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CONE PRESERVING MAPPINGS FOR QUADRATIC CONES OVER ARBITRARY FIELDS

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1. Introduction and terminology. Let V be a non-singular metric vector space, that is, a vector space over a field F not of characteristic two, upon which is defined a non-singular symmetric bilinear form (,). For any $a \in V$, we define the *cone with vertex* a to be the set

$$C(a) = \{x | (x - a, x - a) = 0, x \in V\}.$$

A mapping $f : V \to V$ will be said to *preserve cones* if $f[C(a)] = C[f(a)]$.

Mappings which preserve Minkowskian cones ($F = R$ and $(x, y) = x_1y_1 - \sum_{i=2}^{n} x_iy_i$ with respect to some basis) have been examined by many authors: we mention only Alexandrov [**1**] and Zeeman [**6**] as two notable examples. We are interested here in a result of Borchers and Hegerfeldt [**4**]. These authors showed that bijections of Minkowski space (of dimension $\geqq 3$) which preserve cones are, up to dilatations and translations, Lorentz transformations; this result was proven by first demonstrating the linearity of the cone-preserving mappings (up to a translation). We shall generalize their result to cones over arbitrary fields by first showing that the cone-preserving mappings are, up to a translation, semi-linear, as defined in [**2**, **3**]:

A *semi-linear bijection* of a vector space W over a field F is a pair of bijections $L : W \to W$, $\tau : F \to F$ such that for all $x, y \in W$, $\alpha \in F$,

i) $L(x + y) = Lx + Ly$

ii) $L(\alpha x) = \alpha^\tau Lx$.

(These two properties imply that τ is an automorphism of F).

Our main result is the following.

THEOREM. *Let V be a non-singular metric vector space over the field F, with bilinear form* (,); *assume that* dim $V \geqq 3$ *and that V is not anisotropic* (*i.e.* $(x, x) = 0$ *for some* $x \neq 0$). *Let $f : V \to V$ be a bijection of V which preserves cones. Then $f(x) = Lx + f(0)$, where (L, τ) is a semi-linear bijection of V satisfying $(Lx, Ly) = \lambda(x, y)^\tau$ for some non-zero $\lambda \in F$ and for all $x, y \in V$.*

Note 1. If $F = R$, then $\tau = \mathrm{id}_R$ (since R has no non-trivial automorphisms) and for some $\mu \in R$, either $\lambda = \mu^2$ or $\lambda = -\mu^2$. In the first case, $i = \mu^{-1}L$ satisfies $(ix, iy) = (x, y)$, i.e. i is an isometry of V. In the second case, $j = \mu^{-1}L$ satisfies $(jx, jy) = -(x, y)$; this statement implies that the bilinear forms

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(,) and $-($, $)$ have the same signature. By Sylvester's law of inertia, this is possible only if the dimension of V is twice its Witt index (to be defined below). Thus the first case includes the Minkowskian case of Borchers and Hegerfeldt.

Note 2. For general F, if $\lambda = \mu^2$ for some $\mu \in F$, then $s = \mu^{-1}L$ satisfies $(sx, sy) = (x, y)^\tau$ for all $x, y \in V$. Semi-linear bijections (s, τ) which satisfy this property are generalizations of isometries, and may be called "semi-isometries." The fact that such mappings arise naturally from transformations which preserve the cones generated by the metric structure of the space indicates that further study of semi-isometries might well prove worthwhile.

The proof of our theorem relies heavily on the geometry of metric vector spaces; a very readable presentation of this appears in Snapper and Troyer [**5**] (see also Artin [**2**]). Those features of this geometry relevant to our discussion are outlined below.

Let V denote any non-singular metric vector space. A bijection of V which preserves the bilinear form (,) of V is called an *isometry*.

The vectors $x, y \in V$ are said to be *orthogonal* if $(x, y) = 0$; this notion extends to orthogonality of subspaces of V. Self-orthogonal vectors are called *null*, or *isotropic*, while self-orthogonal subspaces are called *totally isotropic*. Any subspace U has an *orthogonal complement* $U^\perp$, consisting of all vectors orthogonal to U; U is then said to be *non-singular* if and only if the subspace rad $U = U \cap U^\perp$, called the *radical* of U, is $\{0\}$. There always exists a non-singular subspace $\bar{U}$ of U, unique up to isometry, such that U can be decomposed as the orthogonal direct sum $U = \bar{U} \oplus \text{rad } U$.

The space V has an orthogonal direct sum decomposition of the form

$$V = U \oplus H_1 \oplus \ldots \oplus H_k$$

where U is *anisotropic* (it contains no non-zero null vectors) and each H_i is a *hyperbolic 2-space* (called *Artininan plane* in [**5**]), i.e. it is spanned by two non-orthogonal null vectors. Such decompositions are unique up to isometry, thus the non-negative integer k, called the *Witt index* of V, is an invariant of V. It can be shown that the largest totally isotropic subspace of V has dimension k.

Some notation: for $u, v, w, \ldots \in V$, $\langle u, v, w, \ldots \rangle$ denotes the subspace spanned by $u, v, w, \ldots$. For any subset S of V, S^c denotes the (set-theoretic) complement of S in V.

2. Some geometric properties of cones and subspaces of V. Throughout this section, V denotes a non-singular non-anisotropic metric vector space (thus V has Witt index at least 1).

Definition. A basis of null vectors of any metric vector space will be called a *null basis.*

LEMMA 2.1. *Any non-singular, non-anisotropic metric vector space has a null basis.*

Proof. Let W be such a space; then $W = U \oplus H_1 \oplus \ldots \oplus H_k$ where U is anisotropic and $H_1, \ldots, H_k$ are hyperbolic 2-spaces. If $U = \{0\}$, the required basis can be constructed by taking pairs of non-orthogonal null vectors $n_i, m_i \in H_i$, $i = 1, 2, \ldots, k$. If $U \neq \{0\}$, let $\{u_j\}$ be a basis of U and define the vectors k_j by $k_j = u_j + n_1 + \beta_j m_1$, where $\beta_j = -\frac{1}{2}(u_j, u_j)(n_1, m_1)^{-1}$. Then $\{k_j; n_i, m_i\}$ is the required basis.

LEMMA 2.2. *If V has Witt index at least 2, then for any $x \in V$, $\langle x \rangle^{\perp}$ has a null basis.*

Proof. If x is not null, $\langle x \rangle^{\perp}$ is non-singular and non-anisotropic and Lemma 2.1 applies; the same is true if $x = 0$. If x is null and non-zero, it can be chosen as the vector n_2 of a decomposition of V like that of Lemma 2.1; thus for k_j, n_i, m_i as in Lemma 2.1, $\{k_j, n_i, m_r, r \neq 2\}$ is the required basis.

LEMMA 2.3. *For non-zero $x \in V$, if $\langle x \rangle^{\perp}$ has a null basis,*

$$\langle x \rangle = \bigcap_{n \in \langle x \rangle^{\perp} \cap C(0)} \langle n \rangle^{\perp}.$$

Proof. Because $n \in \langle x \rangle^{\perp} \cap C(0)$ implies $\langle x \rangle \subset \langle n \rangle^{\perp}$, we have

$$\langle x \rangle \subseteq \bigcap_{n \in \langle x \rangle^{\perp} \cap C(0)} \langle n \rangle^{\perp}.$$

But $\langle x \rangle^{\perp}$ has a null basis, so $\langle x \rangle^{\perp} \cap C(0)$ contains $(\dim V) - 1$ linearly independent vectors; hence $\bigcap_{n \in \langle x \rangle^{\perp} \cap C(0)} \langle n \rangle^{\perp}$ must be a line, specifically, the line $\langle x \rangle$.

LEMMA 2.4. *If $m \neq 0$ in V is null, then*

$$C(m) \cap C(0) = \langle m \rangle^{\perp} \cap C(0).$$

Proof. The equations of $C(m)$, $C(0)$ and $\langle m \rangle^{\perp}$ are $(x, x) - 2(x, m) = 0$, $(x, x) = 0$, and $(x, m) = 0$ respectively. Any two of these implies the third, proving our claim.

LEMMA 2.5. *If $m \neq 0$ in V is null, then*

$$\langle m \rangle = \bigcap_{n \in C(m) \cap C(0)} C(n)$$

Proof. Case i). V has Witt index 1. Lemma 2.4 implies that for any non-zero null vector k, $C(0) \cap C(k) = \langle k \rangle$ (since otherwise V would contain a totally isotropic 2-space, an impossibility in spaces of Witt index 1). We obtain

$$\langle m \rangle = \bigcap_{\substack{n \in C(m) \cap C(0) \\ n \neq 0}} [C(n) \cap C(0)] = \bigcap_{n \in C(m) \cap C(0)} C(n).$$

Case ii). V has Witt index >1. Since $\langle m \rangle^{\perp}$ has a null basis by Lemma 2.2, Lemma 2.3 implies

$$\langle m \rangle = \bigcap_{n \in \langle m \rangle^{\perp} \cap C(0)} \langle n \rangle^{\perp}.$$

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But $\langle m\rangle \subset C(0)$, so

$$\langle m\rangle = \bigcap_{n\in\langle m\rangle^{\perp}\cap C(0)} [\langle n\rangle^{\perp} \cap C(0)]$$

$$= \bigcap_{n\in C(m)\cap C(0)} [C(n) \cap C(0)] \quad \text{(by Lemma 2.4)}$$

$$= \bigcap_{n\in C(m)\cap C(0)} C(n)$$

LEMMA 2.6. *If $m \neq 0$ in V is null, then*

$$\bigcup_{n\in\langle m\rangle} C(n) = C(0) \cup (\langle m\rangle^{\perp})^{c}.$$

Proof. i) Assume that $x \in \bigcup_{n\in\langle m\rangle} C(n)$; then $x \in C(n)$ for some $n \in \langle m\rangle$, implying $(x, x) - 2(x, n) = 0$. If $x \notin C(0)$, then $(x, x) \neq 0$, so $(x, n) \neq 0$, i.e. $x \notin \langle n\rangle^{\perp}$. Since $\langle m\rangle^{\perp} = \langle n\rangle^{\perp}$, $x \in (\langle m\rangle^{\perp})^{c}$. Thus $x \in C(0) \cup (\langle m\rangle^{\perp})^{c}$.

ii) Assume that $y \in C(0) \cup (\langle m\rangle^{\perp})^{c}$. If $y \in C(0)$, then $y \in \bigcup_{n\in\langle m\rangle} C(n)$. If $y \notin C(0)$, then $y \notin \langle m\rangle^{\perp}$, so $(y, m) \neq 0$. Consequently, for $\alpha = \frac{1}{2}(y, y)(y, m)^{-1}$, we have $(y - \alpha m, y - \alpha m) = 0$, i.e. $y \in C(\alpha m) \subseteq \bigcup_{n\in\langle m\rangle} C(n)$.

Our claim is proven from i) and ii).

LEMMA 2.7. *If $m \neq 0$ in V is null, then*

$$\langle m\rangle^{\perp} = \left[\left\{\bigcap_{n\in\langle m\rangle} C(n)^{c}\right\} \cup C(0)\right] \cap [C(m) \cup C(0)^{c}].$$

Proof. Put $A = \langle m\rangle^{\perp}$ and $B = C(0)$ in the set-theoretic identity

$$A = [\{A^{c} \cup B\}^{c} \cup B] \cap [(A \cap B) \cup B^{c}]$$

and use Lemmas 2.4 and 2.6.

3. Proof of the theorem. For any $a \in V$, define the mapping $f^{a} : V \rightarrow V$ by

$$f^{a} = T_{-f(a)} \circ f \circ T_{a}$$

where for any $b \in V$, T_{b} denotes the translation $T_{b}x = x + b$. Then f^{a} is bijective, $f^{a}(0) = 0$, and f^{a} preserves cones: $f^{a}[C(x)] = C[f^{a}(x)]$.

In the next three lemmas, where a is fixed, denote the image of any $x \in V$ under f^{a} by $\bar{x} : \bar{x} = f^{a}(x)$.

LEMMA 3.1. *For any non-zero null vector m,*

$$f^{a}(\langle m\rangle) = \langle f^{a}(m)\rangle.$$

Proof. By Lemma 2.5

$$\langle m\rangle = \bigcap_{n\in C(m)\cap C(0)} C(n),$$

thus, since f^{a} preserves cones,

$$f^{a}(\langle m\rangle) = \bigcap_{n\in C(m)\cap C(0)} C(\bar{n}).$$

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But $n \in C(m) \cap C(0)$ if and only if $\bar{n} \in C(\bar{m}) \cap C(0)$, thus

$$f^a(\langle m \rangle) = \bigcap_{\bar{n} \in C(\bar{m}) \cap C(0)} C(\bar{n}) = \langle \bar{m} \rangle = \langle f^a(m) \rangle,$$

using Lemma 2.5 again.

LEMMA 3.2. *For non-zero null vectors m,*

$$f^a(\langle m \rangle^\perp) = \langle f^a(m) \rangle^\perp.$$

Proof. From Lemma 2.7,

$$\langle m \rangle^\perp = \left[\left\{ \bigcap_{n \in \langle m \rangle} C(n)^c \right\} \cup C(0) \right] \cap [C(m) \cup C(0)^c].$$

Since f^a is bijective, it preserves unions, intersections and complements of subsets of V; hence

$$f^a(\langle m \rangle^\perp) = \left[\left\{ \bigcap_{n \in \langle m \rangle} C(\bar{n})^c \right\} \cup C(0) \right] \cap [C(\bar{m}) \cup C(0)^c]$$

But by Lemma 3.1, $n \in \langle m \rangle$ if and only if $\bar{n} \in \langle \bar{m} \rangle$, so

$$f^a(\langle m \rangle^\perp) = \left[\left\{ \bigcap_{\bar{n} \in \langle \bar{m} \rangle} C(\bar{n})^c \right\} \cup C(0) \right] \cap [C(\bar{m}) \cup C(0)^c] = \langle \bar{m} \rangle^\perp = \langle f^a(m) \rangle^\perp$$

by another application of Lemma 2.7.

LEMMA 3.3. *For any non-zero $x \in V$, if $\langle x \rangle^\perp$ has a null basis, then $f^a(\langle x \rangle) = \langle f^a(x) \rangle$.*

Proof. For null x, Lemma 3.1 applies. For non-null x, Lemma 2.3 yields

$$\langle x \rangle = \bigcap_{n \in \langle x \rangle^\perp \cap C(0)} \langle n \rangle^\perp,$$

thus, using Lemma 3.2

$$f^a(\langle x \rangle) = \bigcap_{n \in \langle x \rangle^\perp \cap C(0)} \langle \bar{n} \rangle^\perp.$$

But $n \in \langle x \rangle^\perp$ if and only if $x \in \langle n \rangle^\perp$, which, by Lemma 3.2, is true if and only if $\bar{x} \in \langle \bar{n} \rangle^\perp$, or $\bar{n} \in \langle \bar{x} \rangle^\perp$. Thus

$$f^a(\langle x \rangle) = \bigcap_{\bar{n} \in \langle \bar{x} \rangle^\perp \cap C(0)} \langle \bar{n} \rangle^\perp .$$

If $\langle \bar{x} \rangle^\perp$ has no null basis, it is anisotropic by Lemma 2.1 (it is not singular, since $\bar{x}$ is not null), and the above equation gives the contradiction $f^a(\langle x \rangle) =$

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$\langle 0\rangle^\perp = V$. Thus $\langle \bar{x}\rangle^\perp$ has a null basis, and Lemma 2.3 yields $f^a(\langle x\rangle) = \langle \bar{x}\rangle = \langle f^a(x)\rangle$.

COROLLARY. *If $x \neq 0$ is such that $\langle x\rangle^\perp$ has a null basis, then f maps lines parallel to x into lines.*

Proof. Such a line is a coset of the form $a + \langle x\rangle$ for some $a \in V$; thus $f(a + \langle x\rangle) = f \circ T_a(\langle x\rangle) = T_{f(a)} \circ f^a(\langle x\rangle) = f(a) + \langle f^a(x)\rangle$.

From the above corollary, Lemma 2.2 and the fundamental theorem of projective geometry (see [**2**] or [**3**]) if V has Witt index at least 2, $f(x) = Lx + f(0)$ for some semi-linear bijection (L, τ) of V. We now consider the case where V has Witt index 1. The following is an algebraic generalization of results in [**4**] for Minkowskian spaces.

LEMMA 3.4. *If V has Witt index 1, then for any $a \in V$, f^a maps hyperbolic 2-spaces into hyperbolic 2-spaces.*

Proof. Let $P = \langle m, n\rangle$ be a hyperbolic 2-space in V, where m and n are non-orthogonal null vectors. Then $P' = \langle f^a(m), f^a(n)\rangle$ is also a hyperbolic 2-space (it contains the distinct null lines $\langle f^a(m)\rangle = f^a(\langle m\rangle)$ and $\langle f^a(n)\rangle = f^a(\langle n\rangle)$, which are not orthogonal, since V has Witt index 1). Let k be any non-zero null vector not contained in P, and pick $s \in \langle k\rangle^\perp \cap P$, $s \neq 0$. Because V has Witt index 1, s is not null (else $\langle s, k\rangle$ would be a totally isotropic 2-space in V) thus s is not parallel to m or n. By Lemma 2.1 and the corollary to Lemma 3.3, since $\langle s\rangle^\perp$ is non-singular and non-anisotropic (it contains k), f maps lines parallel to s into lines. It follows that f^a does the same.

Now any $y \in P\backslash\langle s\rangle$ lies on a line parallel to s which intersects $\langle n\rangle$ and $\langle m\rangle$ at distinct points. Hence $f^a(y)$ lies on a line intersecting $\langle f^a(n)\rangle$ and $\langle f^a(m)\rangle$ at distinct points, so $f^a(y) \in P'$.

Any $y \in \langle s\rangle$ lies on some line l parallel to n. Since $l\backslash\{y\} \subset P\backslash\langle s\rangle$, $f^a(l\backslash\{y\}) \subset f^a(P\backslash\langle s\rangle) \subset P'$. But $f^a(l)$ is a line (Lemma 3.1 implies f^a maps lines parallel n into lines), thus $f^a(y) \in P'$.

Therefore $f^a(P) \subseteq P'$.

LEMMA 3.5. *If $\langle x\rangle^\perp$ has no null basis for some $x \neq 0$, then f maps lines parallel to x into lines.*

Proof. By Lemma 2.2, V has Witt index 1. If x is null, Lemma 3.1 yields the required result. If x is not null, then $\langle x\rangle^\perp$ is non-singular and thus anisotropic by Lemma 2.1. For some two null vectors $n, m \in V$, x, n and m are linearly independent, and neither of (x, n), (x, m) is zero (since $\langle x\rangle^\perp$ is anisotropic); thus $\langle x, m\rangle$ and $\langle x, n\rangle$ are hyperbolic 2-spaces. From $\langle x\rangle = \langle x, m\rangle \cap \langle x, n\rangle$ and Lemma 3.4 follows that $f^a(\langle x\rangle)$ is contained in a line, and from this, our desired conclusion.

Now Lemma 3.5 and the fundamental theorem imply that $f(x) = Lx + f(0)$ for some semi-linear bijection (L, τ) of V.

The following lemma completes the proof of our theorem.

LEMMA 3.6. *If (L, τ) is a cone-preserving semi-linear bijection, then for some non-zero $\lambda \in F$,*

$$(Lx, Ly) = \lambda(x, y)^\tau$$

for all $x, y \in V$.

Proof. Set $V = U \oplus H_1 \oplus \ldots \oplus H_k$ as in § 1; then $H_i = \langle m_i, n_i \rangle$ where m_i and n_i are non-orthogonal null vectors. For any $x \in V$, x is null if and only if Lx is null.

i) For $i, j = 1, 2, \ldots, k; i \neq j$, the vectors $n_i, m_i, n_i + n_j, m_i + m_j, n_i + m_j$ are null, implying that their images under L are null. It follows that $(Ln_i, Ln_i) = (Lm_i, Lm_i) = (Ln_i, Ln_j) = (Lm_i, Lm_j) = (Ln_i, Lm_j) = 0$. For any $i = 1, 2, \ldots, k$, $n_i + m_i$ is not null, thus neither is $Ln_i + Lm_i$. Hence $(Ln_i, Lm_i) \neq 0$. Set $(Lm_i, Ln_i) = \lambda_i(m_i, n_i)^\tau, i = 1, 2, \ldots, k$.

ii) We show that all the λ_i's are equal ($k \geqq 2$). For $\beta = -(n_i, m_i)(n_j, m_j)^{-1}$, $b = n_i + m_i + n_j + \beta m_j$ is null, implying that Lb is null, i.e. $(Ln_i, Lm_i) + \beta^\tau(Ln_j, Lm_j) = 0$. Using i), then, $\lambda_i(m_i, n_i)^\tau + \lambda_j \beta^\tau(n_j, m_j)^\tau = 0$, which implies $\lambda_i = \lambda_j$. Thus $(Ln_i, Lm_i) = \lambda(n_i, m_i)^\tau, i = 1, 2, \ldots, k$.

iii) If $U = \{0\}$, we are done; thus assume $U \neq \{0\}$. For non-zero $u \in U$, define $c = \gamma u + n_i$, where $\gamma^\tau = -2(Lu, Ln_i)(Lu, Lu)^{-1}$. Since Lc is null, c is null, implying that $\gamma^2 = 0$. Thus $\gamma^\tau = 0$ and $(Lu, Ln_i) = 0$. Similarly, $(Lu, Lm_i) = 0$.

If $\delta = -\frac{1}{2}(u, u)(n_i, m_i)^{-1}$ for non-zero $u \in U$, the vector $d = u + n_i + \delta m_i$ is null. Then Ld is null: $(Lu, Lu) + 2\delta^\tau(Ln_i, Lm_i) = 0$, which implies via i) and ii), $(Lu, Lu) = \lambda(u, u)^\tau$.

For distinct $v, w \in V$, $u = v + w$ satisfies $(Lu, Lu) = \lambda(u, u)^\tau$; this yields $(Lv, Lw) = \lambda(v, w)^\tau$.

Parts i), ii) and iii) prove our result.

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