$ is best to f with respect to $\| \|_k$, then by arguments of the preceding example $F(A_k, 0)<-1$, which is a contradiction.

There appears to be no simple treatment of the behaviour of $\rho_r(f)$ as a function of r. The possible non-existence of best approximations complicates analyses greatly. The following example shows that we can have discontinuities even with fixed degree.

EXAMPLE 3. Let $F(a, .)=a$ and $P=\{a : a\notin[0, 1]\}$. We have $\rho_r(0)=0$ for $r<\infty$ but $\rho_\infty(0)=1$.

The major theorem of this paper ensures that $\rho_r(f)\rightarrow\rho_\infty(f)$ if the best one-sided approximation is of maximum degree.

The case where F is merely an alternating approximating function, as considered in [**7**; **9**, section 7-7], is also of interest. The uniform convergence part of the theorem applies by Lemma 1 and 2, but no existence result holds.

As best biased approximations can be computed by the Remez algorithm for approximation with respect to a generalized weight function [**5**, 228] the theorem suggests use of a large bias to get an approximation close to a best one-sided approximation from above.

Let us also consider what happens when the bias factor r tends to zero. Positive deviations are weighted by r and negative deviations weighted by 1.

This is equivalent to weighting positive deviations by 1 and negative deviations by $1/r$, which increases both deviations by a factor of $1/r$. We get by similar arguments

THEOREM. *Let F be unisolvent of variable degree. Let f have a best one-sided approximation from below $F(A, .)$ and F be of degree n (maximal) at A. There exists f such that $r < \varepsilon$ implies that there is a best approximation to f with respect to $\| \|_r$. Let $r(k)$ be a decreasing sequence with limit 0 and $F(A_k, .)$ be best with respect to $\| \|_{r(k)}$ then $\{F(A_k, .)\}$ converges uniformly to $F(A, .)$.*

REFERENCES

1. R. Barrar and H. Loeb, *On N-parameter and unisolvent families*, J. Approx. Theory **1** (1968), 180–181.
2. R. Barrar and H. Loeb, *On the continuity of the nonlinear Tschebyscheff operator*, Pac. J. Math **32** (1970), 593–601.
3. C. Dunham, *Necessity of alternation*, Can. Math. Bull. **11** (1968), 743–744.
4. C. Dunham, *Continuity of the varisolvent Chebyshev operator*, Bull. Amer. Math. Soc. **74** (1968), 606–608.
5. C. Dunham, *Chebyshev approximation with respect to a weight function*, J. Approx. Theory **2** (1969), 223–232.
6. C. Dunham, *Alternating minimax approximation with unequal restraints*, J. Approx. Theory **10** (1974), 199–205.
7. C. Dunham, *Alternating Chebyshev approximation*, Trans. Amer. Math. Soc. **178** (1973), 95–109.
8. G. Meinardus and D. Schwedt, *Nicht-lineare approximationen*, Arch. Rat. Mech. Anal. **17** (1964), 297–326.
9. J. Rice, *The Approximation of Functions*, Volume 2, Addison-Wesley, Reading, Mass., 1969.

COMPUTER SCIENCE DEPARTMENT
UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO N6A 5B9 CANADA